ECE 3793
Test 2

Tuesday, November 23, 1999
12:00 PM - 1:15 PM

Fall 1999 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded. You have 75 minutes to complete the test. All work must be your own. You may use two 8.5 × 11 inch two-sided note sheets as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. **2.** **3.** **4.** **5.**

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

1. **25 pts.** The input $x(t)$ and output $y(t)$ of a causal LTI system are related by the differential equation

$$y''(t) + 6y'(t) + 8y(t) = 2x(t).$$

(a) **10 pts.** Find the system frequency response $H(\omega)$.

$$(j\omega)^2 Y(\omega) + 6j\omega Y(\omega) + 8Y(\omega) = 2X(\omega)$$

$$\left[-\omega^2 + 6j\omega + 8\right] Y(\omega) = 2X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{-\omega^2 + 6j\omega + 8}$$

(b) **10 pts.** Find the system impulse response $h(t)$.

$$H(\omega) = \frac{2}{(j\omega)^2 + 6j\omega + 8} = \frac{2}{(j\omega+4)(j\omega+2)}$$

$$= \frac{A}{j\omega+4} + \frac{B}{j\omega+2}$$

$$A = \left.\frac{2}{\theta+2}\right|_{\theta=-4} = \frac{2}{-2} = -1$$

$$B = \left.\frac{2}{\theta+4}\right|_{\theta=-2} = \frac{2}{2} = 1$$

$$H(\omega) = \frac{1}{j\omega+2} - \frac{1}{j\omega+4}$$

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

Problem 1, cont...

(c) **10 pts.** Find the system output $y(t)$ if the input is given by

$$x(t) = \delta'(t) + \frac{1}{2}\delta''(t).$$

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$\delta'(t) \overset{\mathcal{F}}{\longleftrightarrow} j\omega \quad \text{(derivative property)}$$

$$\delta''(t) \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^2$$

$$X(\omega) = j\omega + \tfrac{1}{2}(j\omega)^2$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{2\left[j\omega + \frac{1}{2}(j\omega)^2\right]}{(j\omega+4)(j\omega+2)}$$

$$= \frac{2j\omega + (j\omega)^2}{(j\omega+2)(j\omega+4)}$$

$$= j\omega \frac{j\omega+2}{(j\omega+2)(j\omega+4)} = j\omega \frac{1}{j\omega+4}$$

$$\frac{1}{j\omega+4} \overset{\mathcal{F}}{\longleftrightarrow} e^{-4t}u(t)$$

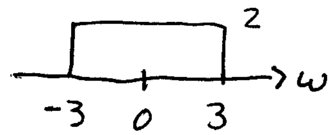
$$y(t) = \mathcal{F}^{-1}\left\{\frac{j\omega}{j\omega+4}\right\} = \frac{d}{dt} e^{-4t}u(t)$$

$$= e^{-4t}\delta(t) + (-4)e^{-4t}u(t)$$

$$= \underline{\underline{\delta(t) - 4e^{-4t}u(t)}}$$

2. **25 pts**. The Fourier transform of $x(t)$ is given by $|X(\omega)|e^{j\sphericalangle X(\omega)}$, where

$$|X(\omega)| = 2\{u(\omega+3) - u(\omega-3)\}$$



and

$$\sphericalangle X(\omega) = -\frac{3}{2}\omega.$$

Find $x(t)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{\sphericalangle X(\omega)} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-3}^{3} 2e^{-\frac{3}{2}j\omega} e^{j\omega t}\, d\omega$$

$$= \frac{1}{\pi}\int_{-3}^{3} e^{j\omega(t-\frac{3}{2})}\, d\omega$$

$$= \frac{1}{\pi}\frac{1}{j(t-\frac{3}{2})}\left[e^{j\omega(t-\frac{3}{2})}\right]_{\omega=-3}^{3}$$

$$= \frac{1}{j\pi(t-\frac{3}{2})}\left[e^{j3(t-\frac{3}{2})} - e^{-j3(t-\frac{3}{2})}\right]$$

$$= \frac{2}{\pi(t-\frac{3}{2})}\left[\frac{e^{j3(t-\frac{3}{2})} - e^{-j3(t-\frac{3}{2})}}{2j}\right]$$

$$= \underline{\underline{\frac{2\sin 3(t-\frac{3}{2})}{\pi(t-\frac{3}{2})}}}$$

3. **25 pts.** Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n].$$

(a) **5 pts.** Determine the frequency response $H(e^{j\omega})$ of this system.

$$Y(e^{j\omega}) + \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

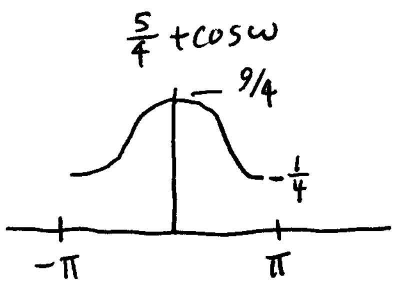
$$\left[1 + \frac{1}{2}e^{-j\omega}\right]Y(e^{j\omega}) = X(e^{j\omega})$$

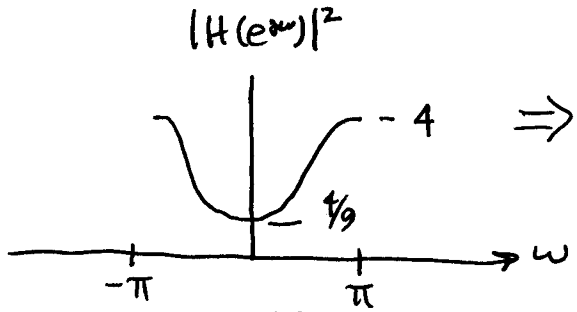
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{\underline{\frac{1}{1+\frac{1}{2}e^{-j\omega}}}}$$

(b) **4 pts.** Give a rough sketch the squared-magnitude response $|H(e^{j\omega})|^2 = H(e^{j\omega})H(e^{-j\omega})$ for the fundamental period $-\pi \le \omega \le \pi$. Is this system a low-pass filter, a high-pass filter, or a bandpass filter?

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H(e^{-j\omega}) = \frac{1}{(1+\frac{1}{2}e^{-j\omega})(1+\frac{1}{2}e^{j\omega})}$$

$$= \frac{1}{1+\frac{1}{2}e^{j\omega}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}} = \frac{1}{\frac{5}{4}+\cos\omega}$$





$\Rightarrow$ <u>High-Pass Filter</u>

(c) **4 pts.** Find the system impulse response $h[n]$.

$$h[n] = \mathcal{F}^{-1}\{H(e^{j\omega})\} = \underline{\underline{\left(-\frac{1}{2}\right)^n u[n]}}$$

Problem 3, cont...

(d) **7 pts**. Find the system response $y[n]$ when the input is

$$x[n] = \left(-\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(-\frac{1}{2}\right)^{n-1} u[n-1].$$

$$X(e^{j\omega}) = \mathcal{F}\left\{\left(-\tfrac{1}{2}\right)^n u[n]\right\} - \mathcal{F}\left\{\tfrac{1}{4}\left(-\tfrac{1}{2}\right)^{n-1} u[n-1]\right\}$$

$$= \frac{1}{1+\frac{1}{2}e^{-j\omega}} - \frac{1}{4}e^{-j\omega}\frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})^2}$$

$$= \frac{1}{(1+\frac{1}{2}e^{-j\omega})^2} - \frac{1}{4}e^{-j\omega}\frac{1}{(1+\frac{1}{2}e^{-j\omega})^2}$$

$$\frac{1}{(1+\frac{1}{2}e^{-j\omega})^2} \xleftrightarrow{\mathcal{F}} (n+1)\left(-\tfrac{1}{2}\right)^n u[n]$$

$$\frac{e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})^2} \xleftrightarrow{\mathcal{F}} n\left(-\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$\underline{\underline{y[n] = (n+1)\left(-\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{4}n\left(-\tfrac{1}{2}\right)^{n-1} u[n-1]}}$$

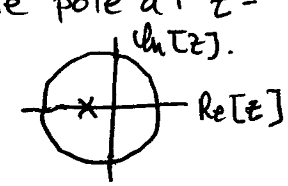
(e) **5 pts**. Determine the transfer function $H(z)$ of the system. Be sure to specify the ROC. Is the system stable? *(Justify your answer)*

$$Y(z) + \tfrac{1}{2}z^{-1}Y(z) = X(z)$$

$$\left[1+\tfrac{1}{2}z^{-1}\right]Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \underline{\underline{\frac{1}{1+\frac{1}{2}z^{-1}}}}$$

One pole at $z = -\frac{1}{2}$.



Causal $\Rightarrow$ ROC is exterior.

$\underline{\underline{\text{ROC: } |z| > \tfrac{1}{2}}}$

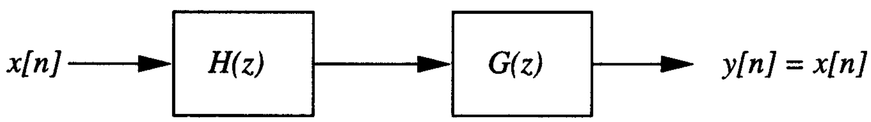
The system is stable because it is causal and all poles are inside the unit circle.

4. **25 pts**. Suppose that Joebob uses his cellular phone to call up Hidea on her cellular phone and ask for a date. The call gets digitized, and Joebob's voice ends up as a discrete-time signal $x[n]$. Because of interference being put out by Joebob's microwave oven and electric toilet paper dispenser, his voice is distorted during transmission so that the signal $r[n]$ received by Hidea's phone is not exactly equal to $x[n]$. In fact, the received signal is given by

$$r[n] = \frac{1}{2}\left(\frac{3}{4}r[n-1] + x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]\right).$$

Since this date is **really important** to Joebob, you have been hired to design an LTI channel equalizer that will "undo" the distortion. The equalizer is required to be both **causal** and **stable**.

Let $G(z)$ be the transfer function of the distortion and $H(z)$ be the transfer function of the equalizer. The equalizer will be placed in Joebob's phone so that it is in series with the distortion:



Note that, since the order does not matter when two linear systems are connected in series, this is equivalent to processing the received signal $r[n]$ above with an identical equalizer installed in Hidea's phone instead of in Joebob's phone.

(a) **10 pts**. Find $G(z)$, the transfer function of the distortion.

$$2R(z) = \frac{3}{4}z^{-1}R(z) + X(z) - \frac{3}{4}z^{-1}X(z) + \frac{1}{8}z^{-2}X(z)$$

$$\left[2 - \frac{3}{4}z^{-1}\right]R(z) = \left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right]X(z)$$

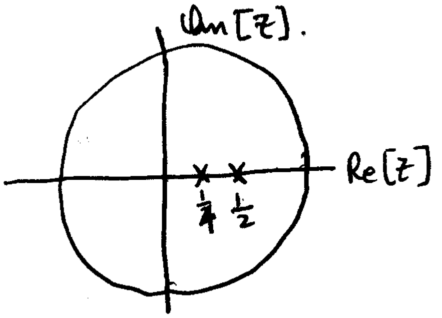
$$G(z) = \frac{R(z)}{X(z)} = \underline{\underline{\frac{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}{2 - \frac{3}{4}z^{-1}}}}$$

Problem 4, cont...

(b) **5 pts.** Find the equalizer transfer function $H(z) = 1/G(z)$ so that the signal received by Hidea's phone will equal $x[n]$. Be sure to specify the region of convergence so that the equalizer will be causal and stable.

$$H(z) = \frac{1}{G(z)} = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \underline{\underline{\frac{2 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}}}$$

$H(z)$ has poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$



For the system to be causal and stable, the ROC must be exterior and the poles must be inside the unit circle.

ROC: $|z| > \frac{1}{2}$.

(c) **10 pts.** Find the equalizer impulse response $h[n]$.

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - \frac{1}{4}z^{-1}}$$

$$A = \left.\frac{2 - \frac{3}{4}\theta}{1 - \frac{1}{4}\theta}\right|_{\theta = 2} = \frac{2 - \frac{3}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$B = \left.\frac{2 - \frac{3}{4}\theta}{1 - \frac{1}{2}\theta}\right|_{\theta = 4} = \frac{2 - 3}{1 - 2} = \frac{-1}{-1} = 1$$

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$\underline{\underline{h[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{4}\right)^n u[n]}}$$

5. **25 pts.** The input $x(t)$ and output $y(t)$ of a continuous-time LTI system are related by the differential equation

$$y(t) + y'(t) = x'(t) - x(t).$$

(a) **10 pts.** Find the system transfer function $H(s)$.

$$Y(s) + sY(s) = sX(s) - X(s)$$

$$[1+s]Y(s) = [s-1]X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \underline{\underline{\frac{s-1}{s+1}}}$$

(b) **15 pts.** The system output is observed to be $y(t) = e^{-2t}u(t)$, and it is known that the Fourier transform of the input $x(t)$ exists. Find $x(t)$.

$$Y(s) = \frac{1}{s+2}, \quad Re[s] > -2.$$

$$H(s) = \frac{Y(s)}{X(s)} \Rightarrow X(s) = \frac{Y(s)}{H(s)}$$

$$X(s) = \frac{1}{s+2}\,\frac{s+1}{s-1} = \frac{s+1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$A = \left.\frac{s+1}{s-1}\right|_{s=-2} = \frac{-1}{-3} = \frac{1}{3}$$

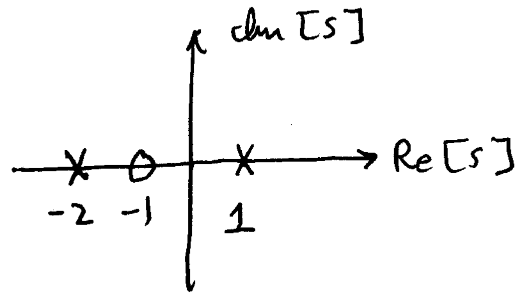
$$B = \left.\frac{s+1}{s+2}\right|_{s=1} = \frac{2}{3}$$

$$X(s) = \frac{1}{3}\frac{1}{s+2} + \frac{2}{3}\frac{1}{s-1}$$

$\longrightarrow$

More Workspace for Problem 5...

Pole-zero plot for $X(s)$:



From the pole-zero plot, there are three possible ROC's:

$$\mathrm{Re}[s] < -2$$
$$-2 < \mathrm{Re}[s] < 1$$
$$\mathrm{Re}[s] > 1.$$

Since $x(t)$ has a Fourier transform, the ROC must include the imaginary axis $\mathrm{Re}[s]=0$.

So the ROC must be $-2 < \mathrm{Re}[s] < 1$.

So the term $\frac{1}{3}\frac{1}{s+2}$ must have ROC $\mathrm{Re}[s] > -2$ and the term $\frac{2}{3}\frac{1}{s-1}$ must have ROC $\mathrm{Re}[s] < 1$.

Then $x(t) = \mathcal{L}^{-1}\{X(s)\}$

$$= \underbrace{\tfrac{1}{3}\mathcal{L}^{-1}\left\{\tfrac{1}{s+2}\right\}}_{\mathrm{Re}[s]>-2} + \underbrace{\tfrac{2}{3}\mathcal{L}^{-1}\left\{\tfrac{1}{s-1}\right\}}_{\mathrm{Re}[s]<1}$$

$$x(t) = \tfrac{1}{3}e^{-2t}u(t) - \tfrac{2}{3}e^{t}u(-t).$$