ECE 3793
Test 2

Friday, December 5, 2003
7:00 PM - 10:00 PM

Fall 2003 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work all **four**.

▸ You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

1. **25 pts.** Suppose $x(t)$ is a cable television signal that is transmitted from Atlanta to Oklahoma City via a satellite communications channel (a network "feed" that supplies the signal to the local cable franchise). Let $r(t)$ be the signal that is actually received in Oklahoma City.

Because of cosmic rays, solar radiation, water vapor in the lower atmosphere, and other factors, the signal is **distorted** during transmission. Thus, the received signal $r(t)$ is not equal to the transmitted signal $x(t)$. Suppose that the distortion can be modeled according to

$$r(t) = x(t) * d(t)$$

where

$$d(t) = \delta(t) + 2e^{-4t}u(t).$$

Since this distortion is highly undesirable, you have been hired to design an LTI system H that can be used in Oklahoma City to recover the original transmitted signal from the received signal.

Therefore, you must design a system H that will input the received signal $r(t)$ and output a signal $y(t)$ that is equal to the orignal transmitted signal $x(t)$.

For your designed system H, give both the frequency response $H(\omega)$ and the impulse response $h(t)$.

$$x(t) \longrightarrow \boxed{D} \longrightarrow r(t) \longrightarrow \boxed{H} \longrightarrow y(t) = x(t).$$

- To "undo" the distortion, we need H to be the inverse system of D. That is, $H(\omega) = 1/D(\omega)$.

$$D(\omega) = \mathcal{F}\{\delta(t) + 2e^{-4t}u(t)\} \overset{\text{Table}}{=\!=} 1 + \frac{2}{4+j\omega}$$

$$H(\omega) = \frac{1}{D(\omega)} = \frac{1}{1 + \frac{2}{4+j\omega}} \cdot \frac{4+j\omega}{4+j\omega} = \frac{4+j\omega}{4+j\omega+2} = \frac{4+j\omega}{6+j\omega}$$

$$\boxed{H(\omega) = \frac{4+j\omega}{6+j\omega}}$$

More Work Space for Problem 1...

$$H(\omega) = \frac{4}{6+j\omega} + \frac{j\omega}{6+j\omega}$$

$$= 4\,\frac{1}{6+j\omega} + j\omega\,\frac{1}{6+j\omega}$$

Table: $\frac{1}{6+j\omega} \overset{\mathcal{F}}{\longleftrightarrow} e^{-6t}u(t)$

Property: $j\omega \cdot \frac{1}{6+j\omega} \overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{dt}\left[e^{-6t}u(t)\right]$

$$= \left[\frac{d}{dt}e^{-6t}\right]u(t) + e^{-6t}\left[\frac{d}{dt}u(t)\right]$$

$$= -6e^{-6t}u(t) + e^{-6t}\delta(t)$$

$$= e^{-6t}\Big|_{t=0}\delta(t) - 6e^{-6t}u(t)$$

$$= \delta(t) - 6e^{-6t}u(t)$$

$$\boxed{\begin{aligned} \Rightarrow\ h(t) &= 4e^{-6t}u(t) + \delta(t) - 6e^{-6t}u(t) \\ &= \delta(t) - 2e^{-6t}u(t) \end{aligned}}$$

2. **25 pts.** A continuous-time LTI system H has input $x(t)$ and output $y(t)$ related by the differential equation

$$\sum_{i=0}^{N} a_i \frac{d^i}{dt^i} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t),$$

where N, M, a_i, and b_k are real constants such that $N > M$.

(a) **8 pts.** Find the system frequency response $H(\omega)$.

$$\mathcal{F}\left\{\sum_{i=0}^{N} a_i \frac{d^i}{dt^i} y(t)\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)\right\}$$

$$\sum_{i=0}^{N} a_i \mathcal{F}\left\{\frac{d^i}{dt^i} y(t)\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k}{dt^k} x(t)\right\}$$

derivative property: $\sum_{i=0}^{N} a_i (j\omega)^i Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$

$$Y(\omega) \sum_{i=0}^{N} a_i (j\omega)^i = X(\omega) \sum_{k=0}^{M} b_k (j\omega)^k$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{i=0}^{N} a_i (j\omega)^i}}$$

Problem 2, cont...

(b) **8 pts.** Now consider the special case where $N = 2$, $a_0 = 8$, $a_1 = 6$, $a_2 = 1$, $M = 1$, $b_0 = 3$, and $b_1 = 1$. Find $H(\omega)$ and the impulse response $h(t)$.

From part (a): $H(\omega) = \dfrac{\sum_{k=0}^{1} b_k (j\omega)^k}{\sum_{i=0}^{2} a_i (j\omega)^i} = \dfrac{b_0 + b_1 (j\omega)}{a_0 + a_1(j\omega) + a_2(j\omega)^2}$

$$= \frac{3 + j\omega}{8 + 6j\omega + (j\omega)^2}$$

$$\boxed{H(\omega) = \frac{3 + j\omega}{(j\omega + 4)(j\omega + 2)}}$$

$$A = \frac{3-4}{2-4} = \frac{-1}{-2} = \frac{1}{2}$$

$$\frac{3+\theta}{(4+\theta)(2+\theta)} = \frac{A}{4+\theta} + \frac{B}{2+\theta} \qquad B = \frac{3-2}{4-2} = \frac{1}{2}$$

$$H(\omega) = \frac{1}{2}\,\frac{1}{4+j\omega} + \frac{1}{2}\,\frac{1}{2+j\omega}$$

$$\boxed{h(t) \overset{\text{Table}}{=} \frac{1}{2} e^{-4t} u(t) + \frac{1}{2} e^{-2t} u(t)}$$

Problem 2, cont...

(c) **9 pts.** For the system in part (b), find the output $y(t)$ when the input is given by $x(t) = e^{-2t}u(t) - e^{-3t}u(t)$.

$$X(\omega) \overset{\text{Table}}{=\!=} \frac{1}{2+j\omega} - \frac{1}{3+j\omega} = \frac{3+j\omega - (2+j\omega)}{(2+j\omega)(3+j\omega)} = \frac{1}{(2+j\omega)(3+j\omega)}$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(2+j\omega)(3+j\omega)} \cdot \frac{3+j\omega}{(2+j\omega)(4+j\omega)} = \frac{1}{(4+j\omega)(2+j\omega)^2}$$

$$\frac{1}{(4+\theta)(2+\theta)^2} = \frac{A}{4+\theta} + \frac{B}{(2+\theta)^2} + \frac{C}{2+\theta}$$

$$A = \frac{1}{(2-4)^2} = \frac{1}{(-2)^2} = \frac{1}{4}; \quad B = \frac{1}{4-2} = \frac{1}{2}$$

$$\frac{\partial}{\partial\theta}\left[(4+\theta)^{-1}\right]_{\theta=-2} = \frac{\partial}{\partial\theta}(2+\theta)C\Big|_{\theta=-2}$$

$$-(4+\theta)^{-2}\Big|_{\theta=-2} = C = -(4-2)^{-2} = -(2)^{-2} = -\frac{1}{4}$$

$$Y(\omega) = \frac{1}{4}\frac{1}{4+j\omega} + \frac{1}{2}\frac{1}{(2+j\omega)^2} - \frac{1}{4}\frac{1}{2+j\omega}$$

$$\boxed{y(t) \overset{\text{Table}}{=\!=} \frac{1}{4}e^{-4t}u(t) + \frac{1}{2}te^{-2t}u(t) - \frac{1}{4}e^{-2t}u(t)}$$

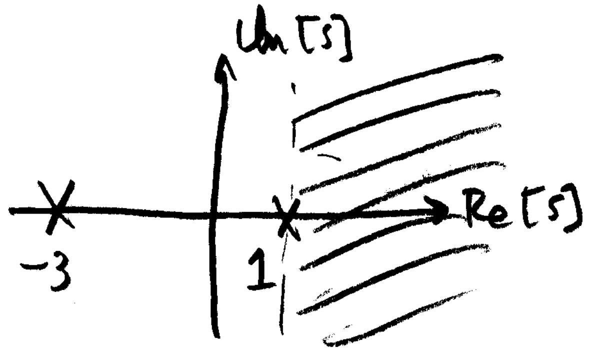
3. **25 pts.** A causal LTI system F has impulse response

$$f(t) = \tfrac{1}{4}e^{t}u(t) - \tfrac{1}{4}e^{-3t}u(t).$$

(a) **3 pts.** Find the transfer function $F(s)$ (be sure to specify the ROC). Give a pole-zero plot for $F(s)$.

$$F(s) = \mathcal{L}\{f(t)\} = \underbrace{\frac{1}{4}\frac{1}{s-1}}_{Re[s]>1} - \underbrace{\frac{1}{4}\frac{1}{s+3}}_{Re[s]>-3} = \underbrace{\frac{1}{4}\left[\frac{1}{s-1} - \frac{1}{s+3}\right]}_{\{Re[s]>1\}\cap\{Re[s]>-3\} = Re[s]>1}$$

$$= \frac{1}{4}\left[\frac{s+3 - (s-1)}{(s-1)(s+3)}\right] = \frac{1}{4}\frac{4}{(s-1)(s+3)} = \frac{1}{(s-1)(s+3)}, \quad Re[s] > 1$$



(b) **2 pts.** Is the system F BIBO stable? Justify your answer.

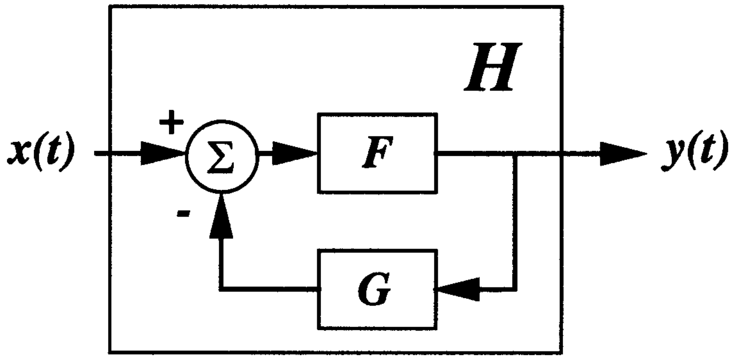
NO. A causal LTI system with rational transfer function is BIBO stable iff all poles of the system transfer fcn. are in the left half of the s-plane. This system has a right half-plane pole at $s=1$.

(c) **2 pts.** Does $f(t)$ have a Fourier transform? Justify your answer.

NO. The ROC of $F(s)$ does not include the $j\omega$-axis. Since $F(\omega)$ is equal to $F(s)$ evaluated along the $j\omega$-axis, this means that $F(\omega)$ is divergent.

Problem 3, cont...

(d) **11 pts.** A new system H is formed by adding negative feedback to F as shown in the figure below. H is both LTI and causal.



System G is LTI and causal and has transfer function $G(s) = 4$. Find the transfer function $H(s)$ of the overall system H.

$$H(s) = \frac{F(s)}{1+F(s)G(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1 + 4\frac{1}{(s-1)(s+3)}} \cdot \frac{(s-1)(s+3)}{(s-1)(s+3)}$$

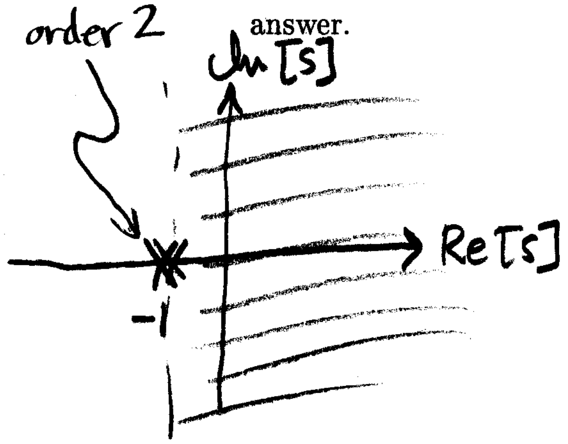
$$= \frac{1}{(s-1)(s+3)+4} = \frac{1}{s^2+2s-3+4}$$

$$= \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2}$$

$$\boxed{H(s) = \frac{1}{(s+1)^2}}$$

Problem 3, cont...

(e) **3 pts.** Give a pole-zero plot for $H(s)$. What is the ROC of $H(s)$? Justify your answer.



ROC: $Re[s] > -1$.

Because this is a causal LTI system with a rational transfer function, the ROC is the right half-plane to the right of the rightmost pole.

(f) **2 pts.** Is the system H BIBO stable? Justify your answer.

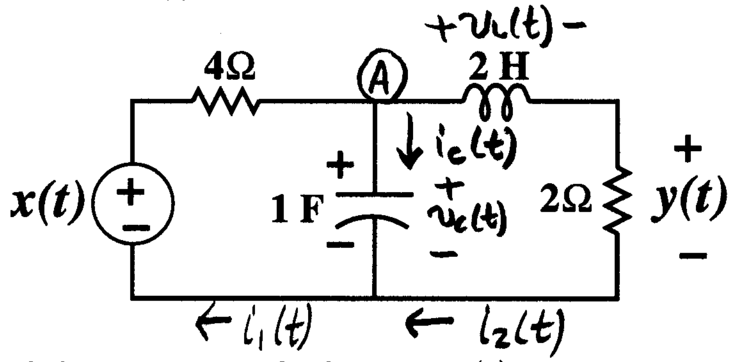
Yes; addition of the feedback moved the poles so that they are now both in the left half-plane. A causal LTI system with rational transfer fcn. that has all poles in the LHP is BIBO stable.

(g) **2 pts.** Consider the system impulse response $h(t)$. Does $h(t)$ have a Fourier transform? Justify your answer.

Yes. The Fourier transform $H(\omega)$ is equal to the Laplace transform $H(s)$ evaluated along the line $Re[s]=0$ ($j\omega$-axis).

Since this line is included in the ROC of $H(s)$, $H(\omega)$ is convergent.

4. **25 pts**. Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.



(a) **15 pts**. Find the system transfer function $H(s)$.

KVL around outside loop: $x(t) = 4i_1(t) + v_L(t) + y(t)$ — ①

KVL around right loop: $v_c(t) = v_L(t) + y(t)$ — ②

2Ω : $y(t) = 2i_2(t) \Rightarrow i_2(t) = \frac{1}{2}y(t)$ — ③

L: $v_L(t) = 2i_2'(t) = 2\cdot\frac{1}{2}y'(t) = y'(t)$ — ④

④→②: $v_c(t) = y'(t) + y(t) \Rightarrow v_c'(t) = y''(t) + y'(t)$ — ⑤

KCL at Ⓐ: $i_1(t) = i_c(t) + i_2(t) \Rightarrow i_c(t) = i_1(t) - i_2(t)$ — ⑥

C: $i_c(t) = Cv_c'(t) = v_c'(t)$

plug in ⑥ and ⑤: $i_1(t) - i_2(t) = y''(t) + y'(t)$

$i_1(t) = y''(t) + y'(t) + i_2(t)$

plug in ③ : $i_1(t) = y''(t) + y'(t) + \frac{1}{2}y(t)$ — ⑦

④,⑦→①: $x(t) = 4y''(t) + 4y'(t) + 2y(t) + y'(t) + y(t)$

$= 4y''(t) + 5y'(t) + 3y(t)$

$\mathcal{L}$: $X(s) = 4s^2Y(s) + 5sY(s) + 3Y(s) = [4s^2+5s+3]Y(s)$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{1}{4s^2+5s+3}}$$

Problem 4, cont...

(b) **10 pts.** Consider another causal, stable LTI system G with impulse response

$$g(t) = \tfrac{1}{4} \cdot \tfrac{8}{\sqrt{23}} \cdot e^{-\frac{5}{8}t} \sin\left(\tfrac{\sqrt{23}}{8}t\right) u(t).$$

Find the transfer function $G(s)$ and show that it is the same as the frequency response $H(s)$ obtained in part (a); this shows that $g(t)$ above is the impulse response of the system in part (a).

Table: $\alpha = 5/8$, $\omega_0 = \sqrt{23}/8$

$$G(s) = \frac{1}{4} \cdot \frac{8}{\sqrt{23}} \cdot \frac{\sqrt{23}/8}{(s+\frac{5}{8})^2 + (\frac{\sqrt{23}}{8})^2} = \frac{1}{4} \frac{1}{s^2 + \frac{10}{8}s + \frac{25}{64} + \frac{23}{64}}$$

$$= \frac{1}{4} \frac{1}{s^2 + \frac{5}{4}s + \frac{48}{64}} = \frac{1}{4} \frac{1}{s^2 + \frac{5}{4}s + \frac{3}{4}}$$

$$= \frac{1}{4s^2 + 5s + 3} \quad \checkmark$$