# ECE 3793
## Test 2

Wednesday, April 27, 2016
7:00 PM - 10:00 PM

Spring 2016 Name: SOLUTION

Dr. Havlicek Student Num:__________

**Directions:** This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

## GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

TOTAL (100):

*On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.*

Name:__________ Date:__________

1. **25 pts.** A causal continuous-time LTI system $H$ has input $x(t)$ and output $y(t)$ related by

$$y''(t) + 5y'(t) + 6y(t) = x(t) - x'(t).$$

(a) **6 pts.** Find the frequency response $H(\omega)$.

$$\mathcal{F}: \; (j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega) = X(\omega) - j\omega X(\omega)$$

$$\left[(j\omega)^2 + 5j\omega + 6\right] Y(\omega) = \left[1 - j\omega\right] X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - j\omega}{(j\omega)^2 + 5j\omega + 6} \; //$$

$$= \frac{1 - j\omega}{(3 + j\omega)(2 + j\omega)}$$

(b) **8 pts.** Find the impulse response $h(t)$.

$$\text{PFE}: \quad H(\theta) = \frac{1-\theta}{(3+\theta)(2+\theta)} = \frac{A}{3+\theta} + \frac{B}{2+\theta}$$

$$A = \left.\frac{1-\theta}{2+\theta}\right|_{\theta=-3} = \frac{1+3}{2-3} = \frac{4}{-1} = -4$$

$$B = \left.\frac{1-\theta}{3+\theta}\right|_{\theta=-2} = \frac{1+2}{3-2} = \frac{3}{1} = 3$$

$$H(\omega) = \frac{-4}{3+j\omega} + \frac{3}{2+j\omega}$$

$$\text{TABLE}: \quad \underline{\underline{h(t) = 3e^{-2t}u(t) - 4e^{-3t}u(t)}}$$

Problem 1, cont...

(c) **11 pts.** The input is given by $x(t) = e^{-t}u(t)$. Find the output signal $y(t)$.

TABLE: $X(\omega) = \dfrac{1}{1+j\omega}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1-j\omega}{(1+j\omega)(2+j\omega)(3+j\omega)}$$

$$= \frac{A}{1+j\omega} + \frac{B}{2+j\omega} + \frac{C}{3+j\omega}$$

$$A = \left.\frac{1-\theta}{(2+\theta)(3+\theta)}\right|_{\theta=-1} = \frac{2}{1\cdot 2} = 1$$

$$B = \left.\frac{1-\theta}{(1+\theta)(3+\theta)}\right|_{\theta=-2} = \frac{3}{(-1)(1)} = -3$$

$$C = \left.\frac{1-\theta}{(1+\theta)(2+\theta)}\right|_{\theta=-3} = \frac{4}{(-2)(-1)} = \frac{4}{2} = 2$$

$$Y(\omega) = \frac{1}{1+j\omega} - \frac{3}{2+j\omega} + \frac{2}{3+j\omega}$$

TABLE: $\underline{\underline{y(t) = e^{-t}u(t) - 3e^{-2t}u(t) + 2e^{-3t}u(t)}}$

2. **25 pts.** The continuous-time system $H$ has input $x(t)$ and output $y(t)$ related by

$$y(t) = x(t-2)\cos(100t).$$

The input is given by

$$x(t) = \begin{cases} 1, & |t| < 2, \\ 0, & |t| > 2. \end{cases}$$

Find $Y(\omega)$, the Fourier transform of $y(t)$.

**Hint:** $\cos(100t) = \frac{1}{2}e^{j100t} + \frac{1}{2}e^{-j100t}$.

TABLE (with $T_1 = 2$): $X(\omega) = \dfrac{2\sin 2\omega}{\omega}$

Let $v(t) = x(t-2)$.

Then $V(\omega) = e^{-j2\omega}X(\omega) = \dfrac{2e^{-j2\omega}\sin 2\omega}{\omega}$ (Time Shift Property)

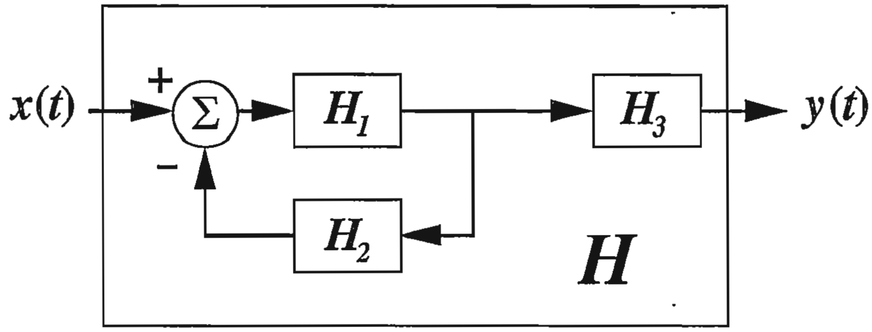
Now, $y(t) = v(t)\cos 100t$

$= \frac{1}{2}e^{j100t}v(t) + \frac{1}{2}e^{-j100t}v(t)$

Frequency Shift Property: $Y(\omega) = \frac{1}{2}V(\omega - 100) + \frac{1}{2}V(\omega + 100)$

$$Y(\omega) = \frac{e^{-j2(\omega-100)}\sin[2(\omega-100)]}{\omega - 100} + \frac{e^{-j2(\omega+100)}\sin[2(\omega+100)]}{\omega + 100}$$

3. **25 pts.** Consider the **stable** continuous-time LTI system $H$ shown below. Systems $H_1$, $H_2$, and $H_3$ are all LTI.




LTI system $H_1$ has impulse response $h_1(t) = e^t u(t)$.

LTI system $H_2$ has impulse response $h_2(t) = 4\delta(t)$.

LTI system $H_3$ has input $x_3(t)$ and output $y_3(t)$ related by

$$y_3''(t) - y_3'(t) - 6y_3(t) = x_3''(t) - 3x_3'(t) + 2x_3(t).$$

(a) **8 pts.** Find the overall transfer function $H(s)$ and give a pole-zero plot. Don't forget to specify the ROC.

TABLE: $H_1(s) = \frac{1}{s-1}$, $Re\{s\} > 1$ TABLE: $H_2(s) = 4$, all $s$.

For $H_3$: $\mathcal{L}$: $[s^2 - s - 6]\, Y_3(s) = [s^2 - 3s + 2]\, X(s)$

$$H_3(s) = \frac{Y_3(s)}{X_3(s)} = \frac{s^2 - 3s + 2}{s^2 - s - 6} = \frac{(s-1)(s-2)}{(s-3)(s+2)}$$

Note: the ROC of $H_3(s)$ is unclear at this point.

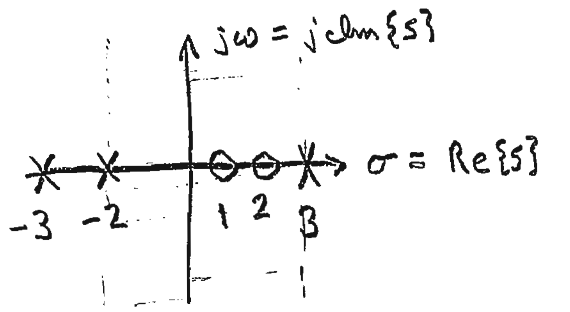
For the feedback connection, the transfer function is:

$$\frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{1/s-1}{1 + \frac{4}{s-1}} \cdot \underbrace{\frac{s-1}{s-1}}_{\text{one}} = \frac{1}{s-1+4} = \frac{1}{s+3} \quad \left(\begin{matrix}\text{ROC}\\ \text{unclear}\end{matrix}\right)$$

The overall system $H$ is a series connection of $\frac{1}{s+3}$ and $H_3(s)$:

$$H(s) = \frac{1}{s+3} H_3(s) = \underline{\underline{\frac{(s-1)(s-2)}{(s+3)(s+2)(s-3)}}}$$

Pole-zero plot:




Because the system is STABLE, the ROC must include the $j\omega$-axis:

$$\underline{\underline{\text{ROC: } -2 < Re\{s\} < 3}}$$

(b) **3 pts.** Is the overall system $H$ causal? *Justify your answer.*

Because the ROC of $H(s)$ is a STRIP, the impulse response must be TWO-SIDED. This means that $h(t)$ cannot be zero for all $t<0$. Therefore, the system $H$ is not causal.

$$\boxed{\text{NOT CAUSAL}}$$

(c) **5 pts.** Find the overall system impulse response $h(t)$.

$$H(s) = \frac{(s-1)(s-2)}{(s+3)(s+2)(s-3)} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$A = \left.\frac{(s-1)(s-2)}{(s+2)(s-3)}\right|_{s=-3} = \frac{(-4)(-5)}{(-1)(-6)} = \frac{20}{6} = \frac{10}{3}$$

$$B = \left.\frac{(s-1)(s-2)}{(s+3)(s-3)}\right|_{s=-2} = \frac{(-3)(-4)}{(1)(-5)} = \frac{12}{-5} = -\frac{12}{5}$$

$$C = \left.\frac{(s-1)(s-2)}{(s+3)(s+2)}\right|_{s=3} = \frac{(2)(1)}{(6)(5)} = \frac{2}{30} = \frac{1}{15}$$

$$H(s) = \underbrace{\frac{10/3}{s+3}}_{Re[s]>-3} - \underbrace{\frac{12/5}{s+2}}_{Re[s]>-2} + \underbrace{\frac{1/15}{s-3}}_{Re[s]<3}$$

TABLE: $$h(t) = \frac{10}{3}e^{-3t}u(t) - \frac{12}{5}e^{-2t}u(t) - \frac{1}{15}e^{3t}u(-t)$$

Problem 3, cont...

(d) **9 pts.** The input is given by $x(t) = e^{-4t}u(t)$. Find the output signal $y(t)$.

TABLE: $X(s) = \frac{1}{s+4}, \quad Re[s] > -4$

$$Y(s) = X(s)H(s) = \frac{(s-1)(s-2)}{(s+4)(s+3)(s+2)(s-3)}, \quad -2 < Re[s] < 3$$

$$= \frac{A}{s+4} + \frac{B}{s+3} + \frac{C}{s+2} + \frac{D}{s-3}$$

$$A = \left.\frac{(s-1)(s-2)}{(s+3)(s+2)(s-3)}\right|_{s=-4} = \frac{(-5)(-6)}{(-1)(-2)(-7)} = \frac{(-5)(3)}{7} = \frac{-15}{7}$$

$$B = \left.\frac{(s-1)(s-2)}{(s+4)(s+2)(s-3)}\right|_{s=-3} = \frac{(-4)(-5)}{(1)(-1)(-6)} = \frac{20}{6} = \frac{10}{3}$$

$$C = \left.\frac{(s-1)(s-2)}{(s+4)(s+3)(s-3)}\right|_{s=-2} = \frac{(-3)(-4)}{(2)(1)(-5)} = \frac{6}{-5} = -\frac{6}{5}$$

$$D = \left.\frac{(s-1)(s-2)}{(s+4)(s+3)(s+2)}\right|_{s=3} = \frac{(2)(1)}{(7)(6)(5)} = \frac{1}{(7)(3)(5)} = \frac{1}{21 \cdot 5} = \frac{1}{105}$$

$$Y(s) = \underbrace{-\frac{15}{7}\frac{1}{s+4}}_{Re[s]>-4} + \underbrace{\frac{10}{3}\frac{1}{s+3}}_{Re[s]>-3} - \underbrace{\frac{6}{5}\frac{1}{s+2}}_{Re[s]>-2} + \underbrace{\frac{1}{105}\frac{1}{s-3}}_{Re[s]<3}$$

TABLE: $$\underline{\underline{y(t) = -\frac{15}{7}e^{-4t}u(t) + \frac{10}{3}e^{-3t}u(t) - \frac{6}{5}e^{-2t}u(t) - \frac{1}{105}e^{3t}u(-t)}}$$

4. **25 pts.** The input $x(t)$ and output $y(t)$ of a **causal** LTI system $H$ are related by

$$y''(t) + 2y'(t) - 3y(t) = x'(t) + 2x(t).$$

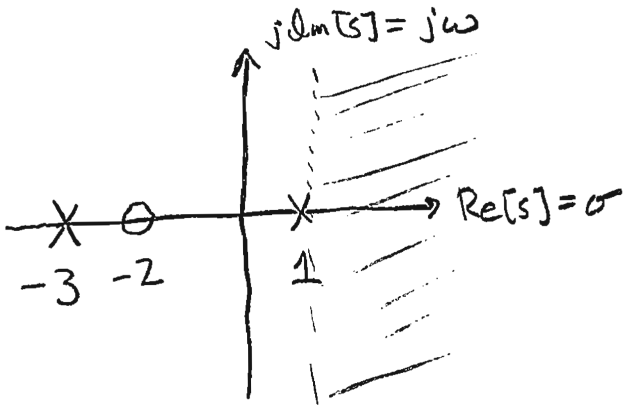
(a) **7 pts.** Find the transfer function $H(s)$ and give a pole-zero plot. Be sure to specify the ROC.

$\mathcal{L}$: $s^2 Y(s) + 2s Y(s) - 3Y(s) = sX(s) + 2X(s)$

$[s^2+2s-3]\, Y(s) = [s+2]\, X(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \underline{\underline{\frac{s+2}{s^2+2s-3}}} = \frac{s+2}{(s+3)(s-1)}$$

Pole-zero plot:




Because the system is causal, $h(t)$ must be right-sided.

Therefore, the ROC of $H(s)$ must be right-sided.

$\underline{\underline{\text{ROC: } \mathrm{Re}[s] > 1.}}$

Problem 4, cont...

(b) **4 pts.** Is the system $H$ stable? *Justify your answer.*

The ROC of $H(s)$ does <u>not</u> contain the $j\omega$-axis.

$$\boxed{\Longrightarrow \text{ NOT STABLE}}$$

(c) **7 pts.** Find the impulse response $h(t)$.

$$H(s) = \frac{s+2}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}, \quad \mathrm{Re}[s] > 1$$

$$A = \left.\frac{s+2}{s-1}\right|_{s=-3} = \frac{-1}{-4} = \frac{1}{4}$$

$$B = \left.\frac{s+2}{s+3}\right|_{s=1} = \frac{3}{4}$$

$$H(s) = \underbrace{\frac{1/4}{s+3}}_{\mathrm{Re}[s]>-3} + \underbrace{\frac{3/4}{s-1}}_{\mathrm{Re}[s]>1}$$

TABLE: $h(t) = \frac{1}{4}e^{-3t}u(t) + \frac{3}{4}e^{t}u(t)$

Problem 4, cont...

(d) **7 pts.** The system input is $x(t) = 2u(t)$ with initial condition $x(0^-) = 0$. The initial conditions on the output $y(t)$ are $y(0^-) = 1$ and $y'(0^-) = 1$. Use the unilateral Laplace transform to find $y(t)$.

TABLE: $\mathcal{X}_u(s) = 2/s$

Given: $y''(t) + 2y'(t) - 3y(t) = x'(t) + 2x(t)$

$$\mathcal{UL}: \left[s^2 \mathcal{Y}_u(s) - s\overbrace{y(0^-)}^{1} - \overbrace{y'(0^-)}^{1}\right] + 2\left[s\mathcal{Y}_u(s) - \overbrace{y(0^-)}^{1}\right] - 3\mathcal{Y}_u(s)$$

$$= \left[s\mathcal{X}_u(s) - \overbrace{x(0^-)}^{0}\right] + 2\mathcal{X}_u(s)$$

$$s^2\mathcal{Y}_u(s) - s - 1 + 2s\mathcal{Y}_u(s) - 2 - 3\mathcal{Y}_u(s) = s\mathcal{X}_u(s) + 2\mathcal{X}_u(s)$$

$$[s^2 + 2s - 3]\,\mathcal{Y}_u(s) - s - 1 - 2 = [s+2]\,\mathcal{X}_u(s)$$

$$[s^2 + 2s - 3]\,\mathcal{Y}_u(s) - s - 3 = [s+2]\,\frac{2}{s}$$

$$[s^2 + 2s - 3]\,\mathcal{Y}_u(s) = 2 + \frac{4}{s} + s + 3 = s + 5 + \frac{4}{s}$$

$$\mathcal{Y}_u(s) = \frac{s + 5 + \frac{4}{s}}{s^2 + 2s - 3} \cdot \frac{s}{s} = \frac{s^2 + 5s + 4}{s(s^2 + 2s - 3)}$$

$$= \frac{(s+4)(s+1)}{s(s+3)(s-1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-1}$$

$$A = \left.\frac{(s+4)(s+1)}{(s+3)(s-1)}\right|_{s=0} = \frac{(4)(1)}{3(-1)} = -\frac{4}{3}$$

$$B = \left.\frac{(s+4)(s+1)}{s(s-1)}\right|_{s=-3} = \frac{(1)(-2)}{(-3)(-4)} = \frac{-2}{12} = -\frac{1}{6}$$

$$C = \left.\frac{(s+4)(s+1)}{s(s+3)}\right|_{s=1} = \frac{(5)(2)}{(1)(4)} = \frac{10}{4} = \frac{5}{2}$$

$$\mathcal{Y}_u(s) = \underbrace{-\frac{4}{3}\frac{1}{s}}_{Re[s]>0} - \underbrace{\frac{1}{6}\frac{1}{s+3}}_{Re[s]>-3} + \underbrace{\frac{5}{2}\frac{1}{s-1}}_{Re[s]>1}$$

TABLE: $\boxed{y(t) = -\frac{4}{3}u(t) - \frac{1}{6}e^{-3t}u(t) + \frac{5}{2}e^{t}u(t)}$