ECE 3793
Test 2

Monday, April 12, 2004
6:30 PM - 9:30 PM

Spring 2004 Name: SOLUTION
Dr. Havlicek Student Num: _______________________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are four problems. Work all four.

► You are allowed to use the separate formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) _______

2. (25) _______

3. (25) _______

4. (25) _______

_____________________

TOTAL (100):
1. 25 pts. Consider the causal, stable LTI system $H$ shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.

\[ x(t) \quad + \quad i(t) \quad + \quad y(t) \quad \text{+} \quad \frac{1}{6} F \quad \text{+} \quad y(t) \text{--} \]

(a) 10 pts. Find the system frequency response $H(\omega)$.

KVL: \[ \chi(t) = 5i(t) + u_L(t) + y(t) \quad \text{(1)} \]

Cap: \[ i(t) = \frac{1}{6} y'(t) \quad \text{(2)} \]

L: \[ u_L(t) = 1i'(t) = \frac{1}{6} y''(t) \quad \text{(3)} \]

\[ (2,3) \rightarrow (1): \quad \chi(t) = \frac{5}{6} y'(t) + \frac{1}{6} y''(t) + y(t) \]

\[ \frac{1}{6} y''(t) + \frac{5}{6} y'(t) + y(t) = \chi(t) \]

\[ y''(t) + 5y'(t) + 6y(t) = 6\chi(t) \]

\[ (j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega) = 6X(\omega) \]

\[ \left( (j\omega)^2 + 5j\omega + 6 \right) Y(\omega) = 6X(\omega) \]

\[ H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{6}{(j\omega)^2 + 5j\omega + 6} \]
Problem 1, cont...

(b) 5 pts. Find the system impulse response \( h(t) \).

\[
H(j\omega) = \frac{6}{(j\omega)^2 + 5j\omega + 6} = \frac{6}{(j\omega + 3)(j\omega + 2)} = \frac{A}{2 + j\omega} + \frac{B}{3 + j\omega}
\]

\[
A = \left. \frac{6}{3 + \theta} \right|_{\theta = -2} = \frac{6}{3 - 2} = 6
\]

\[
B = \left. \frac{6}{2 + \theta} \right|_{\theta = -3} = \frac{6}{-1} = -6
\]

\[
H(j\omega) = \frac{6}{2 + j\omega} - \frac{6}{3 + j\omega}
\]

\[
h(t) = 6 e^{-2t} u(t) - 6 e^{-3t} u(t)
\]
Problem 1, cont...

(c) 10 pts. Find the system output \( y(t) \) when the input is given by \( x(t) = \frac{1}{2} e^{-3t} u(t) \).

Table: \( X(\omega) = \frac{1/2}{3+j\omega} \)

\[
Y(\omega) = X(\omega)H(\omega) = \frac{1/2}{3+j\omega} \cdot \frac{6}{(3-j\omega)(2+j\omega)} = \frac{3}{(2+j\omega)(3+j\omega)^2}
\]

\[
\frac{3}{(2+\theta)(3+\theta)^2} = \frac{A}{2+\theta} + \frac{B}{(3+\theta)^2} + \frac{C}{3+\theta}
\]

\[
A = \frac{3}{(3+\theta)^2} \bigg|_{\theta = -2} = \frac{3}{1^2} = 3
\]

\[
B = \frac{3}{2+\theta} \bigg|_{\theta = -3} = \frac{3}{-1} = -3
\]

\[
\frac{d}{d\theta} \left[ 3(2+\theta)^{-1} \right] \bigg|_{\theta = -3} = \frac{d}{d\theta} \left[ C(3+\theta) \right] \bigg|_{\theta = -3}
\]

\[-3(2+\theta)^{-2} \bigg|_{\theta = -3} = C = -3(-1)^{-2} = -3(1) = -3
\]

\[
Y(\omega) = \frac{3}{2+j\omega} - \frac{3}{(3+j\omega)^2} - \frac{3}{3+j\omega}
\]

Table: \( y(t) = 3e^{-2t}u(t) - 3te^{-3t}u(t) - 3e^{-3t}u(t) \)
2. 25 pts. Consider the simple digital transmission system shown in the figure below. The signal $x[n]$ is transmitted through a communications channel $H$. Because the channel is not ideal, $x[n]$ is distorted as it passes through the channel. The distortion is modeled as a discrete-time LTI system $H$, so that the received signal is $y[n] = x[n] * h[n]$.

The system $H$ is a parallel connection of two discrete-time LTI systems $H_1$ and $H_2$. The impulse response of system $H_1$ is given by $h_1[n] = \delta[n - 1]$. The impulse response of system $H_2$ is given by $h_2[n] = \frac{1}{2}(\frac{1}{4})^{n-2}u[n - 2]$.

(a) 6 pts. Find the frequency response $H(e^{j\omega})$ of the system $H$.

\[ H_1(e^{j\omega}) = e^{-j\omega} \mathcal{F}\{e[n]\} = e^{-j\omega} \]
\[ H_2(e^{j\omega}) = \frac{1}{2} e^{-2j\omega} \mathcal{F}\{(\frac{1}{4})^nu[n]\} = \frac{1}{2} e^{-2j\omega} \frac{1}{1-\frac{1}{4}e^{-j2\omega}} \]

\[ H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega}) \]
\[ = e^{-j\omega} + \frac{1}{2} e^{-2j\omega} \frac{1}{1-\frac{1}{4}e^{-j2\omega}} \]
\[ = \frac{e^{-j\omega}(1 - \frac{1}{4}e^{-j\omega})}{1-\frac{1}{4}e^{-j2\omega}} + \frac{\frac{1}{2} e^{-j2\omega}}{1-\frac{1}{4}e^{-j2\omega}} \]
\[ = \frac{e^{-j\omega} - \frac{1}{4}e^{-j2\omega} + \frac{1}{2} e^{-j2\omega}}{1-\frac{1}{4}e^{-j2\omega}} = \frac{e^{-j\omega} + \frac{1}{4} e^{-j2\omega}}{1-\frac{1}{4}e^{-j2\omega}} \]

\[ H(e^{j\omega}) = \frac{e^{-j\omega} (1 + \frac{1}{4} e^{-j\omega})}{1-\frac{1}{4}e^{-j\omega}} \]
Problem 2, cont...

(b) 8 pts. You have been hired to design a channel equalizer $G$ that will be implemented at the receiver to "undo" the distortion. Your system $G$ will input the received signal $y[n]$ and output a signal $r[n]$ that is identical to the original transmitted signal $x[n]$. Your channel equalizer $G$ is required to be a discrete-time LTI system. **Hint:** for this to work, you must design $G$ so that it is the inverse system of $H$. Find the equalizer frequency response $G(e^{j\omega})$.

\[
G(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{1 - \frac{1}{4}e^{-j\omega}}{e^{j\omega}(1 + \frac{1}{4}e^{-j\omega})} \cdot \frac{e^{j\omega}}{e^{j\omega}}
\]

\[
G(e^{j\omega}) = \frac{e^{j\omega}(1 - \frac{1}{4}e^{-j\omega})}{1 + \frac{1}{4}e^{-j\omega}}
\]

\[
= \frac{e^{j\omega}}{1 + \frac{1}{4}e^{-j\omega}} - \frac{\frac{1}{4}}{1 + \frac{1}{4}e^{-j\omega}}
\]
Problem 2, cont...

(c) 5 pts. Use the inverse DTFT to find the equalizer impulse response $g[n]$. Is the equalizer $G$ a causal discrete-time LTI system? Justify your answer.

\[
G(e^{j\omega}) = \frac{e^{j\omega}}{1 + \frac{1}{4}e^{-j\omega}} - \frac{1}{4} \frac{1}{1 + \frac{1}{4}e^{-j\omega}}
\]

Table:

\[
\frac{1}{1 + \frac{1}{4}e^{-j\omega}} \leftrightarrow \mathcal{F} \rightarrow (-\frac{1}{4})^n u[2n]
\]

\[
g[n] = (-\frac{1}{4})^{n+1} u[n+1] - \frac{1}{4} (-\frac{1}{4})^n u[2n]
\]

**NOT CAUSAL** because $g[-1] = 1 \neq 0$.

(d) 6 pts. Find the time domain difference equation that relates the input $y[n]$ and output $r[n]$ of the equalizer $G$.

\[
G(e^{j\omega}) = \frac{R(e^{j\omega})}{Y(e^{j\omega})} = \frac{e^{j\omega}(1 - \frac{1}{4}e^{-j\omega})}{1 + \frac{1}{4}e^{-j\omega}} = \frac{e^{j\omega} - \frac{1}{4}}{1 + \frac{1}{4}e^{-j\omega}}
\]

\[
Y(e^{j\omega})[e^{j\omega} - \frac{1}{4}] = R(e^{j\omega})[1 + \frac{1}{4}e^{-j\omega}]
\]

\[
e^{j\omega}Y(e^{j\omega}) - \frac{1}{4}Y(e^{j\omega}) = R(e^{j\omega}) + \frac{1}{4}e^{-j\omega}R(e^{j\omega})
\]

\[
y[n+1] - \frac{1}{4}y[n] = r[n] + \frac{1}{4}r[n-1]
\]
3. 25 pts. Consider a causal continuous-time LTI system \( H \). As shown in the figure below, \( H \) is a negative feedback connection of two causal continuous-time LTI systems \( F \) and \( G \). The impulse response of system \( F \) is given by \( f(t) = 3e^{-4t}u(t) \).

![Diagram of system](image)

The input-output relation for the overall system \( H \) is given by

\[
y''(t) + 4y'(t) + 3y(t) = 3x'(t).
\]

(a) 5 pts. Find the overall transfer function \( H(s) \). Be sure to specify the ROC. Give a pole-zero plot for \( H(s) \).

\[
5^2 Y(s) + 4sY(s) + 3Y(s) = 3sX(s)
\]

\[
\left[ s^2 + 4s + 3 \right] Y(s) = 3sX(s)
\]

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{3s}{s^2 + 4s + 3} = \frac{3s}{(s+3)(s+1)}
\]

\[
\text{System is causal} \rightarrow \text{ROC is Right-Sided}
\]

\( \text{Roc: } \Re[s] > -1 \).

(b) 2 pts. Is the system \( H \) BIBO stable? Justify your answer.

Yes. It is a causal LTI system with a rational transfer function, so it is stable iff all poles are in the LHP, which is the case here.
Problem 3, cont...

(c) 2 pts. Does the overall system impulse response \( h(t) \) have a Fourier transform? Justify your answer.

Yes: because the \( j\omega \)-axis of the \( s \)-plane is included in the ROC of \( H(s) \).

(d) 5 pts. Use the inverse Laplace transform to find the impulse response \( h(t) \).

\[
H(s) = \frac{3s}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}
\]

\[
A = \left. \frac{3s}{s+1} \right|_{s=-3} = -\frac{9}{-2} = \frac{9}{2}; \quad B = \left. \frac{3s}{s+3} \right|_{s=-1} = \frac{-3}{2}
\]

\[
H(s) = \frac{9}{2} \frac{1}{s+3} - \frac{3}{2} \frac{1}{s+1} \quad \text{for } \text{Re}[s]>\begin{cases} -3 & \text{for } \text{Re}[s]>-1 \end{cases}
\]

Table: \( h(t) = \frac{9}{2} e^{-3t} u(t) - \frac{3}{2} e^{-t} u(t) \)

(e) 3 pts. Find the transfer function \( F(s) \) of system \( F \). Be sure to specify the ROC.

\[ f(t) = 3e^{-4t} u(t) \]

Table: \( F(s) = \frac{3}{s+4}, \quad \text{Re}[s]>-4 \)
Problem 3, cont...

(f) 8 pts. Find the impulse response \( g(t) \) of system \( G \).

\[
H(s) = \frac{F(s)}{1 + F(s)G(s)}
\]

\[
\frac{3s}{(s+3)(s+1)} = \frac{3}{s+4 + 3G(s)} \cdot \frac{s+4}{s+4}
\]

\[
\frac{3s}{(s+3)(s+1)} = \frac{3}{s^2 + 4s + 3G(s)}
\]

\[
3s(s+4+3G(s)) = 3(s+3)(s+1) = 3(s^2 + 4s + 3)
\]

\[
3s^2 + 12s + 9sG(s) = 3s^2 + 12s + 9
\]

\[
9sG(s) = 9
\]

\[
G(s) = \frac{1}{s}, \quad \text{Re}[s] > 0 \quad \{ \text{because } G \text{ is causal} \}
\]

Table: \( g(t) = u(t) \)
4. **25 pts.** A discrete-time LTI system $H$ has input $x[n]$ and output $y[n]$ related by the linear constant coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

(a) **6 pts.** Find the transfer function $H(z)$.

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z) \left[1 - \frac{1}{2}z^{-1}\right] = X(z) \left[1 + \frac{1}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

(b) **3 pts.** Give a pole-zero plot for $H(z)$.
Problem 4, cont…

(c) 6 pts. Assume that the system frequency response $H(e^{j\omega})$ exists. Give the ROC for $H(z)$ and find the system impulse response $h[n]$. 

\[ H(e^{j\omega}) \text{ exists} \rightarrow \text{ROC for } H(z) \text{ includes unit circle} \]

\[ \rightarrow \text{ROC is } \Re\{z\} > \frac{1}{2} \]

\[ H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{3}z^{-1} \frac{1}{1 - \frac{1}{2}z^{-1}} \]

Table: 

\[ h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-1] \]

(d) 4 pts. Under the assumption of part (c) — that $H(e^{j\omega})$ exists — is the system causal? Is it BIBO stable? Justify your answers.

Exterior ROC $\rightarrow$ right-sided $h[n]$.

Moreover, from (c) we have $h[n] = 0$ for $n < 0$.

$\Rightarrow$ System is causal.

It is a causal LTI system with a rational transfer function that has all poles inside the unit circle. It is BIBO stable.
(e) 6 pts. Now assume that the system $H$ is unstable and is not causal. Give the ROC for $H(z)$ and find the system impulse response $h[n]$.

Unstable: ROC cannot contain the unit circle.
Not causal: ROC must be $\text{Re}[z] < \frac{1}{2}$.

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{3}z^{-1} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Table:

$$h[n] = -\left(\frac{1}{2}\right)^n u[n-n-1] - \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-(n-1)-1]$$

$$h[n] = -\left(\frac{1}{2}\right)^n u[n-n-1] - \frac{1}{3} \left(\frac{1}{2}\right)^n u[n-n-1]$$