ECE 3793
Test 2
Friday, December 5, 2003
7:00 PM - 10:00 PM

Fall 2003
Dr. Havlicek

Name: SOLUTION
Student Num: _______________________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are four problems. Work all four.

- You are allowed to use the separate formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) _______
2. (25) _______
3. (25) _______
4. (25) _______

TOTAL (100): 

_________________________________
1. **25 pts.** Suppose \( x(t) \) is a cable television signal that is transmitted from Atlanta to Oklahoma City via a satellite communications channel (a network "feed" that supplies the signal to the local cable franchise). Let \( r(t) \) be the signal that is actually received in Oklahoma City.

Because of cosmic rays, solar radiation, water vapor in the lower atmosphere, and other factors, the signal is **distorted** during transmission. Thus, the received signal \( r(t) \) is not equal to the transmitted signal \( x(t) \). Suppose that the distortion can be modeled according to

\[
r(t) = x(t) * d(t)
\]

where

\[
d(t) = \delta(t) + 2e^{-4t}u(t).
\]

Since this distortion is highly undesirable, you have been hired to design an LTI system \( H \) that can be used in Oklahoma City to recover the original transmitted signal from the received signal.

Therefore, you must design a system \( H \) that will input the received signal \( r(t) \) and output a signal \( y(t) \) that is equal to the original transmitted signal \( x(t) \).

For your designed system \( H \), give both the frequency response \( H(\omega) \) and the impulse response \( h(t) \).

\[
\begin{align*}
\chi(t) & \rightarrow \boxed{D} \rightarrow r(t) \rightarrow \boxed{H} \rightarrow y(t) = \chi(t).
\end{align*}
\]

To "undo" the distortion, we need \( H \) to be the inverse system of \( D \). That is, \( H(\omega) = \frac{1}{D(\omega)} \).

\[
D(\omega) = \mathcal{F}\left\{ \delta(t) + 2e^{-4t}u(t) \right\} \xrightarrow{\text{Table}} \frac{1}{1 + \frac{2}{4+j\omega}}
\]

\[
H(\omega) = \frac{1}{D(\omega)} = \frac{1}{\frac{1}{1 + \frac{2}{4+j\omega}}} = \frac{1}{4+j\omega} \cdot \frac{4+j\omega}{4+j\omega} = \frac{4+j\omega}{4+j\omega+2} = \frac{4+j\omega}{6+j\omega}
\]

\[
H(\omega) = \frac{4+j\omega}{6+j\omega}
\]
More Work Space for Problem 1...

\[ H(w) = \frac{4}{6 + jw} + \frac{jw}{6 + jw} \]

\[ = 4 \frac{1}{6 + jw} + jw \frac{1}{6 + jw} \]

Table: \( \frac{1}{6 + jw} \xrightarrow{\mathcal{F}} e^{-6t}u(t) \)

Property: \( jw - \frac{1}{6 + jw} \xrightarrow{\mathcal{F}} \frac{d}{dt} \left[ e^{-6t}u(t) \right] \)

\[ = \left[ \frac{d}{dt} e^{-6t} \right] u(t) + e^{-6t} \left[ \frac{d}{dt} u(t) \right] \]

\[ = -6e^{-6t}u(t) + e^{-6t} \delta(t) \]

\[ = e^{-6t} \bigg|_{t=0} \delta(t) - 6e^{-6t}u(t) \]

\[ = \delta(t) - 6e^{-6t}u(t) \]

\[ h(t) = 4e^{-6t}u(t) + \delta(t) - 6e^{-6t}u(t) \]

\[ = \delta(t) - 2e^{-6t}u(t) \]
2. **25 pts.** A continuous-time LTI system $H$ has input $x(t)$ and output $y(t)$ related by the differential equation

$$\sum_{i=0}^{N} a_i \frac{d^i}{dt^i} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t),$$

where $N$, $M$, $a_i$, and $b_k$ are real constants such that $N > M$.

(a) **8 pts.** Find the system frequency response $H(\omega)$.

$$\mathcal{F}\left\{ \sum_{i=0}^{N} a_i \frac{d^i}{dt^i} y(t) \right\} = \mathcal{F}\left\{ \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \right\}$$

$$\sum_{i=0}^{N} a_i \mathcal{F}\left\{ \frac{d^i}{dt^i} y(t) \right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{ \frac{d^k}{dt^k} x(t) \right\}$$

**derivative property:** $\sum_{i=0}^{N} a_i (j\omega)^i Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$

$$Y(\omega) \sum_{i=0}^{N} a_i (j\omega)^i = X(\omega) \sum_{k=0}^{M} b_k (j\omega)^k$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{i=0}^{N} a_i (j\omega)^i}$$
Problem 2, cont...

(b) 8 pts. Now consider the special case where \( N = 2, a_0 = 8, a_1 = 6, a_2 = 1, \) \( M = 1, b_0 = 3, \) and \( b_1 = 1. \) Find \( H(\omega) \) and the impulse response \( h(t). \)

From part (a): 
\[
H(\omega) = \frac{\sum_{k=0}^{2} b_k (j\omega)^k}{\sum_{i=0}^{2} a_i (j\omega)^i} = \frac{b_0 + b_1 (j\omega)}{a_0 + a_1 (j\omega) + a_2 (j\omega)^2}
\]

\[
= \frac{3 + j\omega}{8 + 6j\omega + (j\omega)^2}
\]

\[
H(\omega) = \frac{3 + j\omega}{(j\omega+4)(j\omega+2)}
\]

\[
A = \frac{3 - 4}{2 - 4} = \frac{-1}{-2} = \frac{1}{2}
\]

\[
\frac{3 + \Theta}{(4+\Theta)(2+\Theta)} = \frac{A}{4+\Theta} + \frac{B}{2+\Theta} \quad B = \frac{3 - 2}{4 - 2} = \frac{1}{2}
\]

\[
H(\omega) = \frac{1}{2} \frac{1}{4+j\omega} + \frac{1}{2} \frac{1}{2+j\omega}
\]

\[
h(t) \xrightarrow{\text{Table}} \frac{1}{2} e^{-4t} u(t) + \frac{1}{2} e^{-2t} u(t)
\]
Problem 2, cont...

(c) 9 pts. For the system in part (b), find the output $y(t)$ when the input is given by $x(t) = e^{-2t}u(t) - e^{-3t}u(t)$.

\[
X(\omega) \overset{\text{Table}}{=} \frac{1}{2+j\omega} - \frac{1}{3+j\omega} = \frac{3+j\omega - (2+j\omega)}{(2+j\omega)(3+j\omega)} = \frac{1}{(2+j\omega)(3+j\omega) - (2+j\omega)}
\]

\[
Y(\omega) = X(\omega)H(\omega) = \frac{1}{(2+j\omega)(3+j\omega)} \cdot \frac{3+j\omega}{(2+j\omega)(4+j\omega)} = \frac{1}{(4+j\omega)(2+j\omega)^2}
\]

\[
\frac{1}{(4+j\omega)(2+j\omega)^2} = \frac{A}{4+\theta} + \frac{B}{(2+\theta)^2} + \frac{C}{2+\theta}
\]

\[
A = \frac{1}{(2-4)^2} = \frac{1}{(-2)^2} = \frac{1}{4}, \quad B = \frac{1}{4-2} = \frac{1}{2}
\]

\[
\frac{d}{d\theta} \left[ (4+\theta)^{-1} \right]_{\theta=-2} = \frac{d}{d\theta} (2+\theta) C \bigg|_{\theta=-2} = (4-2)^{-2} = -\frac{1}{4}
\]

\[
\gamma(\omega) = \frac{1}{4} \frac{1}{4+j\omega} + \frac{1}{2} \frac{1}{(2+j\omega)^2} - \frac{1}{4} \frac{1}{2+j\omega}
\]

\[
y(t) \overset{\text{Table}}{=} \frac{1}{4} e^{-4t}u(t) + \frac{1}{2} t e^{-2t}u(t) - \frac{1}{4} e^{-2t}u(t)
\]
3. **25 pts.** A causal LTI system $F$ has impulse response

$$f(t) = \frac{1}{4} e^t u(t) - \frac{1}{4} e^{-3t} u(t).$$

(a) **3 pts.** Find the transfer function $F(s)$ (be sure to specify the ROC). Give a pole-zero plot for $F(s)$.

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{4} \left[ \frac{1}{s-1} - \frac{1}{s+3} \right]_{\text{Re}[s] > 1} \quad \frac{1}{4} \left[ \frac{1}{s-1} - \frac{1}{s+3} \right]_{\text{Re}[s] > 1}$$

$$= \frac{1}{4} \left[ \frac{s+3 - (s-1)}{(s-1)(s+3)} \right] = \frac{4}{(s-1)(s+3)} = \frac{1}{(s-1)(s+3)}_{\text{Re}[s] > 1}$$

(b) **2 pts.** Is the system $F$ BIBO stable? Justify your answer.

**NO.** A causal LTI system with rational transfer function is BIBO stable iff all poles of the system transfer func. are in the left half of the $s$-plane. This system has a right half-plane pole at $s=1$.

(c) **2 pts.** Does $f(t)$ have a Fourier transform? Justify your answer.

**NO.** The ROC of $F(s)$ does not include the $j\omega$-axis. Since $F(w)$ is equal to $F(s)$ evaluated along the $j\omega$-axis, this means that $F(w)$ is divergent.
(d) **11 pts.** A new system $H$ is formed by adding negative feedback to $F$ as shown in the figure below. $H$ is both LTI and causal.

![Diagram of system](image)

System $G$ is LTI and causal and has transfer function $G(s) = 4$. Find the transfer function $H(s)$ of the overall system $H$.

$$H(s) = \frac{F(s)}{1 + F(s)G(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1 + 4 \frac{1}{(s-1)(s+3)}} \cdot \frac{(s-1)(s+3)}{(s-1)(s+3)}$$

$$= \frac{1}{(s-1)(s+3) + 4} = \frac{1}{s^2 + 2s - 3 + 4}$$

$$= \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}$$

$$\boxed{H(s) = \frac{1}{(s+1)^2}}$$
Problem 3, cont...

(e) 3 pts. Give a pole-zero plot for \( H(s) \). What is the ROC of \( H(s) \)? Justify your answer.

\[ \operatorname{ROC}: \Re\{s\} > -1. \]

Because this is a causal LTI system with a rational transfer function, the ROC is the right half-plane to the right of the rightmost pole.

(f) 2 pts. Is the system \( H \) BIBO stable? Justify your answer.

Yes; addition of the feedback moved the poles so that they are now both in the left half-plane. A causal LTI system with rational transfer fn. that has all poles in the LHP is BIBO stable.

(g) 2 pts. Consider the system impulse response \( h(t) \). Does \( h(t) \) have a Fourier transform? Justify your answer.

Yes. The Fourier transform \( H(w) \) is equal to the Laplace transform \( H(s) \) evaluated along the line \( \Re\{s\} = 0 \) (\( jw \)-axis).

Since this line is included in the ROC of \( H(s) \), \( H(w) \) is convergent.
4. 25 pts. Consider the causal, stable LTI system $H$ shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.

(a) 15 pts. Find the system transfer function $H(s)$.

KVL around outside loop: $x(t) = 4i_1(t) + u(t) + y(t)$ — (1)

KVL around right loop: $v_c(t) = u_c(t) + y(t)$ — (2)

2Ω: $y(t) = 2i_2(t)$ $\Rightarrow$ $i_2(t) = \frac{1}{2} y(t)$ — (3)

$L$: $u_c(t) = 2i_2'(t) = 2 \cdot \frac{1}{2} y'(t) = y'(t)$ — (4)

$\Rightarrow$ (2): $v_c'(t) = y''(t) + y'(t)$ $\Rightarrow$ $v_c'(t) = y''(t) + y'(t)$ — (5)

KCL at A: $i_1(t) = i_c(t) + i_2(t)$ $\Rightarrow$ $i_c(t) = i_1(t) - i_2(t)$ — (6)

$C$: $i_c(t) = Cu_c'(t) = v_c'(t)$

plug in (6) and (5): $i_1(t) - i_2(t) = y''(t) + y'(t)$

plug in (3): $i_1(t) = y''(t) + y'(t) + \frac{1}{2} y(t)$ — (7)

$\Rightarrow$ (4, 7) $\Rightarrow$ (1): $x(t) = 4y''(t) + 4y'(t) + 2y(t) + y'(t) + y(t)$

$= 4y''(t) + 5y'(t) + 3y(t)$

$L$: $X(s) = 4s^2Y(s) + 5sY(s) + 3Y(s) = [4s^2 + 5s + 3]Y(s)$

$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{4s^2 + 5s + 3}$
Problem 4, cont…

(b) 10 pts. Consider another causal, stable LTI system $G$ with impulse response

$$g(t) = \frac{1}{4} \cdot \frac{8}{\sqrt{23}} \cdot e^{-\frac{5}{8}t} \sin \left( \frac{\sqrt{23}}{8} t \right) u(t).$$

Find the transfer function $G(s)$ and show that it is the same as the frequency response $H(s)$ obtained in part (a); this shows that $g(t)$ above is the impulse response of the system in part (a).

Table: $\alpha = \frac{5}{8}, \quad \omega_0 = \frac{\sqrt{23}}{8}$

$$G(s) = \frac{1}{4} \cdot \frac{8}{\sqrt{23}} \cdot \frac{\sqrt{23}/8}{(s + \frac{5}{8})^2 + (\frac{\sqrt{23}}{8})^2}$$

$$= \frac{1}{4} \cdot \frac{1}{s^2 + \frac{5}{4}s + \frac{25}{64}}$$

$$= \frac{1}{4} \cdot \frac{1}{s^2 + \frac{5}{4}s + \frac{3}{4}}$$

$$= \frac{1}{4s^2 + 5s + 3} \checkmark$$