ECE 3793
Test 2

Friday, April 28, 2006
12:30 PM - 1:20 PM

Spring 2006 | Name: SOLUTION

Dr. Havlicek | Student Num: ____________________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 50 minutes to complete the test.

► You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (35) ________

2. (30) ________

3. (35) ________

____

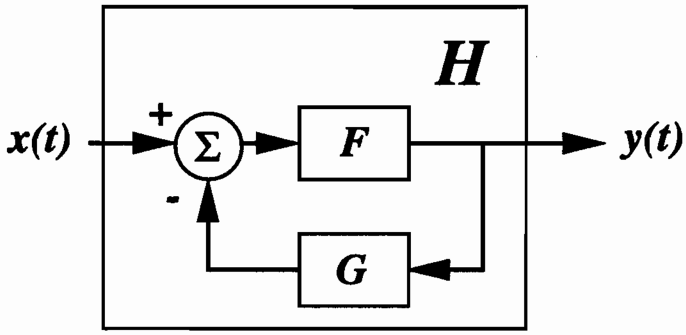
TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______________________________ Date: ______________________________

1. **35 pts.** Consider the continuous-time LTI system H shown below. This system is causal and stable.



The system input-ouptut relation is given by

$$y'(t) + 5y(t) = x(t).$$

The impulse response of system F is given by

$$f(t) = 3e^{-4t}u(t).$$

(a) **15 pts.** Find the system frequency response $H(j\omega)$.

$$y'(t) + 5y(t) = x(t)$$

$$j\omega Y(\omega) + 5Y(\omega) = X(\omega)$$

$$[j\omega + 5]\, Y(\omega) = X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega + 5}}$$

Problem 1, cont...

(b) **10 pts**. Find the system impulse response $h(t)$.

TABLE:

$$\boxed{h(t) = e^{-5t} u(t)}$$

(c) **10 pts**. Find the impulse response $g(t)$ of system G.

TABLE: $F(\omega) = \dfrac{3}{4+j\omega}$

$$\frac{1}{j\omega+5} = H(\omega) = \frac{F(\omega)}{1+F(\omega)G(\omega)} = \frac{\frac{3}{4+j\omega}}{1+\frac{3}{4+j\omega}G(\omega)} \cdot \frac{4+j\omega}{4+j\omega} = \frac{3}{4+j\omega+3G(\omega)}$$

$$4+j\omega+3G(\omega) = 3(j\omega+5) = 3j\omega+15$$

$$3G(\omega) = (3j\omega+15)-(4+j\omega)$$

$$= 11+2j\omega$$

$$G(\omega) = \frac{11}{3} + \frac{2}{3}j\omega$$ use derivative property of F.T.

TABLE: $g(t) = \frac{11}{3}\delta(t) + \frac{2}{3}\frac{d}{dt}\delta(t)$

$$\boxed{g(t) = \frac{11}{3}\delta(t) + \frac{2}{3}\delta'(t)}$$

2. **30 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \frac{2}{5}(n+1)\left(\frac{1}{2}\right)^n u[n] + \frac{24}{25}\left(\frac{1}{2}\right)^n u[n] + \frac{16}{25}\left(-\frac{1}{3}\right)^n u[n].$$

(a) **10 pts.** Find the frequency response $H(e^{j\omega})$.

Hint: the answer is

$$H(e^{j\omega}) = \frac{2-\frac{2}{3}e^{-j\omega}}{1-\frac{2}{3}e^{-j\omega}-\frac{1}{12}e^{-j2\omega}+\frac{1}{12}e^{-j3\omega}} = \frac{2-\frac{2}{3}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2(1+\frac{1}{3}e^{-j\omega})}.$$

TABLE: $H(e^{j\omega}) = \frac{2}{5}\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2} + \frac{24}{25}\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{16}{25}\frac{1}{1+\frac{1}{3}e^{-j\omega}}$

$$= \frac{10}{25}\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2} + \frac{24}{25}\frac{1}{(1-\frac{1}{2}e^{-j\omega})} + \frac{16}{25}\frac{1}{(1+\frac{1}{3}e^{-j\omega})}$$

$$= \frac{1}{25}\left[\frac{10}{(1-\frac{1}{2}e^{-j\omega})^2} + \frac{24}{(1-\frac{1}{2}e^{-j\omega})} + \frac{16}{(1+\frac{1}{3}e^{-j\omega})}\right]$$

$$= \frac{1}{25}\left[\frac{10(1+\frac{1}{3}e^{-j\omega}) + 24(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega}) + 16(1-\frac{1}{2}e^{-j\omega})^2}{(1-\frac{1}{2}e^{-j\omega})^2(1+\frac{1}{3}e^{-j\omega})}\right]$$

$$= \frac{1}{25}\left[\frac{10+\frac{10}{3}e^{-j\omega} + 24(1+\frac{1}{3}e^{-j\omega}-\frac{1}{2}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}) + 16(1-e^{-j\omega}+\frac{1}{4}e^{-j2\omega})}{(1-e^{-j\omega}+\frac{1}{4}e^{-j2\omega})(1+\frac{1}{3}e^{-j\omega})}\right]$$

$$= \frac{1}{25}\left[\frac{10+\frac{10}{3}e^{-j\omega} + 24(1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}) + 16 - 16e^{-j\omega} + 4e^{-j2\omega}}{1+\frac{1}{3}e^{-j\omega} - e^{-j\omega} - \frac{1}{3}e^{-j2\omega} + \frac{1}{4}e^{-j2\omega} + \frac{1}{12}e^{-j3\omega}}\right]$$

$$= \frac{1}{25}\left[\frac{10+\frac{10}{3}e^{-j\omega} + 24 - 4e^{-j\omega} - 4e^{-j2\omega} + 16 - 16e^{-j\omega} + 4e^{-j2\omega}}{1-\frac{2}{3}e^{-j\omega} - \frac{1}{12}e^{-j2\omega} + \frac{1}{12}e^{-j\omega 3}}\right]$$

$$= \frac{1}{25}\left[\frac{50 + (\frac{10}{3}-4-16)e^{-j\omega} + (4-4)e^{-j2\omega}}{1-\frac{2}{3}e^{-j\omega} - \frac{1}{12}e^{-j2\omega} + \frac{1}{12}e^{-j3\omega}}\right]$$

$$= \frac{1}{25}\left[\frac{50 - \frac{50}{3}e^{-j\omega}}{1-\frac{2}{3}e^{-j\omega} - \frac{1}{12}e^{-j2\omega} + \frac{1}{12}e^{-j3\omega}}\right]$$

$$\boxed{H(e^{j\omega}) = \frac{2-\frac{2}{3}e^{-j\omega}}{1-\frac{2}{3}e^{-j\omega}-\frac{1}{12}e^{-j2\omega}+\frac{1}{12}e^{-j3\omega}}}$$

$$= \frac{2-\frac{2}{3}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2(1+\frac{1}{3}e^{-j\omega})}$$

More Workspace for Problem 2(a)...

Problem 2, cont...

(b) **10 pts.** Find the difference equation that relates the input $x[n]$ and output $y[n]$.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2 - \frac{2}{3}e^{-j\omega}}{1 - \frac{2}{3}e^{-j\omega} - \frac{1}{12}e^{-j2\omega} + \frac{1}{12}e^{-j3\omega}}$$

$$Y(e^{j\omega})\left[1 - \tfrac{2}{3}e^{-j\omega} - \tfrac{1}{12}e^{-j2\omega} + \tfrac{1}{12}e^{-j3\omega}\right] = X(e^{j\omega})\left[2 - \tfrac{2}{3}e^{-j\omega}\right]$$

$$Y(e^{j\omega}) - \tfrac{2}{3}e^{-j\omega}Y(e^{j\omega}) - \tfrac{1}{12}e^{-j2\omega}Y(e^{j\omega}) + \tfrac{1}{12}e^{-j3\omega}Y(e^{j\omega})$$

$$= 2X(e^{j\omega}) - \tfrac{2}{3}e^{-j\omega}X(e^{j\omega})$$

$$\boxed{y[n] - \tfrac{2}{3}y[n-1] - \tfrac{1}{12}y[n-2] + \tfrac{1}{12}y[n-3] = 2x[n] - \tfrac{2}{3}x[n-1]}$$

Problem 2, cont...

(c) **10 pts.** Find the output $y[n]$ when the input is given by $x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$.

TABLE: $X(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \left(1-\tfrac{1}{2}e^{-j\omega}\right)\cdot\frac{2-\frac{2}{3}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2\left(1+\frac{1}{3}e^{-j\omega}\right)}$$

$$= \frac{2-\frac{2}{3}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}}$$

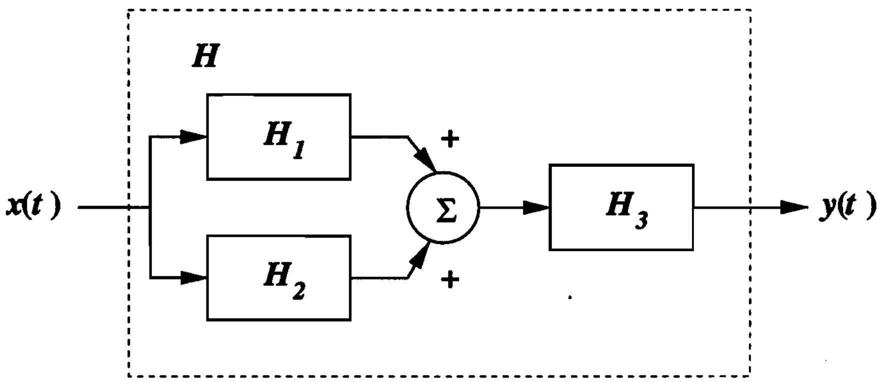
$$A = \left.\frac{2-\frac{2}{3}\theta}{1+\frac{1}{3}\theta}\right|_{\theta=2} = \frac{2-\frac{4}{3}}{1+\frac{2}{3}} = \frac{6-4}{3+2} = \frac{2}{5}$$

$$B = \left.\frac{2-\frac{2}{3}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-3} = \frac{2+2}{1+\frac{3}{2}} = \frac{4}{5/2} = \frac{8}{5}$$

$$Y(e^{j\omega}) = \frac{2}{5}\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{8}{5}\frac{1}{1+\frac{1}{3}e^{-j\omega}}$$

TABLE: $\boxed{y[n] = \frac{2}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{8}{5}\left(-\frac{1}{3}\right)^n u[n]}$

3. **35 pts**. Consider the continuous-time LTI system H shown below:



The impulse response of LTI system H_1 is given by $h_1(t) = -\frac{1}{2}e^{4t}u(-t)$.

LTI system H_2 is causal and has input $x_2(t)$ and output $y_2(t)$ related by

$$4y_2(t) - 2y_2'(t) = x_2(t).$$

LTI system H_3 is causal and stable and has impulse response $h_3(t) = \delta(t-2)$.

(a) **10 pts**. Find the transfer function $H(s)$ for the overall system. Be sure to specify the ROC.

TABLE: $H_1(s) = \frac{1/2}{s-4}$, ROC R_1 : $\underline{\underline{Re[s] < 4}}$

H_2 : $4y_2(t) - 2y_2'(t) = x_2(t)$

$4Y_2(s) - 2sY_2(s) = X_2(s)$

$Y_2(s)[4-2s] = X_2(s)$

$$H_2(s) = \frac{Y_2(s)}{X_2(s)} = \frac{1}{4-2s} = \frac{-1}{2s-4} = -\frac{1}{2}\frac{1}{s-2}$$

Because H_2 is given causal, the ROC is R_2: $\underline{\underline{Re[s] > 2}}$.

TABLE: $H_3(s) = e^{-2s}$, ROC R_3 : All s.

$$H(s) = [H_1(s) + H_2(s)]H_3(s) = \left[\frac{1}{2}\frac{1}{s-4} - \frac{1}{2}\frac{1}{s-2}\right]e^{-2s}$$

$$= \frac{\frac{1}{2}(s-2) - \frac{1}{2}(s-4)}{(s-4)(s-2)}e^{-2s} = \frac{\frac{1}{2}s - 1 - \frac{1}{2}s + 2}{(s-4)(s-2)}e^{-2s} = \frac{e^{-2s}}{(s-4)(s-2)}$$

ROC : $R = R_1 \cap R_2 \cap R_3$

$= \{Re[s] < 4\} \cap \{Re[s] > 2\} \cap \{\text{all } s\}$

$= 2 < Re[s] < 4$

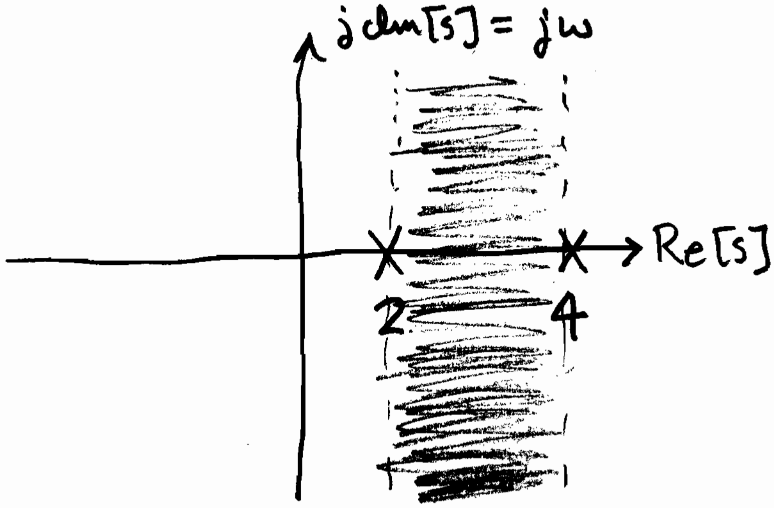
$$\boxed{H(s) = \frac{e^{-2s}}{(s-4)(s-2)},\ 2 < Re[s] < 4}$$

Problem 3, cont...

(b) **5 pts.** Give a pole-zero plot for $H(s)$. Indicate the ROC on the plot.

$$H(s) = \frac{e^{-2s}}{(s-4)(s-2)}$$

$2 < Re[s] < 4$



(c) **5 pts.** Is the system H causal? (Justify your answer)

For an LTI system to be causal, the ROC must be the right half-plane to the right of the rightmost pole.

But here the ROC is the strip $2 < Re[s] < 4$.

So this system is <u>NOT CAUSAL</u>

(d) **5 pts.** Is the system H BIBO stable? (Justify your answer)

For an LTI system to be BIBO stable, the ROC must include the $j\omega$-axis of the s-plane. That's not the case here.

So this system is <u>NOT BIBO STABLE</u>

Problem 3, cont...

(e) **10 pts.** Find the system output $y(t)$ when the input is given by $x(t) = \delta'(t) - 2\delta(t)$.

TABLE : $X(s) = s-2$, ROC : all finite s.

$$Y(s) = X(s)H(s) = (s-2)\frac{e^{-2s}}{(s-4)(s-2)} = \frac{e^{-2s}}{s-4}$$

The ROC of $H(s)$ was $\{Re[s] > 2\} \cap \{Re[s] < 4\}$

But $X(s)$ cancels the pole at $s=2$.

So we are left with $Re[s] < 4$ for the ROC of $Y(s)$.

$$Y(s) = \frac{e^{-2s}}{s-4}, \quad Re[s] < 4.$$

TABLE : $\frac{1}{s-4}$, $Re[s] < 4 \underset{\alpha=-4}{\overset{\mathcal{L}}{\longleftrightarrow}} -e^{4t}u(-t)$

Time Shift Property: $y(t) = -e^{4(t-2)}u(-(t-2))$

$$= -e^{4(t-2)}u(-t+2)$$

$$\boxed{y(t) = -e^{4(t-2)}u(-t+2)}$$