ECE 3793
Test 2

Wednesday, April 23, 2003
7:00 PM - 10:00 PM

Spring 2003

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work all **four**.

▶ You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

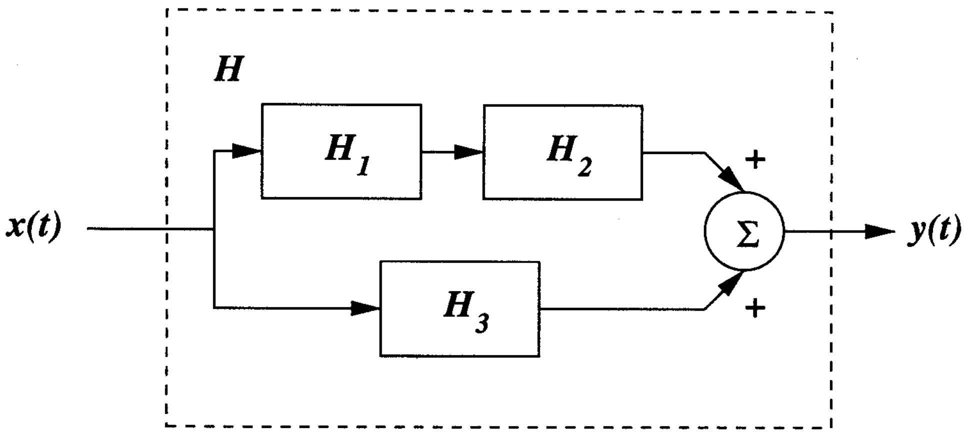
2. (25) ________

3. (25) ________

4. (25) ________

TOTAL (100):

1. **25 pts**. Consider the continuous-time LTI system H shown below.



The systems H_1, H_2, and H_3 are all LTI and causal. The impulse response of system H_1 is given by

$$h_1(t) = te^{-3t}u(t).$$

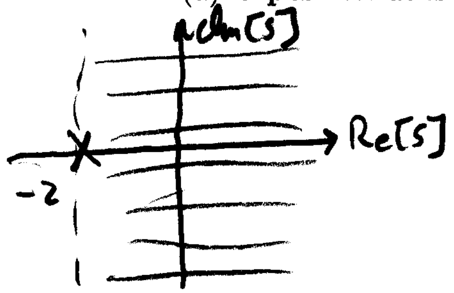
The input/output relation of system H_2 is

$$4y_2(t) + y_2'(t) = x_2(t).$$

The transfer function of system H_3 is given by

$$H_3(s) = \frac{1}{s+2}.$$

(a) **5 pts**. What is the ROC of $H_3(s)$? Justify your answer.



ROC $H_3(s)$: $Re[s] > -2$.

Since H_3 is causal {$h_3(t)$ is right-sided} and $H_3(s)$ is rational, it's every thing to the right of the rightmost pole.

(b) **5 pts**. Is H_3 BIBO stable? Justify your answer.

Yes, because H_3 is a causal LTI system with rational transfer function and all the poles are in the left half-plane.

Problem 1, cont...

(c) **5 pts**. Does the Fourier transform $H_3(\omega)$ exist? Justify your answer. If so, what is it?

Yes, because the ROC of $H_3(s)$ contains the $j\omega$-axis ($s = \sigma + j\omega$).

$$H_3(\omega) = H_3(s)\Big|_{s=j\omega} = \frac{1}{s+2}\Big|_{s=j\omega} = \underline{\underline{\frac{1}{2+j\omega}}}$$

(d) **10 pts**. Find the frequency response $H(\omega)$ of the overall system H.

$h_1(t) = t e^{-3t} u(t)$

Table: $H_1(\omega) = \dfrac{1}{(3+j\omega)^2}$

From (c): $H_3(\omega) = \dfrac{1}{2+j\omega}$

H_2: $4y_2(t) + y_2'(t) = x_2(t)$

$4Y_2(\omega) + j\omega Y_2(\omega) = X_2(\omega)$

$Y_2(\omega)[4+j\omega] = X_2(\omega)$

$H_2(\omega) = \dfrac{Y_2(\omega)}{X_2(\omega)} = \dfrac{1}{4+j\omega}$

$$H(\omega) = H_1(\omega)H_2(\omega) + H_3(\omega) = \frac{1}{(3+j\omega)^2}\frac{1}{4+j\omega} + \frac{1}{2+j\omega}$$

$$= \frac{2+j\omega + (3+j\omega)^2(4+j\omega)}{(3+j\omega)^2(4+j\omega)(2+j\omega)} = \frac{2+j\omega + (9+6j\omega+(j\omega)^2)(4+j\omega)}{\text{denominator}}$$

$$= \frac{2+j\omega+36+24j\omega+4(j\omega)^2+9j\omega+6(j\omega)^2+(j\omega)^3}{\text{denominator}}$$

$$\boxed{H(\omega) = \frac{(j\omega)^3 + 10(j\omega)^2 + 34j\omega + 38}{(4+j\omega)(3+j\omega)^2(2+j\omega)}}$$

2. **25 pts.** A continuous-time LTI system H is causal and has frequency response

$$H(\omega) = \frac{(j\omega)^3 + 10(j\omega)^2 + 33j\omega + 37}{(3+j\omega)^2(4+j\omega)(2+j\omega)}.$$

(a) **15 pts.** The system input is given by $x_a(t) = e^{-4(t-2)}u(t-2)$. Use the Fourier transform to find the resulting system output $y_a(t)$.

Table: $e^{-4t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{4+j\omega}$. Time Shift Property: $e^{-4(t-2)}u(t-2) \overset{\mathcal{F}}{\longleftrightarrow} \frac{e^{-j2\omega}}{4+j\omega}$

$$Y_a(\omega) = H(\omega)X(\omega) = e^{-j2\omega}\frac{(j\omega)^3 + 10(j\omega)^2 + 33j\omega + 37}{(3+j\omega)^2(4+j\omega)^2(2+j\omega)}$$

$$\frac{\theta^3+10\theta^2+33\theta+37}{(3+\theta)^2(4+\theta)^2(2+\theta)} = \frac{A}{(3+\theta)^2} + \frac{B}{3+\theta} + \frac{C}{(4+\theta)^2} + \frac{D}{4+\theta} + \frac{E}{2+\theta}$$

$$A = \left.\frac{\theta^3+10\theta^2+33\theta+37}{(4+\theta)^2(2+\theta)}\right|_{\theta=-3} = \frac{-27+90-99+37}{1\cdot(-1)} = \frac{1}{-1} = -1$$

$$C = \left.\frac{\theta^3+10\theta^2+33\theta+37}{(3+\theta)^2(2+\theta)}\right|_{\theta=-4} = \frac{-64+160-132+37}{1\cdot(-2)} = -\frac{1}{2}$$

$$E = \left.\frac{\theta^3+10\theta^2+33\theta+37}{(3+\theta)^2(4+\theta)^2}\right|_{\theta=-2} = \frac{-8+40-66+37}{1\cdot 4} = \frac{3}{4}$$

B: right side: $\frac{d}{d\theta}(3+\theta)B = \frac{d}{d\theta}\theta B = B$ ✓

$$B = \left.\frac{d}{d\theta}(\theta^3+10\theta^2+33\theta+37)\left[(4+\theta)^2(2+\theta)\right]^{-1}\right|_{\theta=-3}$$

$$= (\theta^3+10\theta^2+33\theta+37)(-1)\left[(4+\theta)^2(2+\theta)\right]^{-2}\left[(4+\theta)^2+2(4+\theta)(2+\theta)\right] + \left.(3\theta^2+20\theta+33)\left[(4+\theta)^2(2+\theta)\right]^{-1}\right|_{\theta=-3}$$

$$= (-27+90-99+37)(-1)\left[1^2\cdot(-1)\right]^{-2}\left[1^2+2\cdot1\cdot(-1)\right] + (27-60+33)\left[1^2(-1)\right]^{-1}$$

$$= 1(-1)(1)(-1)+0 = \boxed{1 = B}$$

More Work Space for Problem 2... D : right side: $\frac{d}{d\theta}(4+\theta)D = \frac{d}{d\theta}\theta D = D$ ✓

$$D = \frac{d}{d\theta}(\theta^3+10\theta^2+33\theta+37)\left[(3+\theta)^2(2+\theta)\right]^{-1}\Big|_{\theta=-4}$$

$$= (\theta^3+10\theta^2+33\theta+37)(-1)\left[(3+\theta)^2(2+\theta)\right]^{-2}\left[2(3+\theta)(2+\theta)+(3+\theta)^2\right]$$
$$+(3\theta^2+20\theta+33)\left[(3+\theta)^2(2+\theta)\right]^{-1}\Big|_{\theta=-4}$$

$$=(-64+160-132+37)(-1)\left[(-1)^2(-2)\right]^{-2}\left[2(-1)(-2)+(-1)^2\right]$$
$$+(48-80+33)\left[(-1)^2(-2)\right]^{-1} = (1)(-1)\left(\tfrac{1}{4}\right)[4+1]+(1)\left(-\tfrac{1}{2}\right)$$
$$= -\frac{5}{4}-\frac{1}{2} = \boxed{\frac{-7}{4} = D}$$

$$\Rightarrow \frac{\theta^3+10\theta^2+33\theta+37}{(3+\theta)^2(4+\theta)^2(2+\theta)} = \frac{-1}{(3+\theta)^2}+\frac{1}{3+\theta}+\frac{-1/2}{(4+\theta)^2}+\frac{-7/4}{4+\theta}+\frac{3/4}{2+\theta}$$

$$Y_a(\omega) = e^{-j2\omega}\left[\frac{-1}{(3+j\omega)^2}+\frac{1}{3+j\omega}+\frac{-1/2}{(4+j\omega)^2}+\frac{-7/4}{4+j\omega}+\frac{3/4}{2+j\omega}\right]$$

$$y_a(t) = \mathcal{F}^{-1}\left\{\frac{-1}{(3+j\omega)^2}+\frac{1}{3+j\omega}+\frac{-1/2}{(4+j\omega)^2}+\frac{-7/4}{4+j\omega}+\frac{3/4}{2+j\omega}\right\}_{t=t-2}$$

$$= \left\{-\frac{1}{2}(t-2)e^{-4(t-2)}-\frac{7}{4}e^{-4(t-2)}-(t-2)e^{-3(t-2)}\right.$$
$$\left.+e^{-3(t-2)}+\frac{3}{4}e^{-2(t-2)}\right\}u(t-2)$$

Problem 2, cont...

(b) **10 pts**. Now the system input is given by $x_b(t) = \cos t$. Find the resulting system output $y_b(t)$.

$$y_b(t) = |H(1)| \cos(t + \arg H(1))$$

$$H(1) = \frac{j^3 + 10j^2 + 33j + 37}{(3+j)^2(4+j)(2+j)} = \frac{-j - 10 + 33j + 37}{(9+6j-1)(8+6j-1)}$$

$$= \frac{27+32j}{(8+6j)(7+6j)} = \frac{27+32j}{56+90j-36} = \frac{27+32j}{20+90j}$$

$$= \frac{41.8688 \exp[j\, 896.942 \times 10^{-3}]}{92.1954 \exp[j\, 1.35213]}$$

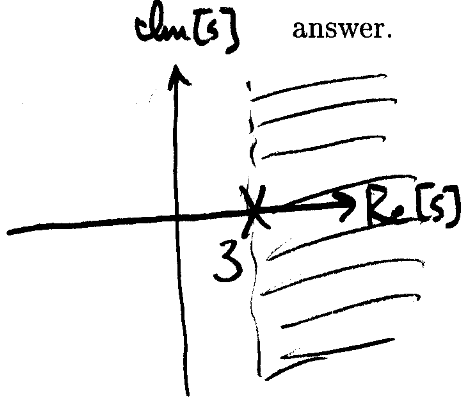
$$= 454.131 \times 10^{-3} \exp[-j\, 482.185 \times 10^{-3}]$$

$$y_b(t) = 454.131 \times 10^{-3} \cos\left[t - 482.185 \times 10^{-3}\right]$$

3. **25 pts**. A causal LTI system F has transfer function

$$F(s) = \frac{1}{s-3}.$$

(a) **3 pts**. Give a pole-zero plot for $F(s)$. What is the ROC of $F(s)$? Justify your answer.



ROC: $Re[s] > 3$.

→ Because the system is causal (impulse response is right-sided) and $F(s)$ is rational, the ROC of $F(s)$ is the right half-plane to the right of the rightmost pole.

(b) **2 pts**. Is the system F BIBO stable? Justify your answer.

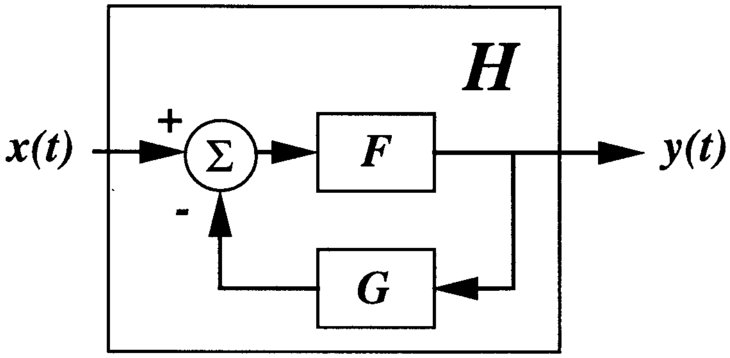
NOT BIBO stable. The system is causal and LTI and has a rational transfer function, but there is a pole in the right half-plane. So it's not stable.

(c) **2 pts**. Consider the system impulse response $f(t)$. Does $f(t)$ have a Fourier transform? Justify your answer.

No, $F(\omega)$ does not exist, because the ROC of $F(s)$ does not contain the $j\omega$-axis of the s-plane.

Problem 3, cont...

(d) **11 pts**. A new system H is formed by adding negative feedback to F as shown in the figure below. H is both LTI and causal.



System G is LTI and causal and has transfer function $G(s) = 6$. Find the transfer function $H(s)$ of the overall system H.

$$F(s) = \frac{1}{s-3} \qquad G(s) = 6$$

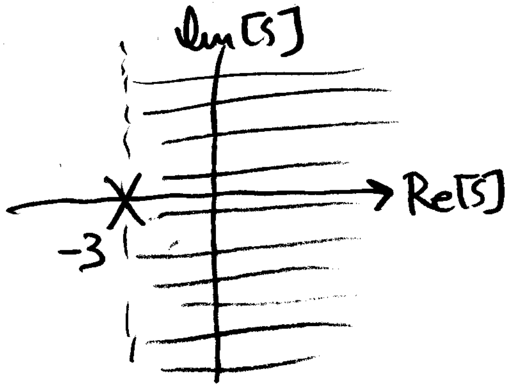
$$H(s) = \frac{F(s)}{1+F(s)G(s)} = \frac{\frac{1}{s-3}}{1+\frac{6}{s-3}} \cdot \frac{s-3}{s-3}$$

$$= \frac{1}{s-3+6} = \frac{1}{s+3}$$

$$\boxed{H(s) = \frac{1}{s+3}}$$

Problem 3, cont...

(e) **3 pts**. Give a pole-zero plot for $H(s)$. What is the ROC of $H(s)$? Justify your answer.



ROC: Re[s] > -3.

Because H is a causal LTI system with rational transfer function, the ROC of H(s) is the right half-plane to the right of the rightmost pole.

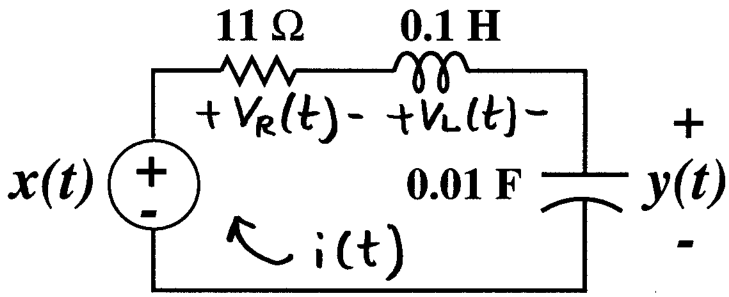
(f) **2 pts**. Is the system H BIBO stable? Justify your answer.

Yes, H is BIBO stable, because it is a causal LTI system with a rational transfer function and all the poles are in the left half-plane.

(g) **2 pts**. Consider the system impulse response $h(t)$. Does $h(t)$ have a Fourier transform? Justify your answer.

Yes, H(ω) exists, because the ROC of H(s) contains the jω-axis of the s-plane.

4. **25 pts**. Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.



(a) **10 pts**. Find the input/output differential equation that relates input $x(t)$ and output $y(t)$.

Constituent equations (Ohm's law):

$$V_R(t) = 11\, i(t) \;;\quad V_L(t) = 0.1\, i'(t);$$

$$i(t) = 0.01\, y'(t).$$

KVL: $x(t) = V_R(t) + V_L(t) + y(t)$

$$= 11\, i(t) + 0.1\, i'(t) + y(t)$$

$$= (11)(.01)\, y'(t) + (0.1)(0.01)\, y''(t) + y(t)$$

$$= 0.11\, y'(t) + 0.001\, y''(t) + y(t)$$

$$\boxed{1000\, x(t) = y''(t) + 110\, y'(t) + 1000\, y(t)}$$

Problem 4, cont...

(b) **15 pts.** The initial conditions on the output are $y(0^-) = 2$ and $y'(0^-) = -1$. Use the unilateral Laplace transform to find the system output for $t > 0$ when the input is $x(t) = e^{-t}u(t)$. $\mathcal{X}_u(s) = \frac{1}{s+1}$

$$1000\mathcal{X}_u(s) = s^2\mathcal{Y}_u(s) - sy(0^-) - y'(0^-) + 110\{s\mathcal{Y}_u(s) - y(0^-)\} + 1000\mathcal{Y}_u(s)$$

$$\frac{1000}{s+1} = s^2\mathcal{Y}_u(s) - 2s + 1 + 110\{s\mathcal{Y}_u(s) - 2\} + 1000\mathcal{Y}_u(s)$$

$$= s^2\mathcal{Y}_u(s) + 110s\mathcal{Y}_u(s) + 1000\mathcal{Y}_u(s) - 2s + 1 - 220$$

$$[s^2 + 110s + 1000]\mathcal{Y}_u(s) = \frac{1000}{s+1} + 2s + 219 = \frac{1000 + 2s(s+1) + 219(s+1)}{s+1}$$

$$(s+1)(s+10)(s+100)\mathcal{Y}_u(s) = 1000 + 2s^2 + 2s + 219s + 219 = 2s^2 + 221s + 1219$$

$$\mathcal{Y}_u(s) = \frac{2s^2 + 221s + 1219}{(s+1)(s+10)(s+100)} = \frac{A}{s+1} + \frac{B}{s+10} + \frac{C}{s+100}$$

$$A = \left.\frac{2s^2 + 221s + 1219}{(s+10)(s+100)}\right|_{s=-1} = \frac{2 - 221 + 1219}{9(99)} = \frac{1000}{891}$$

$$B = \left.\frac{2s^2 + 221s + 1219}{(s+1)(s+100)}\right|_{s=-10} = \frac{200 - 2210 + 1219}{(-9)(90)} = \frac{-791}{-810} = \frac{791}{810}$$

$$C = \left.\frac{2s^2 + 221s + 1219}{(s+1)(s+10)}\right|_{s=-100} = \frac{20{,}000 - 22{,}100 + 1219}{(-99)(-90)} = \frac{-881}{8910}$$

$$\mathcal{Y}_u(s) = \frac{1000}{891}\frac{1}{s+1} + \frac{791}{810}\frac{1}{s+10} - \frac{881}{8910}\frac{1}{s+100}$$

Table: $y(t) = \left\{\frac{1000}{891}e^{-t} + \frac{791}{810}e^{-10t} - \frac{881}{8910}e^{-100t}\right\}u(t)$