ECE 3793
Test 2

Thursday, April 26, 2001
7:00 PM - 10:00 PM

Spring 2001 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **five** problems. Work any **four** of them. Only **four** problems will be graded. Below, you must circle the numbers of the **four** problems you wish to have graded.

▶ Formulas appear **after** problem five.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

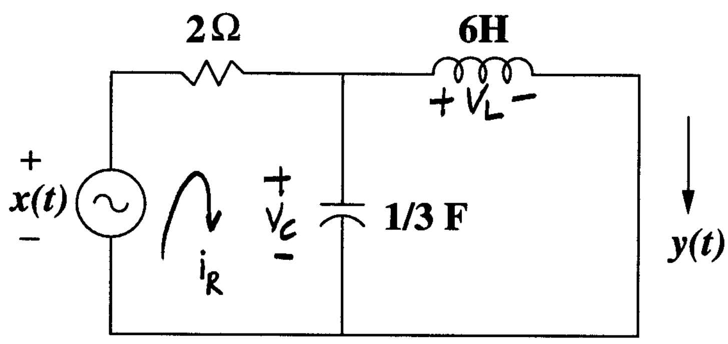
1. 2. 3. 4. 5.

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________
5. (25) ________

TOTAL (100):

1. **25 pts.** Consider the LTI system H shown below, where voltage $x(t)$ is the system input and current $y(t)$ is the system output.



Inductor

$V_L(t) = Ly'(t)$
$= 6y'(t)$

Capacitor

$i_c(t) = i_R(t) - y(t)$
$= \frac{1}{3} V_c'(t)$

(a) **9 pts.** Find the differential equation that relates $x(t)$ and $y(t)$.

KVL left loop: $-x(t) + 2i_R(t) + V_c(t) = 0$ (eq. 1)

KVL right loop: $-V_c(t) + 6y'(t) = 0$ (eq. 2)

→ Solve capacitor equation for $V_c(t)$: $V_c'(t) = 3i_R(t) - 3y(t)$ (eq. 3)

→ Since (eq. 3) is in terms of $V_c'(t)$, differentiate (eq. 1) and (eq. 2):

$-x'(t) + 2i_R'(t) + V_c'(t) = 0$ (eq. 4) ; $-V_c'(t) + 6y''(t) = 0$ (eq. 5).

→ Plug (eq. 3) into (eq. 4) and (eq. 5):

$-x'(t) + 2i_R'(t) + 3i_R(t) - 3y(t) = 0$ (eq. 6)

$-3i_R(t) + 3y(t) + 6y''(t) = 0$ (eq. 7)

→ Solve (eq. 7) for $i_R(t)$: $i_R(t) = y(t) + 2y''(t)$ (eq. 8)

→ Plug (eq. 8) into (eq. 6):

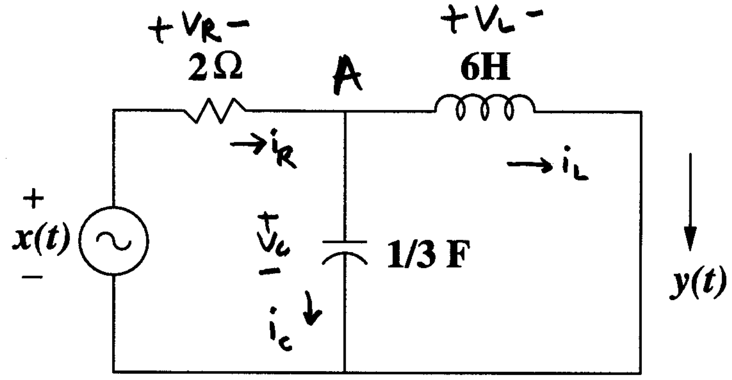
$$-x'(t) + 2y'(t) + 4y'''(t) + 3y(t) + 6y''(t) - 3y(t) = 0$$

$$x'(t) = 2y'(t) + 6y''(t) + 4y'''(t) \quad \text{(eq. 9)}$$

→ Since the $y(t)$ terms cancelled in (eq. 9), integrate once:

$$x(t) = 2y(t) + 6y'(t) + 4y''(t)$$

1. **25 pts**. Consider the LTI system H shown below, where voltage $x(t)$ is the system input and current $y(t)$ is the system output.



(a) **9 pts**. Find the differential equation that relates $x(t)$ and $y(t)$.

A shorter solution for part (a):

inductor: $V_L(t) = L i_L'(t) = L y'(t)$

$\Rightarrow$ Inductor & Capacitor are in parallel, so $V_L(t) = V_C(t)$

Capacitor: $i_C(t) = C V_C'(t) = C V_L'(t) = C L y''(t)$

KCL at node A:

$$-i_R(t) + i_C(t) + i_L(t) = 0$$

$$\frac{V_L(t) - x(t)}{R} + cLy''(t) + y(t) = 0$$

$$\frac{L}{R}y'(t) - \frac{1}{R}x(t) + cLy''(t) + y(t) = 0$$

$$Ly'(t) - x(t) + cRLy''(t) + Ry(t) = 0$$

$$cRLy''(t) + Ly'(t) + Ry(t) = x(t)$$

$$4y''(t) + 6y'(t) + 2y(t) = x(t)$$

Problem 1, cont...

(b) **9 pts**. Find the system frequency response $H(\omega)$.

$$X(\omega) = \left[2 + 6j\omega + 4(j\omega)^2\right] Y(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \underline{\underline{\frac{1}{4(j\omega)^2 + 6j\omega + 2}}}$$

$$H(\omega) = \frac{1}{(2j\omega + 2)(2j\omega + 1)} = \frac{1}{2(j\omega+1)(2)(j\omega+\frac{1}{2})}$$

$$= \frac{\frac{1}{4}}{(j\omega+1)(j\omega+\frac{1}{2})}$$

(c) **7 pts**. Find the system impulse response $h(t)$.

$$H(\omega) = \frac{\frac{1}{4}}{(j\omega+1)(j\omega+\frac{1}{2})} = \frac{A}{j\omega+1} + \frac{B}{j\omega+\frac{1}{2}}$$

$$A = \left.\frac{\frac{1}{4}}{\theta+\frac{1}{2}}\right|_{\theta=-1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$$

$$B = \left.\frac{\frac{1}{4}}{\theta+1}\right|_{\theta=-\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$H(\omega) = \frac{-\frac{1}{2}}{j\omega+1} + \frac{\frac{1}{2}}{j\omega+\frac{1}{2}}$$

$$\boxed{h(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-\frac{1}{2}t}u(t)}$$

2. **25 pts**. LTI System H is neither causal nor stable. The system transfer function is given by

$$H(s) = \frac{2s^2 + 9s + 5}{(s-3)(s^2+4s+4)}.$$

(a) **10 pts**. Find the differential equation that relates the system input $x(t)$ and the system output $y(t)$.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s^2 + 9s + 5}{s^3 + 4s^2 + 4s - 3s^2 - 12s - 12}$$

$$= \frac{2s^2 + 9s + 5}{s^3 + s^2 - 8s - 12}$$

$$Y(s)\left[s^3 + s^2 - 8s - 12\right] = X(s)\left[2s^2 + 9s + 5\right]$$

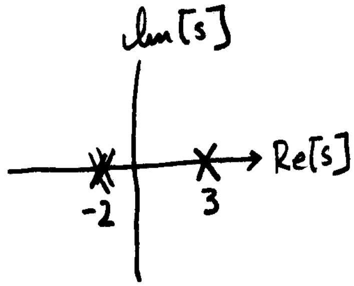
$$y'''(t) + y''(t) - 8y'(t) - 12y(t) = 2x''(t) + 9x'(t) + 5x(t)$$

Problem 2, cont...

(b) **15 pts**. Find the system impulse response $h(t)$.

$$H(s) = \frac{2s^2+9s+5}{(s-3)(s+2)(s+2)} = \frac{2s^2+9s+5}{(s-3)(s+2)^2}$$

$$= \frac{A}{s-3} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$



$$A = \left.\frac{2\theta^2+9\theta+5}{(\theta+2)^2}\right|_{\theta=3} = \frac{2\cdot 9+27+5}{5^2} = \frac{50}{25} = 2$$

$$B = \left.\frac{2\theta^2+9\theta+5}{\theta-3}\right|_{\theta=-2} = \frac{2\cdot 4-18+5}{-5} = \frac{-5}{-5} = 1$$

$$\frac{d}{d\theta}\Big[(2\theta^2+9\theta+5)(\theta-3)^{-1}\Big]_{\theta=-2} = \frac{d}{d\theta}\Big[A(\theta+2)^2(\theta-3)^{-1}\Big]_{\theta=-2} + \underbrace{\frac{d}{d\theta}B}_{\to 0} + \frac{d}{d\theta}\Big[(s+2)C\Big]_{\theta=-2}$$

$$\Big[(2\theta^2+9\theta+5)(-1)(\theta-3)^{-2} + (4\theta+9)(\theta-3)^{-1}\Big]_{\theta=-2} = A\Big[2\cancel{(\theta+2)}(\theta-3)^{-1} + \cancel{(\theta+2)^2}(-1)(\theta-3)^{-2}\Big]_{\theta=-2} + C$$

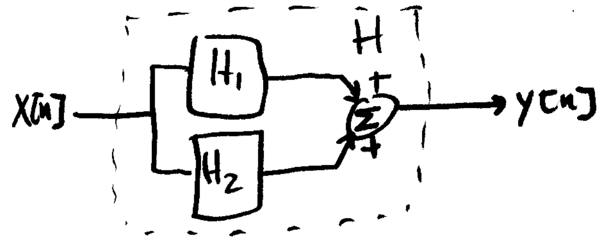
$$C = (2\cdot 4-18+5)(-1)(-5)^{-2} + (-8+9)(-5)^{-1}$$
$$= \frac{5}{25} - \frac{1}{5} = \frac{1}{5} - \frac{1}{5} = 0$$

For H to be <u>not</u> <u>stable</u> and <u>not</u> <u>causal</u>, ROC must be $Re[s] < -2$

$$H(s) = \underbrace{\frac{2}{s-3}}_{Re[s]<3} + \underbrace{\frac{1}{(s+2)^2}}_{Re[s]<-2}$$

$$\boxed{h(t) = -2e^{3t}u(-t) - te^{-2t}u(-t)}$$

3. **25 pts.** An LTI system H_1 with impulse response



$$h_1[n] = \left(\frac{1}{3}\right)^n u[n]$$

is connected in parallel with another LTI system H_2 with impulse response $h_2[n]$ to form the overall system H.

The input $x[n]$ and output $y[n]$ of H are related by the difference equation

$$12y[n] - 7y[n-1] + y[n-2] = -12x[n] + 5x[n-1].$$

(a) **10 pts.** Find the overall system frequency response $H(e^{j\omega})$.

$$Y(e^{j\omega})\left[12 - 7e^{-j\omega} + e^{-j2\omega}\right] = X(e^{j\omega})\left[-12 + 5e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{\underline{\frac{5e^{-j\omega} - 12}{e^{-j2\omega} - 7e^{-j\omega} + 12}}}$$

$$= \frac{5e^{-j\omega} - 12}{(e^{-j\omega} - 3)(e^{-j\omega} - 4)} = \frac{\frac{5}{12}e^{-j\omega} - 1}{(1 - \frac{1}{3}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

Problem 3, cont...

(b) **15 pts**. Find the impulse response $h_2[n]$ of system H_2.

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

$$\rightarrow \quad H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega})$$

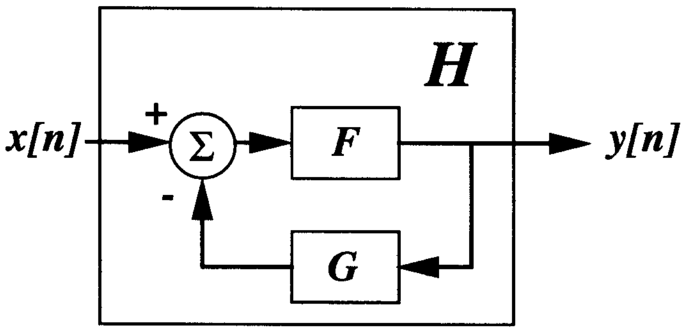
$$= \frac{\frac{5}{12}e^{-j\omega} - 1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} - \frac{1\ (1-\frac{1}{4}e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{\frac{5}{12}e^{-j\omega} - 1 - 1 + \frac{1}{4}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{\frac{2}{3}e^{-j\omega} - 2}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{-2(1-\frac{1}{3}e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{h_2[n] = -2\left(\tfrac{1}{4}\right)^n u[n]}$$

4. **25 pts.** Consider the **causal** LTI system H shown below.



$\delta[n] \overset{z}{\longleftrightarrow} 1, \text{all } z$

The impulse response of LTI system G is given by $g[n] = \frac{3}{2}\delta[n-1]$.

$G(z) = \frac{3}{2} z^{-1}, \; |z| > 0$

When the overall system input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n-1] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1],$$

the overall system output is observed to be

$$y[n] = n\left(\frac{1}{2}\right)^n u[n].$$

(a) **7 pts.** Find the overall system transfer function $H(z)$.

$\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \; |z| > \frac{1}{2}$

$\left(\frac{1}{2}\right)^{n-1} u[n-1] \overset{z}{\longleftrightarrow} \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}, \; |z| > \frac{1}{2}$

Table:

$Y(z) = \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \; |z| > \frac{1}{2}$

$X(z) = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, \; |z| > \frac{1}{2}$

$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \; \frac{1-\frac{1}{2}z^{-1}}{\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}},$

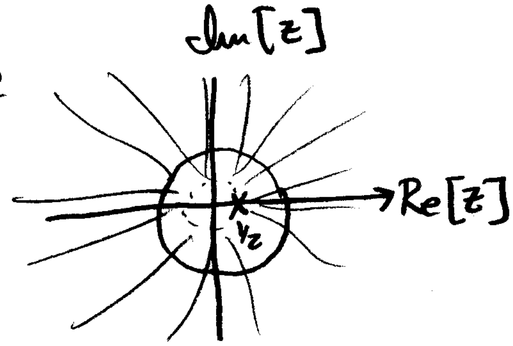
ROC: $|z| > \frac{1}{2}$

$$\boxed{H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \; |z| > \frac{1}{2}}$$

Problem 4, cont...

(b) **4 pts**. Is the system H stable? *Justify your answer.*

The system is stable because it is causal and all poles are inside the unit circle.



(c) **10 pts.** Find the impulse response $f[n]$ of system F.

From previous page, $G(z) = \frac{3}{2} z^{-1}, \quad |z| > 0$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{F(z)}{1+F(z)G(z)} = \frac{F(z)}{1+F(z)\frac{3}{2}z^{-1}}$$

$$\frac{1+\frac{3}{2}z^{-1}F(z)}{1-\frac{1}{2}z^{-1}} = F(z)$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{2}z^{-1}F(z)}{1-\frac{1}{2}z^{-1}} = F(z)$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} = \left[1 - \frac{\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}\right]F(z) = \frac{1-\frac{1}{2}z^{-1}-\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}F(z)$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1-2z^{-1}}{1-\frac{1}{2}z^{-1}}F(z)$$

$$F(z) = \frac{1}{1-2z^{-1}}, \quad |z| > 2 \longrightarrow$$

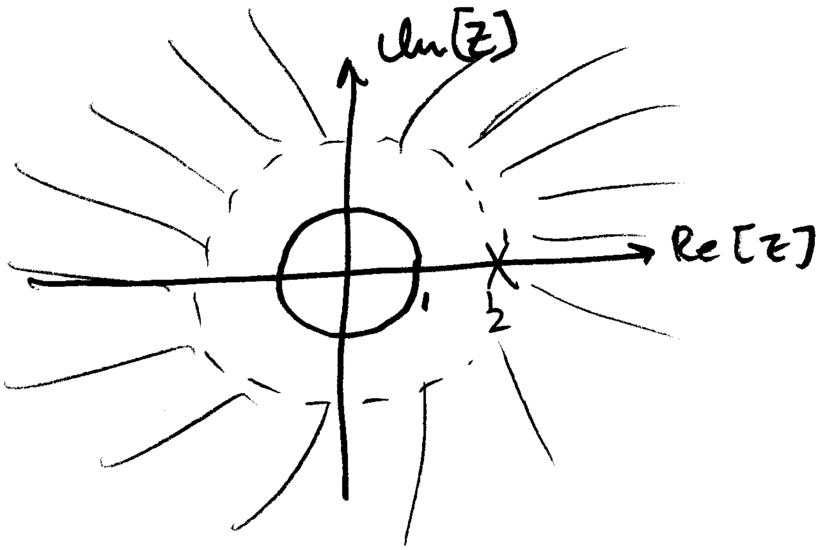
Problem 4, cont...

$$F(z) = \frac{1}{1-2z^{-1}}, \quad |z|>2$$

$$\boxed{f[n] = 2^n u[n]}$$

(d) **4 pts**. Is the system F stable? *Justify your answer.*

$$F(z) = \frac{1}{1-2z^{-1}}, \quad |z|>2$$



The system is not stable because it is causal and the pole at $z=2$ is outside the unit circle.

5. **25 pts.** The input $x(t)$ and output $y(t)$ of LTI system H are related by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t).$$

The system input is given by $x(t) = 2u(t)$. Given the initial conditions $y(0^-) = 3$ and $y'(0^-) = -5$, find the system response $y(t)$ for $t > 0$.

Use the unilateral Laplace transform to account for the initial conditions:

$$s^2 Y_u(s) - sy(0^-) - y'(0^-) + 3[sY_u(s) - y(0^-)] + 2Y_u(s) = \frac{2}{s}$$

$$s^2 Y_u(s) - 3s + 5 + 3sY_u(s) - 9 + 2Y_u(s) = \frac{2}{s}$$

$$Y_u(s)[s^2 + 3s + 2] - 3s - 4 = \frac{2}{s}$$

(all ROCs are right half-planes to the right of the rightmost pole, since all signals are right sided for the unilateral transform)

$$Y_u(s)[s^2 + 3s + 2] = 3s + 4 + \frac{2}{s}$$

$$Y_u(s) = \frac{3s}{s^2+3s+2} + \frac{4}{s^2+3s+2} + \frac{2}{s(s^2+3s+2)}$$

$$= \frac{3s^2+4s+2}{s(s^2+3s+2)} = \frac{3s^2+4s+2}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \left.\frac{3\theta^2+4\theta+2}{(\theta+2)(\theta+1)}\right|_{\theta=0} = \frac{2}{2} = 1$$

$$B = \left.\frac{3\theta^2+4\theta+2}{\theta(\theta+2)}\right|_{\theta=-1} = \frac{3-4+2}{-1} = -1$$

$$C = \left.\frac{3\theta^2+4\theta+2}{\theta(\theta+1)}\right|_{\theta=-2} = \frac{12-8+2}{-2(-1)} = \frac{6}{2} = 3$$

$\longrightarrow$

More Work Space for Problem 5...

$$Y_u(s) = \underbrace{\frac{1}{s}}_{\text{all } s} - \underbrace{\frac{1}{s+1}}_{Re[s] > -1} + \underbrace{\frac{3}{s+2}}_{Re[s] > -2}$$

$$\underline{\underline{y(t) = u(t) - e^{-t}u(t) + 3e^{-2t}u(t)}}$$