# ECE 3793
# Test 2

Thursday, April 28, 2005
6:00 PM - 9:00 PM

Spring 2005 | Name: SOLUTION

Dr. Havlicek | Student Num:

**Directions:** This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**.

► You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

## GOOD LUCK!

SCORE:

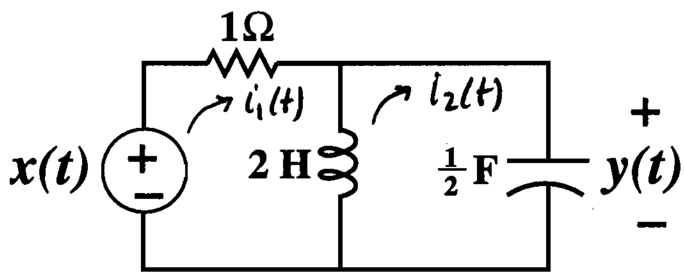
1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

*On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.*

Name: ______________________ Date: ______________________

1. **25 pts**. Consider the causal, stable LTI system $H$ shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.




(a) **10 pts**. Find the system frequency response $H(\omega)$.

cap: $i_2(t) = \frac{1}{2} y'(t)$ ①

Resistor: $V_R(t) = 1\, i_1(t) = i_1(t)$ ②

Inductor: $V_L(t) = 2 i_L'(t)$ ③

KCL @ top node: $i_L(t) = i_1(t) - i_2(t)$ ④

④→③ $V_L(t) = 2 i_1'(t) - 2 i_2'(t)$ ⑤

KVL on right loop: $y(t) = V_L(t)$ ⑥

⑤→⑥: $y(t) = 2 i_1'(t) - 2 i_2'(t)$ ⑦

①→⑦: $y(t) = 2 i_1'(t) - y''(t)$ ⑧

KVL on big loop: $x(t) = V_R(t) + y(t)$ ⑨

②→⑨: $x(t) = i_1(t) + y(t) \Rightarrow i_1(t) = x(t) - y(t)$ ⑩

⑩→⑧: $y(t) = 2x'(t) - 2y'(t) - y''(t)$

$$y''(t) + 2y'(t) + y(t) = 2x'(t)$$

F.T. : $(j\omega)^2 Y(\omega) + 2j\omega Y(\omega) + Y(\omega) = 2j\omega X(\omega)$

$$\left[(j\omega)^2 + 2j\omega + 1\right] Y(\omega) = 2j\omega X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2j\omega}{(j\omega)^2 + 2j\omega + 1} = \frac{2j\omega}{(j\omega+1)^2}}$$

(b) **5 pts.** Find the system impulse response $h(t)$.

METHOD ONE

$$\frac{2\theta}{(\theta+1)^2} = \frac{A}{(\theta+1)^2} + \frac{B}{\theta+1}$$

$$A = 2\theta\Big|_{\theta=-1} = -2$$

$$\frac{\partial}{\partial\theta}\left[2\theta\right]_{\theta=-1} = \frac{\partial}{\partial\theta}A\Big|_{\theta=-1} + \frac{\partial}{\partial\theta}\left[(\theta+1)B\right]_{\theta=-1}$$

$$2 = 0 + B + 0$$

$$B = 2$$

$$H(\omega) = \frac{2}{j\omega+1} - \frac{2}{(j\omega+1)^2}$$

TABLE:

$$\underline{h(t) = 2e^{-t}u(t) - 2te^{-t}u(t)}$$

METHOD TWO

$$H(\omega) = j\omega\,\frac{2}{(1+j\omega)^2}$$

Let $G(\omega) = \dfrac{2}{(1+j\omega)^2}$

Table:

$$g(t) = 2te^{-t}u(t)$$

Time differentiation property:

$$h(t) = g'(t)$$

$$= \left[2te^{-t}\right]\left[\frac{d}{dt}u(t)\right] + \left[\frac{d}{dt}2te^{-t}\right]u(t)$$

$$= 2te^{-t}\delta(t) + \left[2e^{-t} - 2te^{-t}\right]u(t)$$

$$= 2(0)e^{-0}\delta(t) + 2e^{-t}u(t) - 2te^{-t}u(t)$$

$$= \underline{2e^{-t}u(t) - 2te^{-t}u(t)}$$

**Problem 1, cont...**

(c) **10 pts**. Let the system input be $x(t) = \delta(t) - 2e^{-3t}u(t)$. Find the output $y(t)$.

Table: $X(\omega) = 1 - \frac{2}{j\omega+3} = \frac{j\omega+3}{j\omega+3} - \frac{2}{j\omega+3} = \frac{j\omega+1}{j\omega+3}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{j\omega+1}{j\omega+3} \cdot \frac{2j\omega}{(1+j\omega)^2} = \frac{2j\omega}{(j\omega+1)(j\omega+3)}$$

$$\frac{2\theta}{(\theta+1)(\theta+3)} = \frac{A}{\theta+1} + \frac{B}{\theta+3}$$

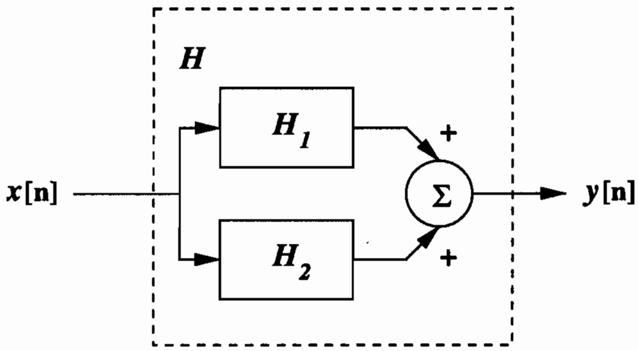
$$A = \left.\frac{2\theta}{\theta+3}\right|_{\theta=-1} = \frac{-2}{2} = -1$$

$$B = \left.\frac{2\theta}{\theta+1}\right|_{\theta=-3} = \frac{-6}{-2} = 3$$

$$Y(\omega) = \frac{3}{j\omega+3} - \frac{1}{j\omega+1}$$

Table: $\boxed{y(t) = 3e^{-3t}u(t) - e^{-t}u(t)}$

2. **25 pts.** Consider a discrete-time LTI system $H_1$ with impulse response $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ and a second discrete-time LTI system $H_2$. The systems $H_1$ and $H_2$ are connected together in parallel to form an overall system $H$ as shown in the figure below.




The frequency response of the overall system $H$ is given by

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}. = \frac{-12+5e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

Find $h_2[n]$, the impulse response of system $H_2$.

Table: $H_1(e^{j\omega}) = \dfrac{1}{1-\frac{1}{3}e^{-j\omega}} = \dfrac{3}{3-e^{-j\omega}}$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega}) \Rightarrow H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega})$$

$$H_2(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})} - \frac{3}{3-e^{-j\omega}} \cdot \frac{4-e^{-j\omega}}{4-e^{-j\omega}}$$

$$= \frac{-12+5e^{-j\omega} - 12 + 3e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-24+8e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$= \frac{-8(3-e^{-j\omega})}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-8}{4-e^{-j\omega}} = \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

Table: $\boxed{h_2[n] = -2\left(\tfrac{1}{4}\right)^n u[n]}$

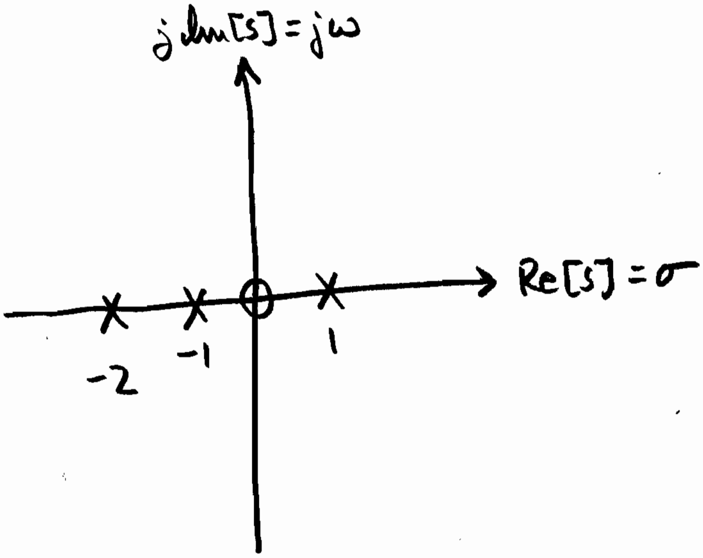
More Workspace for Problem 2...

3. **25 pts.** Consider a **BIBO stable** continuous-time LTI system $H$ with transfer function

$$H(s) = \frac{s}{(s+2)(s+1)(s-1)}.$$

(a) **2 pts.** Give a pole-zero plot for $H(s)$.

one zero at $s=0$, Three poles at $s = -2, -1, 1$




(b) **3 pts.** Carefully use the provided information plus your pole zero plot to determine the ROC of $H(s)$. Justify your answer.

H is a stable LTI system with a rational transfer function $\Rightarrow$ The ROC must include the $j\omega$-axis.

$$\boxed{\text{ROC: } -1 < Re[s] < 1}$$

Problem 3, cont...

(c) **2 pts.** Is the system $H$ causal? Justify your answer.

If the system was causal, the ROC of $H(s)$ would have to be the right half-plane to the right of the rightmost pole, or $Re[s] > 1$. That's not the case here.

$\boxed{\text{NOT CAUSAL}}$

(d) **8 pts.** Find the system impulse response $h(t)$.

$$H(s) = \frac{s}{(s+2)(s+1)(s-1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$A = \left.\frac{s}{(s+1)(s-1)}\right|_{s=-2} = \frac{-2}{(-1)(-3)} = -\frac{2}{3}$$

$$B = \left.\frac{s}{(s+2)(s-1)}\right|_{s=-1} = \frac{-1}{(1)(-2)} = \frac{1}{2}$$

$$C = \left.\frac{s}{(s+2)(s+1)}\right|_{s=1} = \frac{1}{(3)(2)} = \frac{1}{6}$$

$$H(s) = \underbrace{\frac{1/2}{s+1}}_{Re[s]>-1} - \underbrace{\frac{2/3}{s+2}}_{Re[s]>-2} + \underbrace{\frac{1/6}{s-1}}_{Re[s]<1}$$

Table: $\boxed{h(t) = \frac{1}{2}e^{-t}u(t) - \frac{2}{3}e^{-2t}u(t) - \frac{1}{6}e^{t}u(-t)}$

Problem 3, cont...

(e) **10 pts.** Find the system output $y(t)$ when the input is given by $x(t) = e^{-t}u(t)$.

Table: $X(s) = \frac{1}{s+1}$, $\mathrm{Re}[s] > -1$

$$Y(s) = X(s)H(s) = \frac{s}{(s+2)(s+1)^2(s-1)}$$

The ROC for $Y(s)$ is the intersection of the ROCs for $X(s)$ and $H(s)$, or $\{\mathrm{Re}[s] > -1\} \cap \{-1 < \mathrm{Re}[s] < 1\}$

ROC: $-1 < \mathrm{Re}[s] < 1$

$$Y(s) = \frac{s}{(s+2)(s+1)^2(s-1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{s-1}$$

$$A = \frac{s}{(s+1)^2(s-1)}\bigg|_{s=-2} = \frac{-2}{(-1)^2(-3)} = \frac{2}{3}$$

$$C = \frac{s}{(s+2)(s-1)}\bigg|_{s=-1} = \frac{-1}{(1)(-2)} = \frac{1}{2}$$

$$D = \frac{s}{(s+2)(s+1)^2}\bigg|_{s=1} = \frac{1}{(3)(2)^2} = \frac{1}{3\cdot 4} = \frac{1}{12}$$

$$B: \frac{\partial}{\partial s}\left[s(s+2)^{-1}(s-1)^{-1}\right]_{s=-1} = 0\cdot A + \frac{\partial}{\partial s}\left[(s+1)B\right]_{s=-1} + \frac{\partial}{\partial s}C\bigg|_{s=-1} + 0\cdot D$$

$$\left\{s(s+2)^{-1}(-1)(s-1)^{-2} + (s-1)^{-1}\left[s(-1)(s+2)^{-2} + (s+2)^{-1}\right]\right\}_{s=-1} = \frac{\partial}{\partial s}sB\bigg|_{s=-1}$$

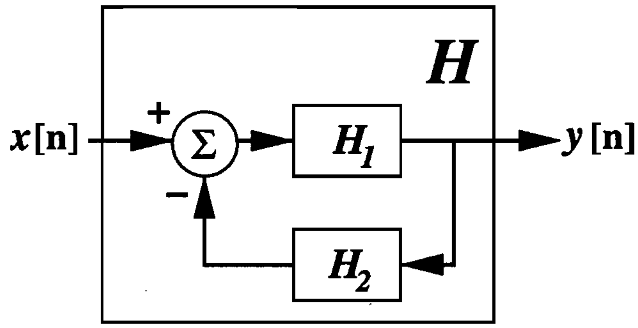
$$\left\{(-1)(1)^{-1}(-1)(-2)^{-2} + (-2)^{-1}\left[(-1)(-1)(1)^2 + (1)^{-1}\right]\right\} = B$$

$$\left\{\frac{1}{4} - \frac{1}{2}(1+1)\right\} = \frac{1}{4} - 1 = -\frac{3}{4} = B$$

$$Y(s) = \underbrace{\frac{2/3}{s+2}}_{\mathrm{Re}[s]>-2} - \underbrace{\frac{3/4}{s+1}}_{\mathrm{Re}[s]>-1} + \underbrace{\frac{1/2}{(s+1)^2}}_{\mathrm{Re}[s]>-1} + \underbrace{\frac{1/12}{s-1}}_{\mathrm{Re}[s]<1}$$

Table: $\boxed{y(t) = \frac{2}{3}e^{-2t}u(t) - \frac{3}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{12}e^{t}u(-t)}$

4. **25 pts**. The **causal** discrete-time system $H$ is formed by connecting two discrete-time LTI systems $H_1$ and $H_2$ in a negative feedback configuration as shown in the figure below.




The impulse response of $H_1$ is given by

$$h_1[n] = \left(\frac{1}{\sqrt{2}}\right)^n u[n].$$

The impulse response of $H_2$ is given by

$$h_2[n] = \left(-\frac{1}{\sqrt{2}}\right)^n u[n].$$

(a) **2 pts**. Find the transfer function $H_1(z)$. Be sure to specify the ROC.

Table: $H_1(z) = \dfrac{1}{1 - \frac{1}{\sqrt{2}} z^{-1}}$ , $|z| > \frac{1}{\sqrt{2}}$

(b) **2 pts**. Find the transfer function $H_2(z)$. Be sure to specify the ROC.

Table: $H_2(z) = \dfrac{1}{1 + \frac{1}{\sqrt{2}} z^{-1}}$ , $|z| > \frac{1}{\sqrt{2}}$

Problem 4, cont...

(c) **6 pts**. Find the transfer function $H(z)$ of the overall system $H$.

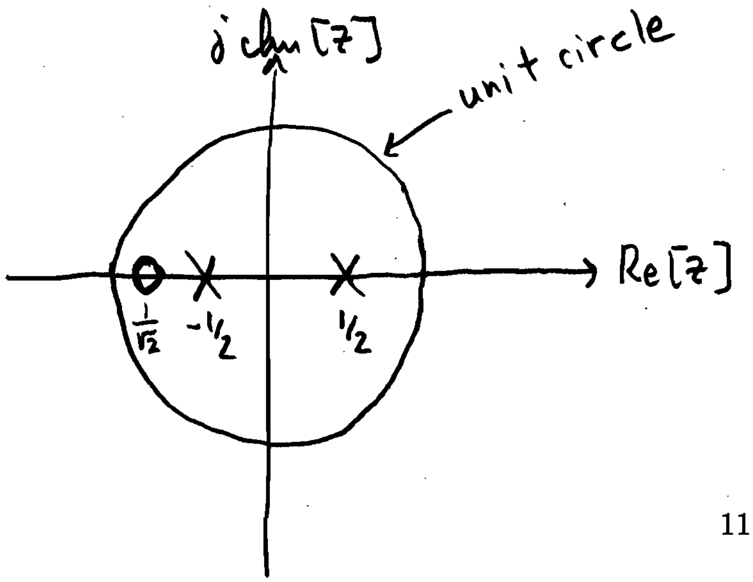
$$H(z) = \frac{H_1(z)}{1+H_1(z)H_2(z)} = \frac{\frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}}}{1+\frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}}\,\frac{1}{1+\frac{1}{\sqrt{2}}z^{-1}}} \cdot \frac{\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}$$

$$= \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)+1} = \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{1-\frac{1}{2}z^{-2}+1} = \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{2-\frac{1}{2}z^{-2}}$$

$$= \frac{1/2}{1/2} \cdot \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{2-\frac{1}{2}z^{-2}} = \frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{1-\frac{1}{4}z^{-2}}$$

$$H(z) = \frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

(d) **3 pts**. Give a pole-zero plot for $H(z)$ and specify the ROC. **Note:** it was given above that the system $H$ is **causal**.




zero: $-\frac{1}{\sqrt{2}}$

poles: $\pm\frac{1}{2}$

Since the system is causal, the ROC is exterior.

ROC: $|z| > \frac{1}{2}$

**Problem 4, cont...**

(e) **2 pts.** Is the system $H$ BIBO stable? Justify your answer.

Yes. Because it has a rational transfer function and the ROC contains the unit circle.

Alternate Answer: Yes, because it's causal and the poles are inside the unit circle.

(f) **10 pts.** Find the overall system impulse response $h[n]$.

$$H(z) = \frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\theta\right)}{1-\frac{1}{2}\theta}\right|_{\theta=-2} = \frac{\frac{1}{2}\left(1+\frac{-2}{\sqrt{2}}\right)}{1+1} = \frac{1}{4}\left(1-\sqrt{2}\right)$$

$$B = \left.\frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\theta\right)}{1+\frac{1}{2}\theta}\right|_{\theta=2} = \frac{\frac{1}{2}\left(1+\frac{2}{\sqrt{2}}\right)}{1+1} = \frac{1}{4}\left(1+\sqrt{2}\right)$$

$$H(z) = \underbrace{\frac{1}{4}\left(1-\sqrt{2}\right)\frac{1}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{1}{4}\left(1+\sqrt{2}\right)\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>1/2}$$

Table: $\boxed{h[n] = \frac{1}{4}\left(1-\sqrt{2}\right)\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{4}\left(1+\sqrt{2}\right)\left(\frac{1}{2}\right)^n u[n]}$