ECE 3793
Test 2

Wednesday, December 1, 2004
7:00 PM - 10:00 PM

Fall 2004 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**.

▶ You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

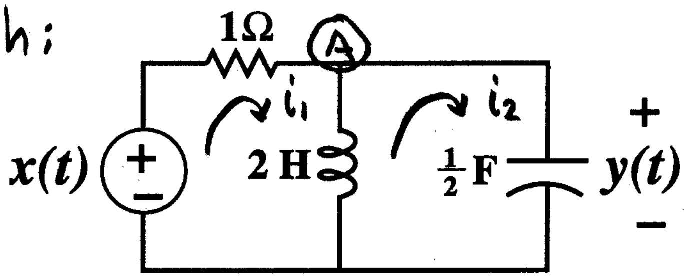
On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.

KVL on right mesh:

$v_L(t) = y(t)$ (1)



KCL at (A):

$i_L(t) = i_1(t) - i_2(t)$ (2)

(a) **10 pts.** Find the system frequency response $H(\omega)$.

R: $v_R(t) = R\,i_R(t) = i_1(t)$ (3)

L: $v_L(t) = L\,i_L'(t)$ (4) ; (1),(2) → (4): $y(t) = 2[i_1'(t) - i_2'(t)]$

$= 2i_1'(t) - 2i_2'(t)$ (5)

KVL on big loop: $v_R(t) + y(t) = x(t)$ (6)

(3) → (6) : $y(t) + i_1(t) = x(t)$ (7)

differentiate (7): $y'(t) + i_1'(t) = x'(t)$ (8)

Solve (5) for $i_1'(t)$: $i_1'(t) = \frac{1}{2}y(t) + i_2'(t)$ (9)

(9) → (8) : $y'(t) + \frac{1}{2}y(t) + i_2'(t) = x'(t)$ (10)

C: $i_c(t) = C\,v_c'(t)$: $i_2(t) = \frac{1}{2}y'(t)$ (11)

differentiate (11): $i_2'(t) = \frac{1}{2}y''(t)$ (12)

(12) → (10) : $y'(t) + \frac{1}{2}y(t) + \frac{1}{2}y''(t) = x'(t)$

$y''(t) + 2y'(t) + y(t) = 2x'(t)$ (13)

$\mathcal{F}[(13)]$: $(j\omega)^2 Y(\omega) + 2j\omega Y(\omega) + Y(\omega) = 2j\omega X(\omega)$

$[(j\omega)^2 + 2j\omega + 1]\,Y(\omega) = 2j\omega X(\omega)$ (14)

SOLVE (14) : $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2j\omega}{(j\omega)^2 + 2j\omega + 1} = \frac{2j\omega}{(j\omega+1)^2}$

Problem 1, cont...

(b) **5 pts.** Find the system impulse response $h(t)$.

$$H(\omega) = \frac{2\theta}{(\theta+1)^2} = \frac{A}{(\theta+1)^2} + \frac{B}{\theta+1}$$

$$A = 2\theta \Big|_{\theta=-1} = -2$$

$$\left[\frac{\partial}{\partial\theta} 2\theta\right]_{\theta=-1} = \left[\frac{\partial}{\partial\theta} A\right]_{\theta=-1} + \left[\frac{\partial}{\partial\theta}(\theta+1)B\right]_{\theta=-1}$$

$$2 = 0 + B \Rightarrow B = 2$$

$$H(\omega) = \frac{2}{j\omega+1} - \frac{2}{(j\omega+1)^2}$$

TABLE: $\boxed{h(t) = 2e^{-t}u(t) - 2te^{-t}u(t)}$

Problem 1, cont...

(c) **10 pts.** Let the system input be $x(t) = \delta(t) - 2e^{-3t}u(t)$. Find the output $y(t)$.

TABLE: $X(\omega) = 1 - \frac{2}{j\omega+3} = \frac{j\omega+3}{j\omega+3} - \frac{2}{j\omega+3} = \frac{j\omega+1}{j\omega+3}$

$Y(\omega) = X(\omega)H(\omega) = \frac{j\omega+1}{j\omega+3} \cdot \frac{2j\omega}{(j\omega+1)^2} = \frac{2j\omega}{(j\omega+3)(j\omega+1)}$

PFE: $\frac{2\theta}{(\theta+3)(\theta+1)} = \frac{A}{\theta+3} + \frac{B}{\theta+1}$

$A = \left.\frac{2\theta}{\theta+1}\right|_{\theta=-3} = \frac{-6}{-2} = 3$

$B = \left.\frac{2\theta}{\theta+3}\right|_{\theta=-1} = \frac{-2}{2} = -1$

$Y(\omega) = \frac{3}{j\omega+3} - \frac{1}{j\omega+1}$

TABLE: $\boxed{y(t) = 3e^{-3t}u(t) - e^{-t}u(t)}$

2. **25 pts.** A discrete-time LTI system H has impulse response $h[n] = (1-n)\left(\frac{1}{4}\right)^{-n} u[-n]$.

(a) **7 pts.** Find the system frequency response $H(e^{j\omega})$.

TABLE: $(1+n)\left(\frac{1}{4}\right)^{n} u[n] \xleftrightarrow{\mathcal{F}} \dfrac{1}{(1-\frac{1}{4}e^{-j\omega})^2}$

Time Reversal: $(1-n)\left(\frac{1}{4}\right)^{-n} u[-n] \xleftrightarrow{\mathcal{F}} \boxed{\dfrac{1}{(1-\frac{1}{4}e^{j\omega})^2} = H(e^{j\omega})}$

NOTE: $H(e^{j\omega})\dfrac{e^{-j2\omega}}{e^{-j2\omega}} = \dfrac{e^{-j2\omega}}{(e^{-j\omega}-\frac{1}{4})^2} = \dfrac{16e^{-j2\omega}}{(4e^{-j\omega}-1)^2} = \dfrac{16e^{-j2\omega}}{(1-4e^{-j\omega})^2}$

(b) **6 pts.** Find the difference equation that relates the system input $x[n]$ and system output $y[n]$. Two equivalent solutions based on the two forms of $H(e^{j\omega})$ shown above:

$H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1}{(1-\frac{1}{4}e^{j\omega})^2}$

$(1-\frac{1}{4}e^{j\omega})^2\, Y(e^{j\omega}) = X(e^{j\omega})$

$[1-\frac{1}{2}e^{j\omega}+\frac{1}{16}e^{2j\omega}]\,Y(e^{j\omega}) = X(e^{j\omega})$

$Y(e^{j\omega})-\frac{1}{2}e^{j\omega}Y(e^{j\omega})+\frac{1}{16}e^{j2\omega} = X(e^{j\omega})$

$\underline{y[n]-\frac{1}{2}y[n+1]+\frac{1}{16}y[n+2] = x[n]}$

$H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{16e^{-j2\omega}}{(1-4e^{-j\omega})^2}$

$(1-4e^{-j\omega})^2\,Y(e^{j\omega}) = 16e^{-j2\omega}X(e^{j\omega})$

$[1-8e^{-j\omega}+16e^{-j2\omega}]\,Y(e^{j\omega}) = 16e^{-j2\omega}X(e^{j\omega})$

$Y(e^{j\omega})-8e^{-j\omega}Y(e^{j\omega})+16e^{-j2\omega}Y(e^{j\omega}) = 16e^{-j2\omega}X(e^{j\omega})$

$\underline{y[n]-8y[n-1]+16y[n-2] = 16x[n-2]}$

Problem 2, cont...

(c) **6 pts.** Suppose that the system output is given by

$$y[n] = \left(\frac{1}{2}n^2 - \frac{3}{2}n + 1\right) 4^n u[-n] = \frac{(-n+3-1)!}{(-n)!(3-1)!}\left(\frac{1}{4}\right)^{-n} u[-n].$$

Find $X(e^{j\omega})$, the DTFT of the system input $x[n]$.

TABLE: $\frac{(n+3-1)!}{n!(3-1)!}\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(1-\frac{1}{4}e^{-j\omega})^3}$

Time Reversal: $\frac{(-n+3-1)!}{(-n)!(3-1)!}\left(\frac{1}{4}\right)^{-n} u[-n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(1-\frac{1}{4}e^{j\omega})^3} = Y(e^{j\omega})$

$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

$X(e^{j\omega}) = Y(e^{j\omega})\left[H(e^{j\omega})\right]^{-1} = \frac{1}{(1-\frac{1}{4}e^{j\omega})^3}(1-\frac{1}{4}e^{j\omega})^2$

$$\boxed{X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{j\omega}}}$$

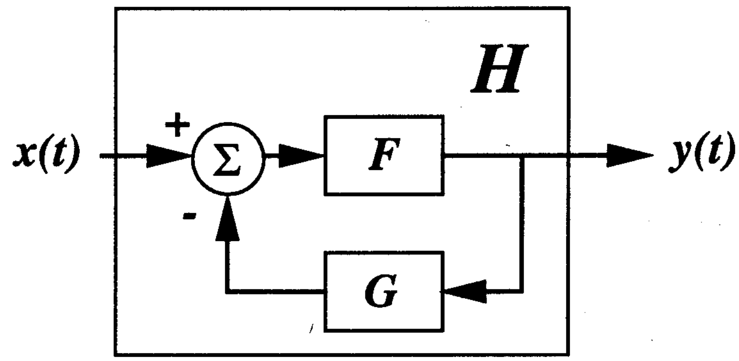
(d) **6 pts.** Find $x[n]$ by inverting the DTFT $X(e^{j\omega})$ that you found above in part (c).

TABLE: $\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}e^{-j\omega}}$

Time Reversal: $\left(\frac{1}{4}\right)^{-n} u[-n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}e^{j\omega}} = X(e^{j\omega})$

$$\boxed{x[n] = \left(\frac{1}{4}\right)^{-n} u[-n]}$$

3. **25 pts.** Consider a causal continuous-time LTI system H. As shown in the figure below, H is a negative feedback connection of two causal continuous-time LTI systems F and G. The impulse response of system F is given by $f(t) = 3e^{-4t}u(t)$.



The input-output relation for the overall system H is given by

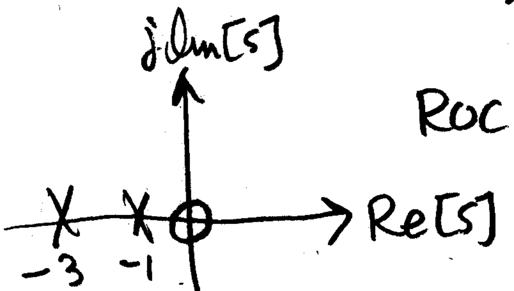
$$y''(t) + 4y'(t) + 3y(t) = 3x'(t).$$

(a) **5 pts.** Find the overall transfer function $H(s)$. Be sure to specify the ROC. Give a pole-zero plot for $H(s)$.

$$\mathcal{L}: \quad s^2 Y(s) + 4s Y(s) + 3Y(s) = 3s X(s)$$

$$[s^2 + 4s + 3] Y(s) = 3s X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s}{s^2+4s+3} = \frac{3s}{(s+1)(s+3)}$$



ROC: $Re[s] > -1$, because it is given that H is causal and we see that H(s) is rational. So the ROC is the right half-plane to the right of the rightmost pole.

(b) **2 pts.** Is the system H BIBO stable? Justify your answer.

Yes, because it is causal and the poles of H(s) are all in the left half-plane.

Problem 3, cont...

(c) **2 pts**. Does the overall system impulse response $h(t)$ have a Fourier transform? Justify your answer.

Yes, because the ROC of $H(s)$ includes the $j\omega$-axis, which is precisely where the Fourier and Laplace transforms coincide.

(d) **5 pts**. Use the inverse Laplace transform to find the impulse response $h(t)$.

$$H(s) = \frac{3s}{(s+1)(s+3)} = \underbrace{\frac{A}{s+1}}_{Re[s]>-1} + \underbrace{\frac{B}{s+3}}_{Re[s]>-3}$$

$$A = \left.\frac{3s}{s+3}\right|_{s=-1} = \frac{-3}{2} = -\frac{3}{2}$$

$$B = \left.\frac{3s}{s+1}\right|_{s=-3} = \frac{-9}{-2} = \frac{9}{2}$$

$$H(s) = \underbrace{\frac{9}{2}\frac{1}{s+3}}_{Re[s]>-3} - \underbrace{\frac{3}{2}\frac{1}{s+1}}_{Re[s]>-1}$$

Table: $\boxed{h(t) = \frac{9}{2}e^{-3t}u(t) - \frac{3}{2}e^{-t}u(t)}$

(e) **3 pts**. Find the transfer function $F(s)$ of system F. Be sure to specify the ROC.

$$f(t) = 3e^{-4t}u(t)$$

Table: $\boxed{F(s) = \frac{3}{s+4},\ Re[s] > -4}$

Problem 3, cont...

(f) **8 pts.** Find the impulse response $g(t)$ of system G.

$$H(s) = \frac{F(s)}{1+F(s)G(s)}$$

$$\frac{3s}{(s+1)(s+3)} = \frac{\frac{3}{s+4}}{1+\frac{3}{s+4}G(s)} \cdot \frac{s+4}{s+4} = \frac{3}{s+4+3G(s)}$$

cross multiply: $3s^2 + 12s + 9sG(s) = (s^2+4s+3)3$

$$= 3s^2 + 12s + 9$$

$$9sG(s) = 9$$

$$G(s) = \frac{1}{s}, \quad \underbrace{\mathrm{Re}[s] > 0}_{\text{because } G \text{ is given to be causal}}$$

Table: $\boxed{g(t) = u(t)}$

4. **25 pts**. A discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by the linear constant coefficient difference equation

$$y[n] + \frac{1}{2}y[n-1] - \frac{3}{16}y[n-2] = x[n-1] - \frac{1}{2}x[n-2].$$

(a) **6 pts**. Find the transfer function $H(z)$.

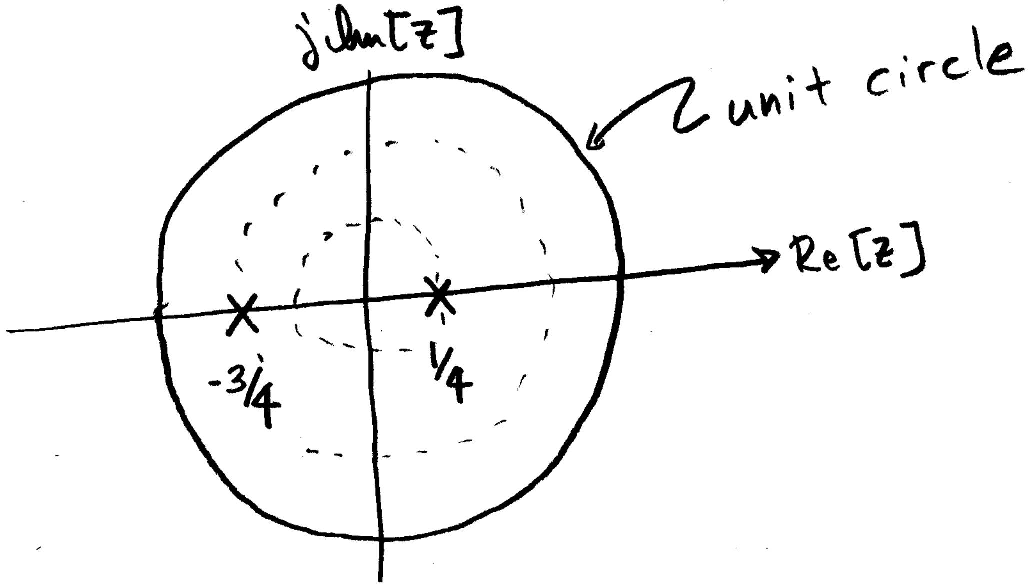
$$\mathcal{Z}: \quad Y(z) + \frac{1}{2}z^{-1}Y(z) - \frac{3}{16}z^{-2}Y(z) = z^{-1}X(z) - \frac{1}{2}z^{-2}X(z)$$

$$\left[1 + \frac{1}{2}z^{-1} - \frac{3}{16}z^{-2}\right]Y(z) = \left[z^{-1} - \frac{1}{2}z^{-2}\right]X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{2}z^{-1} - \frac{3}{16}z^{-2}} = \frac{z^{-1}(1 - \frac{1}{2}z^{-1})}{(1 + \frac{3}{4}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

$$\boxed{H(z) = \frac{z^{-1}(1 - \frac{1}{2}z^{-1})}{(1 + \frac{3}{4}z^{-1})(1 - \frac{1}{4}z^{-1})}}$$

(b) **3 pts**. Give a pole-zero plot for $H(z)$.



Problem 4, cont...

(c) **6 pts.** Assume that the system frequency response $H(e^{j\omega})$ exists. Give the ROC for $H(z)$ and find the system impulse response $h[n]$.

ROC must contain the unit circle: $|z| > \frac{3}{4}$

PFE on the <u>proper</u> fraction $\dfrac{1-\frac{1}{2}z^{-1}}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{4}z^{-1})}$:

$$\frac{1-\frac{1}{2}\theta}{(1+\frac{3}{4}\theta)(1-\frac{1}{4}\theta)} = \frac{A}{1+\frac{3}{4}\theta} + \frac{B}{1-\frac{1}{4}\theta}$$

$$A = \left.\frac{1-\frac{1}{2}\theta}{1-\frac{1}{4}\theta}\right|_{\theta=-\frac{4}{3}} = \frac{1+\frac{4}{6}}{1+\frac{1}{3}} = \frac{1+\frac{2}{3}}{4/3} = \frac{5/3}{4/3} = \frac{5}{4}$$

$$B = \left.\frac{1-\frac{1}{2}\theta}{1+3/4\theta}\right|_{\theta=4} = \frac{1-2}{1+3} = \frac{-1}{4}$$

$$H(z) = \underbrace{\frac{5}{4}\,\frac{z^{-1}}{1+\frac{3}{4}z^{-1}}}_{|z|>3/4} - \underbrace{\frac{1}{4}\,\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}}$$

$\Rightarrow z^{-1}$ gives a time shift

$$h[n] = \left.\frac{5}{4}\left(-\frac{3}{4}\right)^m u[m]\right|_{m=n-1} - \left.\frac{1}{4}\left(\frac{1}{4}\right)^m u[m]\right|_{m=n-1}$$

$$\boxed{h[n] = \frac{5}{4}\left(-\frac{3}{4}\right)^{n-1} u[n-1] - \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u[n-1]}$$

(d) **4 pts.** Under the assumption of part (c) — that $H(e^{j\omega})$ exists — is the system causal? Is it BIBO stable? Justify your answers.

Yes, it is causal, because $h[n]=0 \;\forall n<0$.

Yes, it is BIBO stable, because it is causal and the poles of $H(z)$ are all inside the unit circle.

Problem 4, cont...

(e) **6 pts.** Now assume that the system H is unstable and that the impulse response $h[n]$ is two-sided. Give the ROC for $H(z)$ and find the impulse response $h[n]$.

Because $h[n]$ is two-sided, the ROC of $H(z)$ must be $\frac{1}{4} < |z| < 3/4$.

$$H(z) = \underbrace{\frac{5}{4} z^{-1} \frac{1}{1+\frac{3}{4}z^{-1}}}_{|z| < \frac{3}{4}} - \underbrace{\frac{1}{4} z^{-1} \frac{1}{1-\frac{1}{4}z^{-1}}}_{|z| > \frac{1}{4}}$$

Table, plus recognizing that multiplication by z^{-1} is a time shift by 1:

$$h[n] = \frac{5}{4}(-1)\left(-\frac{3}{4}\right)^m u[-m-1]\Big|_{m=n-1}$$

$$-\frac{1}{4}\left(\frac{1}{4}\right)^m u[m]\Big|_{m=n-1}$$

$$\boxed{h[n] = -\frac{5}{4}\left(-\frac{3}{4}\right)^{n-1} u[-n] - \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u[n-1]}$$