ECE 3793
Test 2

Monday, April 12, 2004
6:30 PM - 9:30 PM

Spring 2004

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work all **four**.

▶ You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

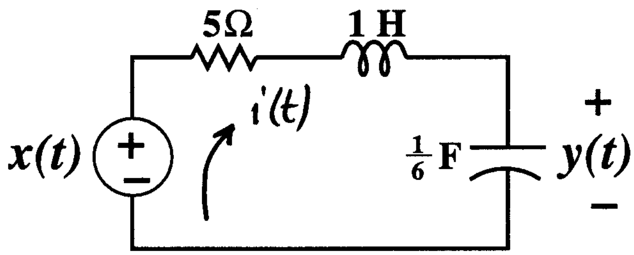
2. (25) ________

3. (25) ________

4. (25) ________

TOTAL (100):

1. **25 pts.** Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.



(a) **10 pts.** Find the system frequency response $H(\omega)$.

KVL: $x(t) = 5i(t) + v_L(t) + y(t)$ ①

Cap: $i(t) = \frac{1}{6}y'(t)$ ②

L ; $v_L(t) = 1i'(t) = \frac{1}{6}y''(t)$ ③

②,③→①: $x(t) = \frac{5}{6}y'(t) + \frac{1}{6}y''(t) + y(t)$

$$\frac{1}{6}y''(t) + \frac{5}{6}y'(t) + y(t) = x(t)$$

$$y''(t) + 5y'(t) + 6y(t) = 6x(t)$$

$$(j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega) = 6X(\omega)$$

$$[(j\omega)^2 + 5j\omega + 6]\, Y(\omega) = 6X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{6}{(j\omega)^2 + 5j\omega + 6}}$$

Problem 1, cont...

(b) **5 pts.** Find the system impulse response $h(t)$.

$$H(\omega) = \frac{6}{(j\omega)^2 + 5j\omega + 6} = \frac{6}{(j\omega+3)(j\omega+2)} = \frac{A}{2+j\omega} + \frac{B}{3+j\omega}$$

$$A = \left.\frac{6}{3+\theta}\right|_{\theta=-2} = \frac{6}{3-2} = 6$$

$$B = \left.\frac{6}{2+\theta}\right|_{\theta=-3} = \frac{6}{-1} = -6$$

$$H(\omega) = \frac{6}{2+j\omega} - \frac{6}{3+j\omega}$$

$$\boxed{h(t) \overset{\text{Table}}{=} 6e^{-2t}u(t) - 6e^{-3t}u(t)}$$

Problem 1, cont...

(c) **10 pts.** Find the system output $y(t)$ when the input is given by $x(t) = \frac{1}{2}e^{-3t}u(t)$.

Table: $X(\omega) = \dfrac{1/2}{3+j\omega}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1/2}{3+j\omega}\,\frac{6}{(3+j\omega)(2+j\omega)} = \frac{3}{(2+j\omega)(3+j\omega)^2}$$

$$\frac{3}{(2+\theta)(3+\theta)^2} = \frac{A}{2+\theta} + \frac{B}{(3+\theta)^2} + \frac{C}{3+\theta}$$

$$A = \frac{3}{(3+\theta)^2}\Big|_{\theta=-2} = \frac{3}{1^2} = 3$$

$$B = \frac{3}{2+\theta}\Big|_{\theta=-3} = \frac{3}{-1} = -3$$

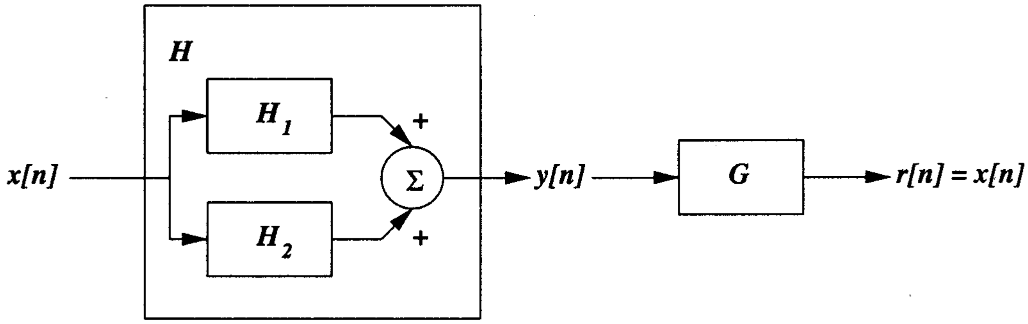
$$\frac{\partial}{\partial\theta}\left[3(2+\theta)^{-1}\right]\Big|_{\theta=-3} = \frac{\partial}{\partial\theta}C(3+\theta)\Big|_{\theta=-3}$$

$$-3(2+\theta)^{-2}\Big|_{\theta=-3} = C = -3(-1)^{-2} = -3(1) = -3$$

$$Y(\omega) = \frac{3}{2+j\omega} - \frac{3}{(3+j\omega)^2} - \frac{3}{3+j\omega}$$

Table: $\boxed{y(t) = 3e^{-2t}u(t) - 3te^{-3t}u(t) - 3e^{-3t}u(t)}$

2. **25 pts.** Consider the simple digital transmission system shown in the figure below. The signal $x[n]$ is transmitted through a communications channel H. Because the channel is not ideal, $x[n]$ is distorted as it passes through the channel. The distortion is modeled as a discrete-time LTI system H, so that the received signal is $y[n] = x[n] * h[n]$.



The system H is a parallel connection of two discrete-time LTI systems H_1 and H_2. The impulse response of system H_1 is given by $h_1[n] = \delta[n-1]$. The impulse response of system H_2 is given by $h_2[n] = \frac{1}{2}(\frac{1}{4})^{n-2}u[n-2]$.

(a) **6 pts.** Find the frequency response $H(e^{j\omega})$ of the system H.

$$H_1(e^{j\omega}) = e^{-j\omega}\mathcal{F}\{\delta[n]\} = e^{-j\omega}$$

$$H_2(e^{j\omega}) = \frac{1}{2}e^{-2j\omega}\mathcal{F}\{(\tfrac{1}{4})^n u[n]\} = \frac{1}{2}e^{-2j\omega}\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

$$= e^{-j\omega} + \frac{1}{2}e^{-2j\omega}\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{e^{-j\omega}(1-\frac{1}{4}e^{-j\omega})}{1-\frac{1}{4}e^{-j\omega}} + \frac{\frac{1}{2}e^{-j2\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{e^{-j\omega} - \frac{1}{4}e^{-j2\omega} + \frac{1}{2}e^{-j2\omega}}{1-\frac{1}{4}e^{-j\omega}} = \frac{e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{H(e^{j\omega}) = \frac{e^{-j\omega}(1+\frac{1}{4}e^{-j\omega})}{1-\frac{1}{4}e^{-j\omega}}}$$

Problem 2, cont...

(b) **8 pts**. You have been hired to design a channel equalizer G that will be implemented at the receiver to "undo" the distortion. Your system G will input the received signal $y[n]$ and output a signal $r[n]$ that is identical to the original transmitted signal $x[n]$. Your channel equalizer G is required to be a discrete-time LTI system. **Hint:** for this to work, you must design G so that it is the *inverse system* of H. Find the equalizer frequency response $G(e^{j\omega})$.

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{1-\frac{1}{4}e^{-j\omega}}{e^{-j\omega}(1+\frac{1}{4}e^{-j\omega})} \cdot \frac{e^{j\omega}}{e^{j\omega}}$$

$$\boxed{G(e^{j\omega}) = \frac{e^{j\omega}(1-\frac{1}{4}e^{-j\omega})}{1+\frac{1}{4}e^{-j\omega}}}$$

$$= \frac{e^{j\omega}}{1+\frac{1}{4}e^{-j\omega}} - \frac{1/4}{1+\frac{1}{4}e^{-j\omega}}$$

Problem 2, cont...

(c) **5 pts.** Use the inverse DTFT to find the equalizer impulse response $g[n]$. Is the equalizer G a causal discrete-time LTI system? Justify your answer.

$$G(e^{j\omega}) = \frac{e^{j\omega}}{1+\frac{1}{4}e^{-j\omega}} - \frac{1/4}{1+\frac{1}{4}e^{-j\omega}}$$

Table: $$\frac{1}{1+\frac{1}{4}e^{-j\omega}} \xleftrightarrow{\mathcal{F}} \left(-\frac{1}{4}\right)^n u[n]$$

$$\boxed{g[n] = \left(-\frac{1}{4}\right)^{n+1} u[n+1] - \frac{1}{4}\left(-\frac{1}{4}\right)^n u[n]}$$

<u>NOT CAUSAL</u> because $g[-1] = 1 \neq 0$.

(d) **6 pts.** Find the time domain difference equation that relates the input $y[n]$ and output $r[n]$ of the equalizer G.

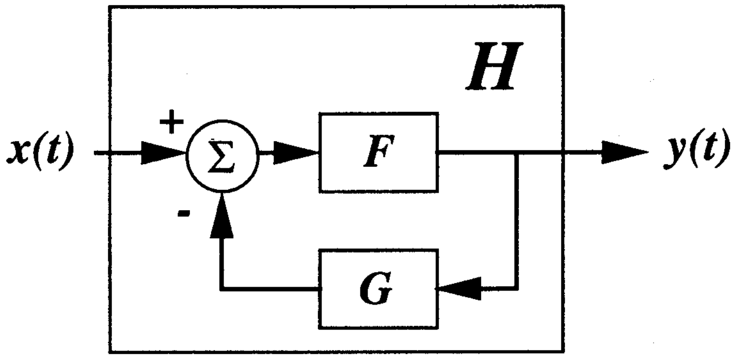
$$G(e^{j\omega}) = \frac{R(e^{j\omega})}{Y(e^{j\omega})} = \frac{e^{j\omega}(1-\frac{1}{4}e^{-j\omega})}{1+\frac{1}{4}e^{-j\omega}} = \frac{e^{j\omega}-\frac{1}{4}}{1+\frac{1}{4}e^{-j\omega}}$$

$$Y(e^{j\omega})\left[e^{j\omega} - \frac{1}{4}\right] = R(e^{j\omega})\left[1+\frac{1}{4}e^{-j\omega}\right]$$

$$e^{j\omega}Y(e^{j\omega}) - \frac{1}{4}Y(e^{j\omega}) = R(e^{j\omega}) + \frac{1}{4}e^{-j\omega}R(e^{j\omega})$$

$$\boxed{y[n+1] - \frac{1}{4}y[n] = r[n] + \frac{1}{4}r[n-1]}$$

3. **25 pts.** Consider a causal continuous-time LTI system H. As shown in the figure below, H is a negative feedback connection of two causal continuous-time LTI systems F and G. The impulse response of system F is given by $f(t) = 3e^{-4t}u(t)$.



The input-output relation for the overall system H is given by

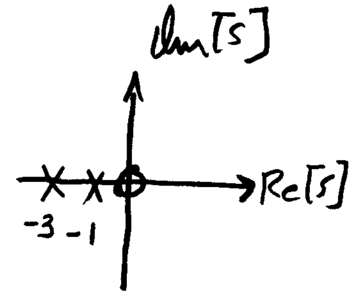
$$y''(t) + 4y'(t) + 3y(t) = 3x'(t).$$

(a) **5 pts.** Find the overall transfer function $H(s)$. Be sure to specify the ROC. Give a pole-zero plot for $H(s)$.

$$s^2 Y(s) + 4s Y(s) + 3Y(s) = 3s X(s)$$

$$[s^2 + 4s + 3] Y(s) = 3s X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s}{s^2+4s+3} = \frac{3s}{(s+3)(s+1)}$$



System is causal → ROC is Right-Sided

ROC: $Re[s] > -1$.

(b) **2 pts.** Is the system H BIBO stable? Justify your answer.

Yes. It is a causal LTI system with a rational transfer function, so it is stable iff all poles are in the LHP, which is the case here.

Problem 3, cont...

(c) **2 pts.** Does the overall system impulse response $h(t)$ have a Fourier transform? Justify your answer.

Yes: because the $j\omega$-axis of the s-plane is included in the ROC of $H(s)$.

(d) **5 pts.** Use the inverse Laplace transform to find the impulse response $h(t)$.

$$H(s) = \frac{3s}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

$$A = \left.\frac{3s}{s+1}\right|_{s=-3} = \frac{-9}{-2} = \frac{9}{2}; \quad B = \left.\frac{3s}{s+3}\right|_{s=-1} = \frac{-3}{2}$$

$$H(s) = \frac{9}{2}\underbrace{\frac{1}{s+3}}_{Re[s]>-3} - \frac{3}{2}\underbrace{\frac{1}{s+1}}_{Re[s]>-1}$$

Table: $\boxed{h(t) = \frac{9}{2}e^{-3t}u(t) - \frac{3}{2}e^{-t}u(t)}$

(e) **3 pts.** Find the transfer function $F(s)$ of system F. Be sure to specify the ROC.

$$f(t) = 3e^{-4t}u(t)$$

Table: $\boxed{F(s) = \frac{3}{s+4}, \; Re[s] > -4}$

Problem 3, cont...

(f) **8 pts.** Find the impulse response $g(t)$ of system G.

$$H(s) = \frac{F(s)}{1 + F(s)G(s)}$$

$$\frac{3s}{(s+3)(s+1)} = \frac{\frac{3}{s+4}}{1 + \frac{3}{s+4} G(s)} \cdot \frac{s+4}{s+4}$$

$$\frac{3s}{(s+3)(s+1)} = \frac{3}{s+4 + 3G(s)}$$

$$3s(s+4+3G(s)) = 3(s+3)(s+1) = 3(s^2+4s+3)$$

$$3s^2+12s+9sG(s) = 3s^2+12s+9$$

$$9sG(s) = 9$$

$$G(s) = \frac{1}{s}, \quad Re[s] > 0 \quad \left\{ \text{because } G \text{ is causal} \right.$$

Table: $\boxed{g(t) = u(t)}$

4. **25 pts**. A discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by the linear constant coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

(a) **6 pts**. Find the transfer function $H(z)$.

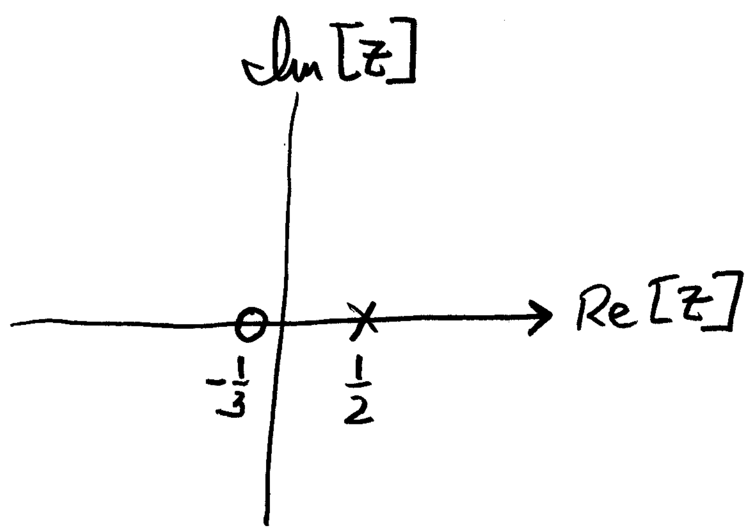
$$Y(z) - \tfrac{1}{2}z^{-1}Y(z) = X(z) + \tfrac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1 - \tfrac{1}{2}z^{-1}\right] = X(z)\left[1 + \tfrac{1}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$\boxed{H(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}}$$

(b) **3 pts**. Give a pole-zero plot for $H(z)$.



Problem 4, cont...

(c) **6 pts**. Assume that the system frequency response $H(e^{j\omega})$ exists. Give the ROC for $H(z)$ and find the system impulse response $h[n]$.

$H(e^{j\omega})$ exists $\rightarrow$ ROC $H(z)$ includes unit circle

$\rightarrow$ ROC is $Re[z] > \frac{1}{2}$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}$$

Table: $\boxed{h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]}$

(d) **4 pts**. Under the assumption of part (c) — that $H(e^{j\omega})$ exists — is the system causal? Is it BIBO stable? Justify your answers.

Exterior ROC $\rightarrow$ right-sided $h[n]$.

Moreover, from (c) we have $h[n] = 0 \;\forall\; n < 0$.

$\Rightarrow$ System is causal.

It is a causal LTI system with a rational transfer function that has all poles inside the unit circle.

It IS BIBO stable.

Problem 4, cont...

(e) **6 pts**. Now assume that the system H is unstable and is not causal. Give the ROC for $H(z)$ and find the system impulse response $h[n]$.

Unstable: ROC cannot contain the unit circle.

Not causal: ROC must be $Re[z] < \frac{1}{2}$.

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}$$

Table: $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-(n-1)-1]$

$$\boxed{h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]}$$