ECE 3793
Test 2

Tuesday, November 29, 2005
6:30 PM - 9:30 PM

Fall 2005 Name: SOLUTION

Dr. Havlicek Student Num: __________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**.

▶ You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

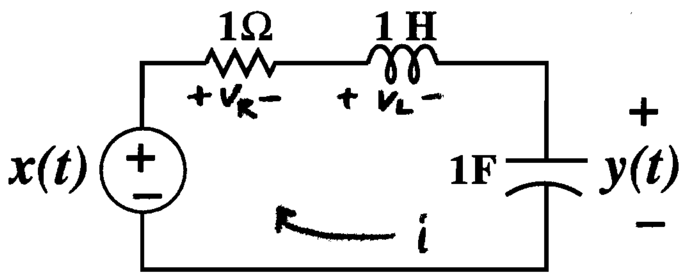
4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: __________ Date: __________

1. **25 pts.** Consider the continuous-time LTI system H shown below:



(a) **7 pts.** Find the differential equation relating the input $x(t)$ and output $y(t)$.

Resistor: $V_R(t) = R i_R(t) = i(t)$

Inductor: $V_L(t) = L i_L'(t) = i'(t)$

Capacitor: $i(t) = C y'(t) = y'(t)$

KVL: $-x(t) + V_R(t) + V_L(t) + y(t) = 0$

$$-x(t) + i(t) + i'(t) + y(t) = 0$$

$$-x(t) + y'(t) + y''(t) + y(t) = 0$$

$$\boxed{y''(t) + y'(t) + y(t) = x(t)}$$

(b) **8 pts.** Find the system frequency response $H(\omega)$.

$$(j\omega)^2 Y(\omega) + j\omega Y(\omega) + Y(\omega) = X(\omega)$$

$$[(j\omega)^2 + j\omega + 1] Y(\omega) = X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + j\omega + 1}}$$

Problem 1, cont...

(c) **10 pts.** Find the system impulse response $h(t)$.

denom: $\theta^2 + \theta + 1$

$a = b = c = 1.$

roots: $\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm j\sqrt{\frac{3}{4}}$

$$\frac{1}{\theta^2+\theta+1} = \frac{1}{\left[\theta - \left(-\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)\right]\left[\theta - \left(-\frac{1}{2} + j\sqrt{\frac{3}{4}}\right)\right]} = \frac{A}{\theta - \left(-\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)} + \frac{B}{\theta - \left(-\frac{1}{2} + j\sqrt{\frac{3}{4}}\right)}$$

$$A = \left.\frac{1}{\theta - \left(-\frac{1}{2} + j\sqrt{\frac{3}{4}}\right)}\right|_{\theta = -\frac{1}{2} - j\sqrt{\frac{3}{4}}} = \frac{1}{-\frac{1}{2} - j\sqrt{\frac{3}{4}} + \frac{1}{2} - j\sqrt{\frac{3}{4}}} = \frac{1}{-j2\sqrt{\frac{3}{4}}} = j\frac{1}{\sqrt{3}}$$

$$B = \left.\frac{1}{\theta - \left(-\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)}\right|_{\theta = -\frac{1}{2} + j\sqrt{\frac{3}{4}}} = \frac{1}{-\frac{1}{2} + j\sqrt{\frac{3}{4}} + \frac{1}{2} + j\sqrt{\frac{3}{4}}} = \frac{1}{j2\sqrt{\frac{3}{4}}} = -j\frac{1}{\sqrt{3}}$$

$$\frac{1}{\theta^2+\theta+1} = \frac{j/\sqrt{3}}{\theta - \left(-\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)} - \frac{j/\sqrt{3}}{\theta - \left(-\frac{1}{2} + j\sqrt{\frac{3}{4}}\right)}$$

$$H(\omega) = \frac{j/\sqrt{3}}{j\omega - \left(-\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)} - \frac{j/\sqrt{3}}{j\omega - \left(-\frac{1}{2} + j\sqrt{3/4}\right)} = \frac{j/\sqrt{3}}{\left(\frac{1}{2} + j\sqrt{\frac{3}{4}}\right) + j\omega} - \frac{j/\sqrt{3}}{\left(\frac{1}{2} - j\sqrt{\frac{3}{4}}\right) + j\omega}$$

$$h(t) = \frac{j}{\sqrt{3}} \exp\left[\left(-\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)t\right] u(t) - \frac{j}{\sqrt{3}} \exp\left[\left(-\frac{1}{2} + j\sqrt{\frac{3}{4}}\right)t\right] u(t)$$

$$= \frac{-1}{\sqrt{3}} \cdot 2 \cdot u(t) \cdot e^{-\frac{1}{2}t} \left[\frac{\exp\left(-j\sqrt{\frac{3}{4}}t\right) - \exp\left(j\sqrt{\frac{3}{4}}t\right)}{2j}\right]$$

$$= \frac{-2}{\sqrt{3}} e^{-\frac{1}{2}t} \left[\sin\left(-\sqrt{\frac{3}{4}}t\right)\right] u(t)$$

$$\boxed{h(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\frac{\sqrt{3}t}{2}\ u(t)}$$

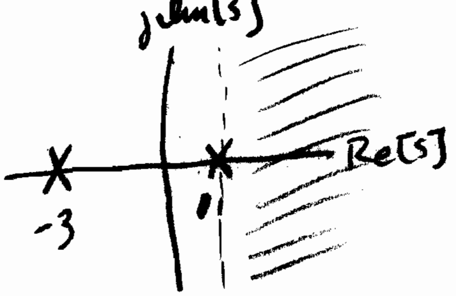
2. **25 pts.** A causal LTI system F has impulse response

$$f(t) = \tfrac{1}{4}e^{t}u(t) - \tfrac{1}{4}e^{-3t}u(t).$$

(a) **3 pts.** Find the transfer function $F(s)$ (be sure to specify the ROC). Give a pole-zero plot for $F(s)$.

$$F(s) = \frac{1}{4}\underbrace{\frac{1}{s-1}}_{Re[s] > 1} - \frac{1}{4}\underbrace{\frac{1}{s+3}}_{Re[s] > -3} = \frac{\frac{1}{4}(s+3) - \frac{1}{4}(s-1)}{(s-1)(s+3)}$$

$$= \frac{\frac{1}{4}s + \frac{3}{4} - \frac{1}{4}s + \frac{1}{4}}{(s-1)(s+3)} = \underline{\underline{\frac{1}{(s-1)(s+3)}, \quad Re[s] > 1}}$$



(b) **2 pts.** Is the system F BIBO stable? Justify your answer.

NO: because $F(s)$ is rational, but the ROC fails to include the $j\omega$-axis.

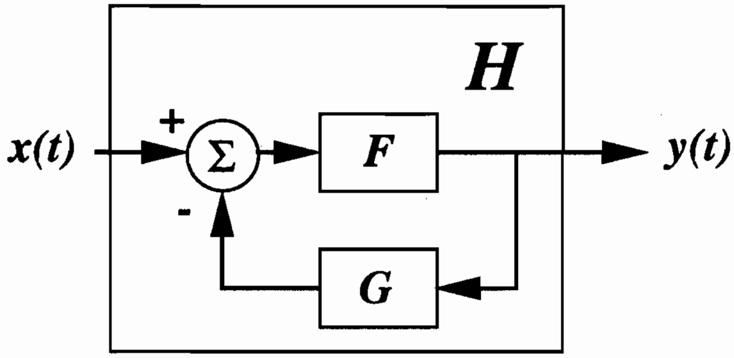
Alternatively: because f is causal, but there is a pole in the left half-plane.

(c) **2 pts.** Does $f(t)$ have a Fourier transform? Justify your answer.

No: For this to be true, the ROC of $F(s)$ would have to include the $j\omega$-axis, but it doesn't $\longrightarrow$ $F(\omega)$ doesn't exist.

Problem 2, cont...

(d) **11 pts.** A new system H is formed by adding negative feedback to F as shown in the figure below. H is both LTI and causal.



System G is LTI and causal and has transfer function $G(s) = 4$. Find the transfer function $H(s)$ of the overall system H.

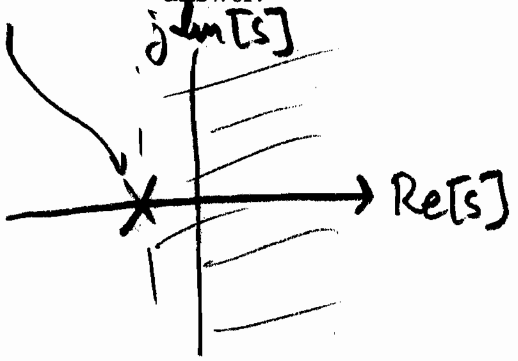
$$H(s) = \frac{F(s)}{1+F(s)G(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1+4\,\frac{1}{(s-1)(s+3)}}$$

$$= \frac{1}{(s-1)(s+3)+4} = \frac{1}{s^2+2s-3+4} = \frac{1}{s^2+2s+1}$$

$$\boxed{H(s) = \frac{1}{(s+1)^2}}$$

Problem 2, cont...

2nd order (e) **3 pts**. Give a pole-zero plot for $H(s)$. What is the ROC of $H(s)$? Justify your answer.



ROC: $\text{Re}[s] > -1$.

ROC is right sided because H is causal $\rightarrow h(t)$ right sided.

(f) **2 pts**. Is the system H BIBO stable? Justify your answer.

Yes, because it is causal and $H(s)$ is rational and all the poles are in the LHP, so the ROC of $H(s)$ includes the $j\omega$-axis.

(g) **2 pts**. Consider the system impulse response $h(t)$. Does $h(t)$ have a Fourier transform? Justify your answer.

Yes. Because the ROC of $H(s)$ includes the $j\omega$-axis, which is where the Laplace transform is equal to the Fourier transform.

3. **25 pts.** The input $x(t)$ and output $y(t)$ of an LTI system G are related by

$$y'(t) + 2y(t) = x''(t) + 2x'(t) - 3x(t).$$

Let LTI system H be the **inverse** system of G.

(a) **10 pts.** Assuming that H is **stable**, find the impulse response $h(t)$.

$$G: \quad sY(s) + 2Y(s) = s^2 X(s) + 2sX(s) - 3X(s)$$

$$Y(s)[s+2] = X(s)[s^2+2s-3]$$

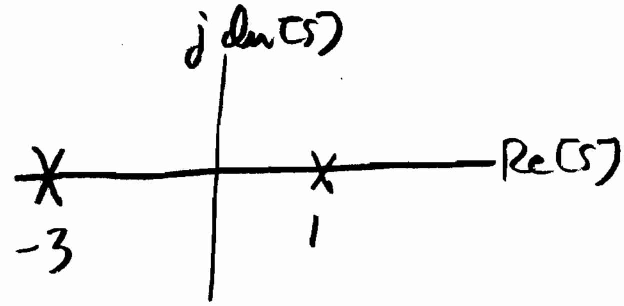
$$G(s) = \frac{Y(s)}{X(s)} = \frac{s^2+2s-3}{s+2}$$

$$H(s) = \frac{1}{G(s)} = \frac{s+2}{s^2+2s-3} = \frac{s+2}{(s-1)(s+3)}$$

$$= \frac{A}{s-1} + \frac{B}{s+3}$$

$$A = \left.\frac{s+2}{s+3}\right|_{s=1} = \frac{3}{4}$$

$$B = \left.\frac{s+2}{s-1}\right|_{s=-3} = \frac{-1}{-4} = \frac{1}{4}$$



- Since H is stable, the ROC must include the $j\omega$-axis.
- So the ROC must be $-3 < Re[s] < 1$.

$$\text{So} \quad H(s) = \frac{3}{4}\underbrace{\frac{1}{s-1}}_{Re[s]<1} + \frac{1}{4}\underbrace{\frac{1}{s+3}}_{Re[s]>-3}$$

$$\boxed{h(t) = -\frac{3}{4}e^{t}u(-t) + \frac{1}{4}e^{-3t}u(t)}$$

Problem 3, cont...

(b) **10 pts**. Now assume that H is **causal**, but not necessarily stable. Find the impulse response $h(t)$.

Since H is now causal, the ROC must be right sided: $Re[s] > 1$.

$$\text{So } H(s) = \frac{3}{4}\underbrace{\frac{1}{s-1}}_{Re[s]>1} + \frac{1}{4}\underbrace{\frac{1}{s+3}}_{Re[s]>-3}$$

$$\boxed{h(t) = \frac{3}{4}e^{t}u(t) + \frac{1}{4}e^{-3t}u(t)}$$

(c) **5 pts**. Does an inverse system with transfer function $H(s)$ exist that is both stable and causal?

No. Since there are poles in both the LHP and RHP, there is no way for the ROC to be right sided and still include the $j\omega$-axis.

4. **25 pts.** A continuous-time LTI system H has input $x(t)$ and output y(t) related by

$$y''(t) + 3y'(t) + 2y(t) = x(t).$$

The input is given by $x(t) = 2u(t)$. The initial conditions on the output are $y(0^-) = 3$ and $y'(0^-) = -5$. Use the unilateral Laplace transform to find the output $y(t)$ for $t > 0$.

$x(t) = 2u(t) \longrightarrow X_u(s) = 2/s.$

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$s^2 Y_u(s) - s\,y(0^-) - y'(0^-) + 3\left[sY_u(s) - y(0^-)\right] + 2Y_u(s) = X_u(s)$$

$$s^2 Y_u(s) - 3s + 5 + 3sY_u(s) - 9 + 2Y_u(s) = \frac{2}{s}$$

$$\left[s^2 + 3s + 2\right]Y_u(s) - 3s - 4 = \frac{2}{s}$$

$$s\left[s^2 + 3s + 2\right] Y_u(s) - 3s^2 - 4s = 2$$

$$s(s+2)(s+1)Y_u(s) = 3s^2 + 4s + 2$$

$$Y_u(s) = \frac{3s^2 + 4s + 2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \left.\frac{3s^2 + 4s + 2}{(s+1)(s+2)}\right|_{s=0} = \frac{2}{1 \cdot 2} = 1; \quad B = \left.\frac{3s^2 + 4s + 2}{s(s+2)}\right|_{s=-1} = \frac{3-4+2}{-1} = -1$$

$$C = \left.\frac{3s^2 + 4s + 2}{s(s+1)}\right|_{s=-2} = \frac{12-8+2}{-2(-1)} = \frac{6}{2} = 3$$

$$Y_u(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

Table:

$$\boxed{y(t) = u(t) - e^{-t}u(t) + 3e^{-2t}u(t), \quad t > 0}$$