ECE 3793
Test 2

Thursday, April 23, 2015
7:00 PM - 10:00 PM

Spring 2015 | Name: SOLUTION

Dr. Havlicek | Student Num:

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

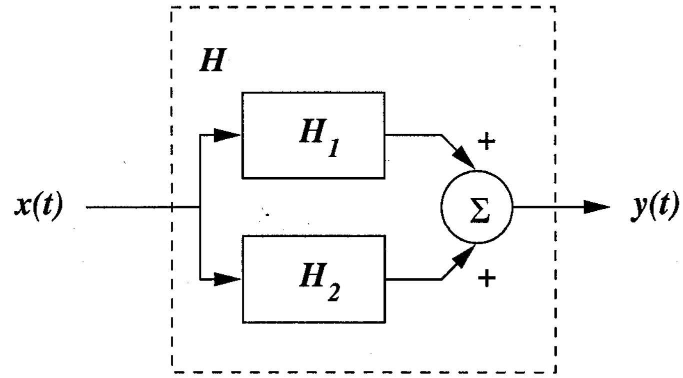
4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________________ Date: ________________

1. **25 pts**. Consider a continuous-time LTI system H formed by connecting two LTI systems H_1 and H_2 in parallel as shown in the figure below.



The impulse response of system H_1 is given by

$$h_1(t) = -e^{-3t}u(t). \qquad H_1(\omega) = \frac{-1}{3+j\omega} \quad \text{(Table)}$$

When the overall system input is

$$x(t) = 4e^{-4t}u(t), \qquad X(\omega) = \frac{4}{4+j\omega} \quad \text{(Table)}$$

the system output is observed to have Fourier transform

$$Y(\omega) = \frac{4}{6+5j\omega-\omega^2}.$$

(a) **15 pts**. Find the impulse response $h_2(t)$ of the system H_2.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{4}{6+5j\omega-\omega^2} \cdot \frac{4+j\omega}{4} = \frac{4+j\omega}{6+5j\omega+(j\omega)^2}$$

$$= \frac{4+j\omega}{(3+j\omega)(2+j\omega)}$$

$$H(\omega) = H_1(\omega) + H_2(\omega)$$

$$H_2(\omega) = H(\omega) - H_1(\omega) = \frac{4+j\omega}{(3+j\omega)(2+j\omega)} + \frac{1}{3+j\omega}$$

$$= \frac{4+j\omega + 2+j\omega}{(3+j\omega)(2+j\omega)} = \frac{6+2j\omega}{(3+j\omega)(2+j\omega)} = \frac{2(3+j\omega)}{(3+j\omega)(2+j\omega)}$$

$$= \frac{2}{2+j\omega}$$

Table: $\boxed{h_2(t) = 2e^{-2t}u(t)}$

Problem 1, cont...

(b) **10 pts**. Find the differential equation relating the input $x(t)$ and output $y(t)$ of the overall cascade system.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{4 + j\omega}{6 + 5j\omega + (j\omega)^2}$$

$$Y(\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = X(\omega)\left[j\omega + 4\right]$$

$$(j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega) = j\omega X(\omega) + 4X(\omega)$$

$$y''(t) + 5y'(t) + 6y(t) = x'(t) + 4x(t)$$

2. **25 pts**. Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.

Capacitor: $i_c(t) = C v_c'(t) = i_2(t)$ ①

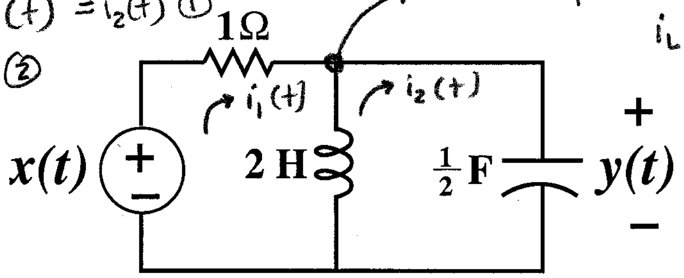
Inductor: $v_L(t) = L i_L'(t)$ ②

Resistor $v_R(t) = R i_R(t)$

$= R i_1(t)$ ③

KCL: $i_1(t) = i_L(t) + i_2(t)$

$i_L(t) = i_1(t) - i_2(t)$ ④



(a) **10 pts**. Find the system frequency response $H(\omega)$.

KVL on right loop: $y(t) = v_L(t)$ ⑤

KVL on left loop: $x(t) = v_R(t) + v_L(t) \overset{③}{=} \overset{1}{R} i_1(t) + v_L(t) \overset{⑤}{=} i_1(t) + y(t)$ ⑥

KVL on big loop: $x(t) = v_R(t) + y(t) \overset{③}{=} R i_1(t) + y(t) = i_1(t) + y(t)$

④→②: $v_L(t) = 2(i_1'(t) - i_2'(t)) = 2i_1'(t) - 2i_2'(t)$

plug in ⑤: $y(t) = 2i_1'(t) - 2i_2'(t)$ ⑦

①: $i_2(t) = C v_c'(t) = \frac{1}{2} y'(t) \mapsto$ ⑦: $y(t) = 2i_1'(t) - 2(\frac{1}{2}) y''(t)$

$y(t) = 2i_1'(t) - y''(t)$ ⑧

⑥: $i_1(t) = x(t) - y(t)$

$i_1'(t) = x'(t) - y'(t) \mapsto$ ⑧: $y(t) = 2x'(t) - 2y'(t) - y''(t)$

$$y''(t) + 2y'(t) + y(t) = 2x'(t)$$

$$\mathcal{F}: \quad (j\omega)^2 Y(\omega) + 2j\omega Y(\omega) + Y(\omega) = 2j\omega X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 2j\omega + 1\right] = X(\omega)\, 2j\omega$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2j\omega}{(j\omega)^2 + 2j\omega + 1} = \underline{\underline{\frac{2j\omega}{(1+j\omega)^2}}}$$

Problem 2, cont...

(b) **5 pts.** Find the system impulse response $h(t)$.

METHOD 1:

$$\frac{2\theta}{(\theta+1)^2} = \frac{A}{(\theta+1)^2} + \frac{B}{\theta+1}$$

$$A = 2\theta\Big|_{\theta=-1} = -2$$

$$\frac{\partial}{\partial\theta}2\theta\Big|_{\theta=-1} = B\frac{\partial}{\partial\theta}(\theta+1)\Big|_{\theta=-1}$$

$$2 = B$$

$$H(\omega) = \frac{2}{1+j\omega} - \frac{2}{(1+j\omega)^2}$$

Table:

$$\boxed{h(t) = 2e^{-t}u(t) - 2te^{-t}u(t)}$$

METHOD 2:

$$H(\omega) = j\omega\,\frac{2}{(1+j\omega)^2}$$

Table: $\dfrac{2}{(1+j\omega)^2} \overset{\mathcal{F}}{\longleftrightarrow} 2te^{-t}u(t)$

Derivative property:

$$h(t) = \frac{d}{dt}\left[2te^{-t}u(t)\right]$$

$$= 2\left[e^{-t}u(t) + t\frac{d}{dt}e^{-t}u(t)\right]$$

$$= 2e^{-t}u(t) + 2t\left[e^{-t}\frac{d}{dt}u(t) - e^{-t}u(t)\right]$$

$$= 2e^{-t}u(t) + 2te^{-t}\delta(t) - 2te^{-t}u(t)$$

$$= 2e^{-t}u(t) + 2\cdot 0\cdot e^{-0}\delta(t) - 2te^{-t}u(t)$$

$$= 2e^{-t}u(t) - 2te^{-t}u(t)$$

Problem 2, cont...

(c) **10 pts.** Let the system input be $x(t) = \delta(t) - 2e^{-3t}u(t)$. Find the output $y(t)$.

TABLE: $$X(\omega) = 1 - \frac{2}{3+j\omega} = \frac{3+j\omega-2}{3+j\omega} = \frac{1+j\omega}{3+j\omega}$$

$$Y(\omega) = X(\omega)\,H(\omega) = \frac{1+j\omega}{3+j\omega} \cdot \frac{2j\omega}{(1+j\omega)^2} = \frac{2j\omega}{(1+j\omega)(3+j\omega)}$$

PFE

$$\frac{2\theta}{(1+\theta)(3+\theta)} = \frac{A}{1+\theta} + \frac{B}{3+\theta}$$

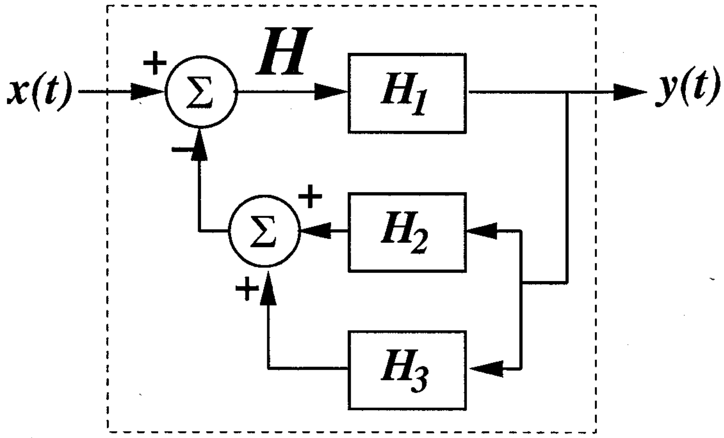
$$A = \left.\frac{2\theta}{3+\theta}\right|_{\theta=-1} = \frac{-2}{2} = -1$$

$$B = \left.\frac{2\theta}{1+\theta}\right|_{\theta=-3} = \frac{-6}{-2} = 3$$

$$H(\omega) = \frac{3}{3+j\omega} - \frac{1}{1+j\omega}$$

Table: $$\boxed{y(t) = 3e^{-3t}u(t) - e^{-t}u(t)}$$

3. **25 pts**. Consider the continuous-time LTI system H shown below. Note that the feedback path consists of a parallel connection of systems H_2 and H_3. Systems H_1, H_2, and H_3 are all LTI.



The system input is given by $x(t) = te^{-t}u(t)$. The output is $y(t) = e^{-t}u(t)$. The impulse response of system H_1 is $h_1(t) = 6\delta(t)$. The transfer function of system H_2 is

$$H_2(s) = \frac{1}{s+2}, \quad \text{Re}\{s\} > -2.$$

For system H_3, find the transfer function $H_3(s)$ and the impulse response $h_3(t)$. Don't forget to specify the ROC of $H_3(s)$.

Table: $X(s) = \frac{1}{(s+1)^2}$, $Re[s] > -1$

Table $Y(s) = \frac{1}{s+1}$, $Re[s] > -1$

$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+1)^2}{s+1} = s+1$, all s.

The feedback path has transfer fcn

$H_2(s) + H_3(s) = \frac{1}{s+2} + H_3(s)$.

Table: The forward path transfer fcn is $H_1(s) = 6$, all s.

More Work Space for Problem 3...

For a negative feedback connection,

$$H(s) = \frac{[\text{Forward Path}]}{1 + [\text{Forward Path}][\text{Reverse Path}]}$$

$$= \frac{H_1(s)}{1 + H_1(s)\left[H_2(s) + H_3(s)\right]}$$

plugging in,

$$s+1 = \frac{6}{1 + 6\left[\frac{1}{s+2} + H_3(s)\right]} = \frac{6}{1 + \frac{6}{s+2} + 6H_3(s)}$$

Cross Multiply:

$$(s+1)\left[1 + \frac{6}{s+2} + 6H_3(s)\right] = 6$$

$$(s+1) + \frac{6(s+1)}{s+2} + 6(s+1)H_3(s) = 6$$

$$6(s+1)H_3(s) = 6 - (s+1) - \frac{6(s+1)}{s+2}$$

$$H_3(s) = \frac{6}{6(s+1)} - \frac{(s+1)}{6(s+1)} - \frac{6(s+1)}{6(s+1)(s+2)}$$

$$= \underbrace{\frac{1}{s+1}}_{Re[s]>-1} - \underbrace{\frac{1}{6}}_{\text{all } s} - \underbrace{\frac{1}{s+2}}_{Re[s]>-2}$$

$$\boxed{H_3(s) = \frac{1}{s+1} - \frac{1}{6} - \frac{1}{s+2}, \quad Re[s] > -1}$$

Table: $\boxed{h_3(t) = e^{-t}u(t) - \frac{1}{6}\delta(t) - e^{-2t}u(t)}$

4. **25 pts**. The input $x(t)$ and output $y(t)$ of a **stable** LTI system H are related by

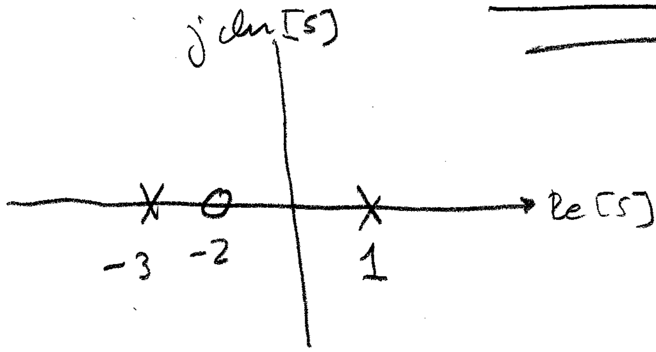
$$y''(t) + 2y'(t) - 3y(t) = x'(t) + 2x(t).$$

(a) **7 pts**. Find the transfer function $H(s)$ and give a pole-zero plot. Be sure to specify the ROC.

$$\mathcal{L}: \quad s^2 Y(s) + 2sY(s) - 3Y(s) = sX(s) + 2X(s)$$

$$Y(s)\left[s^2 + 2s - 3\right] = X(s)\left[s+2\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+2s-3} = \underline{\underline{\frac{s+2}{(s+3)(s-1)}}}$$



Since the system is stable, the ROC must include the $j\omega$-axis.

$$\boxed{\text{ROC}: \ -3 < \mathrm{Re}[s] < 1}$$

Problem 4, cont...

(b) **4 pts.** Is the system H causal? *Justify your answer.*

Because the RoC of $H(s)$ is a strip, $h(t)$ must be two-sided. So the system cannot be causal... because $h(t)$ cannot be zero $\forall\, t<0$.

$\longrightarrow$ NOT CAUSAL

(c) **7 pts.** Find the impulse response $h(t)$.

$$H(s) = \frac{s+2}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$$A = \left.\frac{s+2}{s-1}\right|_{s=-3} = \frac{-1}{-4} = \frac{1}{4}$$

$$B = \left.\frac{s+2}{s+3}\right|_{s=1} = \frac{3}{4}$$

$$H(s) = \underbrace{\frac{1/4}{s+3}}_{Re[s]>-3} + \underbrace{\frac{3/4}{s-1}}_{Re[s]<1}$$

Table: $\boxed{h(t) = \frac{1}{4}e^{-3t}u(t) - \frac{3}{4}e^{t}u(-t)}$

Table: $\mathcal{X}(s) = \frac{2}{s}$

Problem 4, cont...

(d) **7 pts**. Now assume that the system from part (a) is **causal**. The system input is $x(t) = 2u(t)$ with initial condition $x(0^-) = 0$. The initial conditions on the output $y(t)$ are $y(0^-) = 1$ and $y'(0^-) = 1$. Use the unilateral Laplace transform to find $y(t)$ for $t > 0$.

$$y''(t) + 2y'(t) - 3y(t) = x'(t) + 2x(t)$$

UL: $s^2\mathcal{Y}(s) - sy(0^-) - y'(0^-) + 2[s\mathcal{Y}(s) - y(0^-)] - 3\mathcal{Y}(s) = s\mathcal{X}(s) - \cancel{x(0^-)}^{0} + 2\mathcal{X}(s)$

$$s^2\mathcal{Y}(s) - s - 1 + 2s\mathcal{Y}(s) - 2 - 3\mathcal{Y}(s) = s\mathcal{X}(s) + 2\mathcal{X}(s)$$

$$[s^2 + 2s - 3]\mathcal{Y}(s) - s - 3 = (s+2)\mathcal{X}(s) = (s+2)\frac{2}{s} = 2 + \frac{4}{s}$$

$$\mathcal{Y}(s)[s^2 + 2s - 3] = s + 3 + 2 + \frac{4}{s} = s + 5 + \frac{4}{s}$$

$$\mathcal{Y}(s) = \frac{s + 5 + \frac{4}{s}}{s^2 + 2s - 3} \cdot \frac{s}{s} = \frac{s^2 + 5s + 4}{s(s^2 + 2s - 3)}$$

$$= \frac{(s+4)(s+1)}{s(s+3)(s-1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-1}$$

$$A = \left.\frac{(s+4)(s+1)}{(s+3)(s-1)}\right|_{s=0} = \frac{4 \cdot 1}{3 \cdot (-1)} = -\frac{4}{3}$$

$$B = \left.\frac{(s+4)(s+1)}{s(s-1)}\right|_{s=-3} = \frac{1 \cdot (-2)}{(-3)(-4)} = \frac{-2}{12} = -\frac{1}{6}$$

$$C = \left.\frac{(s+4)(s+1)}{s(s+3)}\right|_{s=1} = \frac{5 \cdot 2}{1(4)} = \frac{10}{4} = \frac{5}{2}$$

$$\mathcal{Y}(s) = \underbrace{-\frac{4}{3} \cdot \frac{1}{s}}_{\text{all } s} - \underbrace{\frac{1}{6} \cdot \frac{1}{s+3}}_{Re[s] > -3} + \underbrace{\frac{5}{2}\,\frac{1}{s-1}}_{Re[s] > 1}$$

Table: $\boxed{y(t) = -\frac{4}{3}u(t) - \frac{1}{6}e^{-3t}u(t) + \frac{5}{2}e^{t}u(t)}$