ECE 3793
Test 2

Thursday, April 24, 2008
6:00 PM - 9:00 PM

Spring 2008 — Name: SOLUTION

Dr. Havlicek — Student Num: ______

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test.

► You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ______
2. (25) ______
3. (25) ______
4. (25) ______

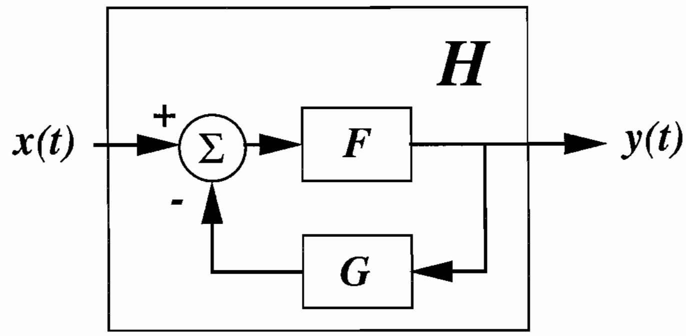
TOTAL (100):

______

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______ Date: ______

1. **25 pts**. Consider the continuous-time LTI system H shown below. This system is causal and stable.



The system input-ouptut relation is given by

$$y'(t) + 5y(t) = x(t).$$

The impulse response of system F is given by

$$f(t) = 3e^{-4t}u(t).$$

(a) **10 pts**. Find the system frequency response $H(j\omega)$.

For the system H, $y'(t) + 5y(t) = x(t)$

$$j\omega Y(\omega) + 5Y(\omega) = X(\omega)$$

$$Y(\omega)[j\omega + 5] = X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{j\omega + 5}}$$

Problem 1, cont...

(b) **5 pts**. Find the system impulse response $h(t)$.

Table: $H(\omega) = \frac{1}{5+j\omega}$

$$\boxed{h(t) = e^{-5t}u(t)}$$

(c) **10 pts**. Find the impulse response $g(t)$ of system G.

TABLE: $f(t) = 3e^{-4t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{3}{4+j\omega} = F(\omega)$

$$H(\omega) = \frac{F(\omega)}{1+F(\omega)G(\omega)} = \frac{1}{5+j\omega}$$

plug In $F(\omega)$ and solve for $G(\omega)$:

$$\frac{\frac{3}{4+j\omega}}{1+\frac{3}{4+j\omega}G(\omega)} = \frac{1}{5+j\omega}$$

$$\frac{3}{4+j\omega+3G(\omega)} = \frac{1}{5+j\omega}$$

$$3(5+j\omega) = 4+j\omega+3G(\omega)$$

$$15+3j\omega = 4+j\omega+3G(\omega)$$

$$11+2j\omega = 3G(\omega)$$

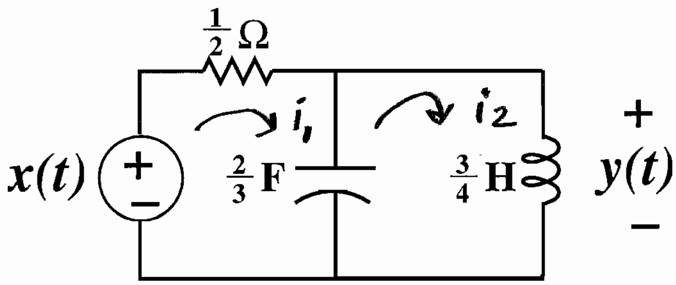
$$G(\omega) = \frac{11}{3} + \frac{2}{3}j\omega$$

TABLE: $\frac{11}{3} \leftrightarrow \frac{11}{3}\delta(t)$

Derivative Property: $\frac{2}{3}j\omega \leftrightarrow \frac{2}{3}\delta'(t)$

$$\boxed{g(t) = \frac{11}{3}\delta(t) + \frac{2}{3}\delta'(t)}$$

2. **25 pts**. Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.



(a) **8 pts**. Find the system input-output equation that relates $x(t)$ and $y(t)$.

R: $v_R(t) = \frac{1}{2} i_1(t)$ (1)

C: $i_c(t) = \frac{2}{3} v_c'(t)$

KCL, KVL

$i_1(t) - i_2(t) = \frac{2}{3} y'(t)$ (2)

L: $y(t) = \frac{3}{4} i_2'(t)$ (3)

KVL on outside loop:

$x(t) = v_R(t) + y(t) = \frac{1}{2} i_1(t) + y(t)$ (4)

solve (2) for $i_1(t)$:

$i_1(t) = i_2(t) + \frac{2}{3} y'(t)$ (5)

(5) → (4):

$x(t) = \frac{1}{2} i_2(t) + \frac{1}{3} y'(t) + y(t)$

$\frac{d}{dt}$: $x'(t) = \frac{1}{2} i_2'(t) + \frac{1}{3} y''(t) + y'(t)$ (6)

(3): $i_2'(t) = \frac{4}{3} y(t)$ (7)

(7) → (6): $x'(t) = \frac{2}{3} y(t) + y'(t) + \frac{1}{3} y''(t)$

$\frac{1}{3} y''(t) + y'(t) + \frac{2}{3} y(t) = x'(t)$

$$\boxed{y''(t) + 3y'(t) + 2y(t) = 3x'(t)}$$

Problem 1, cont...

(b) **5 pts**. Find the system transfer function $H(s)$. Give a pole zero plot and show the region of convergence.

$$y''(t) + 3y'(t) + 2y(t) = 3x'(t)$$

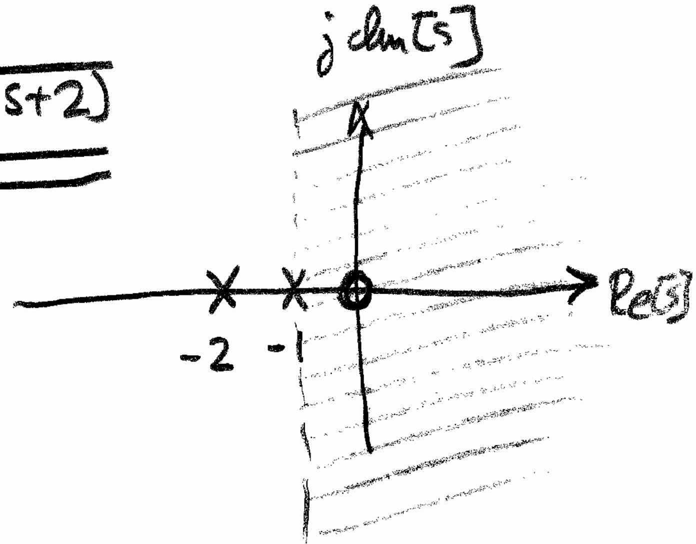
$$s^2 Y(s) + 3sY(s) + 2Y(s) = 3s\,X(s)$$

$$[s^2 + 3s + 2]\,Y(s) = 3s\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s}{s^2+3s+2} = \underline{\underline{\frac{3s}{(s+1)(s+2)}}}$$

ROC: $Re[s] > -1$.

Because the system is causal and stable, the ROC is a right half-plane that includes the $j\omega$-axis. The poles are at $s = -1, -2$. Zero is at $s = 0$.



(c) **4 pts**. Find the system impulse response $h(t)$.

$$H(s) = \frac{3s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \qquad Re[s] > -1$$

$$A = \left.\frac{3s}{s+2}\right|_{s=-1} = \frac{-3}{1} = -3$$

$$B = \left.\frac{3s}{s+1}\right|_{s=-2} = \frac{-6}{-1} = 6$$

$$H(s) = \underbrace{\frac{6}{s+2}}_{Re[s] > -2} - \underbrace{\frac{3}{s+1}}_{Re[s] > -1}$$

TABLE: $\boxed{h(t) = 6e^{-2t}u(t) - 3e^{-t}u(t)}$

Problem 1, cont...

(d) **8 pts**. Let the system input be $x(t) = \frac{1}{3}e^{-t}u(t)$. Find the output $y(t)$.

TABLE: $X(s) = \dfrac{1/3}{s+1}$, $Re[s] > -1$.

$$Y(s) = X(s)H(s) = \frac{1/3}{s+1} \cdot \frac{3s}{(s+1)(s+2)} = \frac{s}{(s+1)^2(s+2)}, \quad Re[s]>-1.$$

$$= \frac{A}{s+2} + \frac{B}{(s+1)^2} + \frac{C}{s+1}$$

$$A = \left.\frac{s}{(s+1)^2}\right|_{s=-2} = \frac{-2}{(-1)^2} = -2$$

$$B = \left.\frac{s}{s+2}\right|_{s=-1} = \frac{-1}{1} = -1$$

$$C: \frac{d}{ds}\left[s(s+2)^{-1}\right]_{s=-1} = \frac{d}{ds}\left[(s+1)^2 A\,(s+2)^{-1}\right]_{s=-1} + \frac{d}{ds}B + \frac{d}{ds}(s+1)C \Big|_{s=-1}$$

$$\left[(s+2)^{-1} + s(-1)(s+2)^{-2}\right]_{s=-1} = \left[2(s+1)(s+2)^{-1} + (s+1)^2(-1)(s+2)^{-1}\right]_{s=-1} A + 0 + C$$

$$\left[(1)^{-1} + (-1)(-1)(1)^{-2}\right] = 0A + 0 + C = C$$

$$C = 1 + 1\cdot 1 = 2$$

$$Y(s) = \underbrace{\frac{2}{s+1} - \frac{1}{(s+1)^2}}_{Re[s]>-1} - \underbrace{\frac{2}{s+2}}_{Re[s]>-2}$$

TABLE: $\boxed{y(t) = 2e^{-t}u(t) - te^{-t}u(t) - 2e^{-2t}u(t)}$

ROC includes $j\omega$-axis

3. **25 pts.** A **BIBO stable** LTI system H has output

$$y(t) = -\tfrac{1}{5}e^{3(t-1)}u(-t+1) - \tfrac{1}{5}e^{-2(t-1)}u(t-1)$$

when the input is given by

$$x(t) = e^{-t}u(t).$$

Find the system impulse response $h(t)$.

Table: $-\frac{1}{5}e^{-(-3)t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{5}\frac{1}{s-3}$, $Re[s] < 3$

Time shift: $-\frac{1}{5}e^{-(-3)(t-1)}u(-(t-1)) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{5}\frac{e^{-s}}{s-3}$, $Re[s] < 3$.

Table: $-\frac{1}{5}e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{1}{5}\frac{1}{s+2}$ $Re[s] > -2$

Time shift: $-\frac{1}{5}e^{-2(t-1)}u(t-1) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{1}{5}\frac{e^{-s}}{s+2}$, $Re[s] > -2$

Put it together:

$$Y(s) = \frac{e^{-s}}{5}\left[\frac{1}{s-3} - \frac{1}{s+2}\right]$$

$$= \frac{e^{-s}}{5}\left[\frac{s+2-(s-3)}{(s+2)(s-3)}\right] = \frac{5e^{-s}}{5(s+2)(s-3)}$$

$$= \frac{e^{-s}}{(s+2)(s-3)}, \quad -2 < Re[s] < 3$$

Table: $X(s) = \frac{1}{s+1}$, $Re[s] > -1$.

$$H(s) = \frac{Y(s)}{X(s)} = Y(s)[X(s)]^{-1} = \frac{(s+1)e^{-s}}{(s+2)(s-3)}, \quad -2 < Re[s] < 3$$

$\Rightarrow e^{-s}$ corresponds to a <u>time</u> <u>shift</u>.

So let $\hat{H}(s) = \frac{s+1}{(s+2)(s-3)}$, $-2 < Re[s] < 3$

More Workspace for Problem 3...

$$\hat{H}(s) = \frac{s+1}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$A = \left.\frac{s+1}{s-3}\right|_{s=-2} = \frac{-1}{-5} = \frac{1}{5}$$

$$B = \left.\frac{s+1}{s+2}\right|_{s=3} = \frac{4}{5}$$

$$\hat{H}(s) = \underbrace{\frac{1/5}{s+2}}_{Re[s] > -2} + \underbrace{\frac{4/5}{s-3}}_{Re[s] < 3}$$

Table: $\hat{h}(t) = \frac{1}{5}e^{-2t}u(t) - \frac{4}{5}e^{3t}u(-t)$

Now, $H(s) = e^{-s}\hat{H}(s)$,

so $h(t) = \hat{h}(t-1)$.

Time Shift Property:

$$\boxed{h(t) = \frac{1}{5}e^{-2(t-1)}u(t-1) - \frac{4}{5}e^{3(t-1)}u(-t+1)}$$

4. **25 pts**. The input $x(t)$ and output $y(t)$ of an LTI system H are related by

$$y''(t) + 5y'(t) + 6y(t) = 2x'(t) + 2x(t).$$

The initial conditions on the output are $y(0^-) = 1$ and $y'(0^-) = 2$. Use the unilateral Laplace transform to find the system output for $t > 0$ when the input is $x(t) = e^{-t}u(t)$.

$x(t) = e^{-t}u(t) \Rightarrow x(0^-) = 0$. Table: $\mathcal{X}(s) = \frac{1}{s+1}$

Apply $\mathcal{UL}$ transform:

$$s^2\mathcal{Y}(s) - sy(0^-) - y'(0^-) + 5[s\mathcal{Y}(s) - y(0^-)] + 6\mathcal{Y}(s)$$
$$= 2[s\mathcal{X}(s) - x(0^-)] + 2\mathcal{X}(s)$$

$$s^2\mathcal{Y}(s) - s(1) - 2 + 5[s\mathcal{Y}(s) - 1] + 6\mathcal{Y}(s)$$
$$= 2[s\mathcal{X}(s) - 0] + 2\mathcal{X}(s)$$

$$s^2\mathcal{Y}(s) - s - 2 + 5s\mathcal{Y}(s) - 5 + 6\mathcal{Y}(s) = 2s\mathcal{X}(s) + 2\mathcal{X}(s)$$

$$\mathcal{Y}(s)[s^2 + 5s + 6] - s - 7 = [2s+2]\mathcal{X}(s)$$
$$= [2s+2]\frac{1}{s+1}$$
$$= 2(s+1)\frac{1}{s+1} = 2$$

$$\mathcal{Y}(s)[s^2+5s+6] - s - 7 = 2$$

$$\mathcal{Y}(s)[s^2+5s+6] = s+9$$

$$\mathcal{Y}(s) = \frac{s+9}{s^2+5s+6} = \frac{s+9}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$A = \left.\frac{s+9}{s+2}\right|_{s=-3} = \frac{6}{-1} = -6$$

$$B = \left.\frac{s+9}{s+3}\right|_{s=-2} = \frac{7}{1} = 7$$

$$\mathcal{Y}(s) = \frac{7}{s+2} - \frac{6}{s+3}$$

TABLE: $\boxed{y(t) = 7e^{-2t}u(t) - 6e^{-3t}u(t)}$