ECE 3793
Test 2

Thursday, April 20, 2017
6:30 PM - 9:30 PM

Spring 2017 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ______

2. (25) ______

3. (25) ______

4. (25) ______

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** A continuous-time LTI system H has impulse response

$$h(t) = 2e^{-t}u(t) - e^{-2t}u(t).$$

(a) **8 pts.** Find the frequency response $H(\omega)$.

Table: $2e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+j\omega}$ $\quad -e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{-1}{2+j\omega}$

$$H(\omega) = \frac{2}{1+j\omega} - \frac{1}{2+j\omega} = \frac{2(2+j\omega)-(1+j\omega)}{(1+j\omega)(2+j\omega)}$$

$$= \frac{4+2j\omega-1-j\omega}{(1+j\omega)(2+j\omega)} = \frac{3+j\omega}{(1+j\omega)(2+j\omega)}$$

$$H(\omega) = \frac{3+j\omega}{(j\omega)^2+3j\omega+2}$$

(b) **8 pts.** The system input is given by $x_1(t) = e^{-4t}u(t)$. Find the system output $y_1(t)$.

Table: $X_1(\omega) = \frac{1}{j\omega+4}$

$$Y_1(\omega) = X_1(\omega)H(\omega) = \frac{3+j\omega}{(1+j\omega)(2+j\omega)(4+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{2+j\omega} + \frac{C}{4+j\omega}$$

$$A = \left.\frac{3+\theta}{(2+\theta)(4+\theta)}\right|_{\theta=-1} = \frac{2}{(1)(3)} = 2/3$$

$$B = \left.\frac{3+\theta}{(1+\theta)(4+\theta)}\right|_{\theta=-2} = \frac{1}{(-1)(2)} = -1/2$$

$$C = \left.\frac{3+\theta}{(1+\theta)(2+\theta)}\right|_{\theta=-4} = \frac{-1}{(-3)(-2)} = -1/6$$

$$Y_1(\omega) = \frac{2/3}{1+j\omega} - \frac{1/2}{2+j\omega} - \frac{1/6}{4+j\omega}$$

Table: $y_1(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) - \frac{1}{6}e^{-4t}u(t)$

(c) **9 pts.** Now the system input is given instead by $x_2(t) = 1 + e^{-3(t-1)}u(t-1)$. Find the new system output $y_2(t)$.

Hint: for any function $H(\omega)$, the product $H(\omega)\delta(\omega)$ is equal to $H(0)\delta(\omega)$. In other words, for the product $H(\omega)\delta(\omega)$ the only part of $H(\omega)$ that matters is it's value at $\omega = 0$.

Table: $1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$ Table: $e^{-3t}u(t) \xleftrightarrow{\mathcal{F}} \dfrac{1}{3+j\omega}$

Time Shift Property: $e^{-3(t-1)}u(t-1) \xleftrightarrow{\mathcal{F}} \dfrac{e^{-j\omega}}{3+j\omega}$

$$X_2(\omega) = 2\pi\delta(\omega) + \frac{e^{-j\omega}}{3+j\omega}$$

$$Y_2(\omega) = X_2(\omega)H(\omega) = 2\pi\delta(\omega)H(\omega) + \frac{e^{-j\omega}}{3+j\omega}H(\omega)$$

$$= 2\pi \left.\frac{3+j\omega}{(1+j\omega)(2+j\omega)}\right|_{\omega=0}\delta(\omega) + \frac{e^{-j\omega}\cancel{(3+j\omega)}}{(1+j\omega)(2+j\omega)\cancel{(3+j\omega)}}$$

$$Y_2(\omega) = 2\pi\frac{3}{(1)(2)}\delta(\omega) + \frac{e^{-j\omega}}{(1+j\omega)(2+j\omega)} = \left(\frac{3}{2}\right)2\pi\delta(\omega) + \frac{e^{-j\omega}}{(1+j\omega)(2+j\omega)}$$

Table:

$\left(\frac{3}{2}\right)2\pi\delta(\omega) \xleftrightarrow{\mathcal{F}} \frac{3}{2}$

$$\frac{1}{(1+j\omega)(2+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{2+j\omega}$$

$$A = \left.\frac{1}{2+\theta}\right|_{\theta=-1} = \frac{1}{1} = 1$$

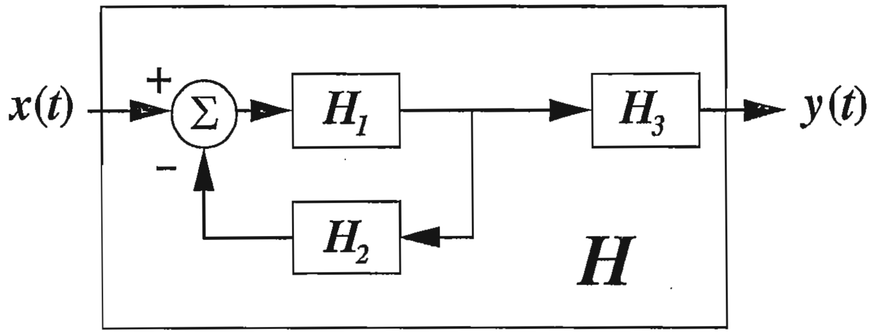
$$B = \left.\frac{1}{1+\theta}\right|_{\theta=-2} = \frac{1}{-1} = -1$$

$$\frac{1}{(1+j\omega)(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega} \xleftrightarrow{\mathcal{F}} e^{-t}u(t) - e^{-2t}u(t)$$

Time Shift: $\dfrac{e^{-j\omega}}{(1+j\omega)(2+j\omega)} \xleftrightarrow{\mathcal{F}} e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)$

All Together: $\boxed{y_2(t) = \frac{3}{2} + e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)}$

2. **25 pts.** Consider the continuous-time LTI system H shown below.



Systems H_1, H_2, and H_3 are all LTI. The impulse responses are given by

$$h_1(t) = e^{-2t}u(t),$$
$$h_2(t) = e^{-4t}u(t),$$
$$h_3(t) = 3\delta(t) + \delta'(t).$$

Table:

$$H_1(\omega) = \frac{1}{2+j\omega}$$
$$H_2(\omega) = \frac{1}{4+j\omega}$$
$$H_3(\omega) = 3+j\omega$$

(a) **9 pts.** Find the frequency response $H(\omega)$.

$$H(\omega) = \frac{H_1(\omega)}{1+H_1(\omega)H_2(\omega)}\, H_3(\omega) = \frac{H_1(\omega)H_3(\omega)}{1+H_1(\omega)H_2(\omega)}$$

$$= \frac{\frac{3+j\omega}{2+j\omega}}{1+\frac{1}{(2+j\omega)(4+j\omega)}} \cdot \underbrace{\frac{(2+j\omega)(4+j\omega)}{(2+j\omega)(4+j\omega)}}_{\text{one}}$$

$$= \frac{(3+j\omega)(4+j\omega)}{(2+j\omega)(4+j\omega)+1} = \frac{(3+j\omega)(4+j\omega)}{(j\omega)^2+6j\omega+8+1}$$

$$= \frac{(3+j\omega)(4+j\omega)}{(j\omega)^2+6j\omega+9} = \frac{\cancel{(3+j\omega)}(4+j\omega)}{\cancel{(3+j\omega)}(3+j\omega)}$$

$$\boxed{H(\omega) = \frac{4+j\omega}{3+j\omega}}$$

Problem 2, cont...

(b) **8 pts.** The system input is given by $x(t) = e^{-2t}u(t) + e^{2t}u(-t)$. Find the Fourier transform $X(\omega)$.

Hint: use the Fourier transform time reversal property for the second term.

Table: $e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j\omega}$

Time Reversal: $e^{2t}u(-t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2-j\omega}$

$$X(\omega) = \frac{1}{2+j\omega} + \frac{1}{2-j\omega}$$

$$= \frac{2-j\omega+2+j\omega}{(2+j\omega)(2-j\omega)} = \underline{\underline{\frac{4}{(2+j\omega)(2-j\omega)}}}$$

(c) **8 pts.** Use the Fourier transform to find the system output $y(t)$. Don't forget about the time reversal property! (it will help you with one of the terms)

$$Y(\omega) = X(\omega)H(\omega) = \frac{4(4+j\omega)}{(2+j\omega)(3+j\omega)(2-j\omega)} = \frac{A}{2+j\omega} + \frac{B}{3+j\omega} + \frac{C}{2-j\omega}$$

$$A = \left.\frac{4(4+\theta)}{(3+\theta)(2-\theta)}\right|_{\theta=-2} = \frac{4(2)}{(1)(4)} = 2$$

$$B = \left.\frac{4(4+\theta)}{(2+\theta)(2-\theta)}\right|_{\theta=-3} = \frac{4(1)}{(-1)(5)} = -\frac{4}{5}$$

$$C = \left.\frac{4(4+\theta)}{(2+\theta)(3+\theta)}\right|_{\theta=2} = \frac{4\cdot 6}{4\cdot 5} = \frac{6}{5}$$

$$Y(\omega) = \frac{2}{2+j\omega} - \frac{4/5}{3+j\omega} + \underbrace{\frac{6/5}{2-j\omega}}_{\text{time reversal}}$$

Table: $\boxed{y(t) = 2e^{-2t}u(t) - \frac{4}{5}e^{-3t}u(t) + \frac{6}{5}e^{2t}u(-t)}$

3. **25 pts.** A continuous-time LTI system H has input $x(t) = e^{-2t}u(t)$. The output is given by

$$y(t) = e^{-t}u(-t) + 3e^{-2t}u(t) - 2e^{-3t}u(t).$$

(a) **8 pts.** Find the transfer function $H(s)$. Don't forget to specify the ROC.
Hint: note that $e^{-t}u(-t) = \{-1\} \times \{-e^{-t}u(-t)\}$.

Table: $X(s) = \frac{1}{s+2}$, $Re[s] > -2$

Table: $Y(s) = \underbrace{\frac{-1}{s+1}}_{Re[s]<-1} + \underbrace{\frac{3}{s+2}}_{Re[s]>-2} - \underbrace{\frac{2}{s+3}}_{Re[s]>-3}$

$$Y(s) = \frac{-(s+2)(s+3) + 3(s+1)(s+3) - 2(s+1)(s+2)}{(s+1)(s+2)(s+3)}$$

$$= \frac{-(s^2+5s+6) + 3(s^2+4s+3) - 2(s^2+3s+2)}{(s+1)(s+2)(s+3)}$$

$$= \frac{(-1+3-2)s^2 + (-5+12-6)s + (-6+9-4)}{(s+1)(s+2)(s+3)}$$

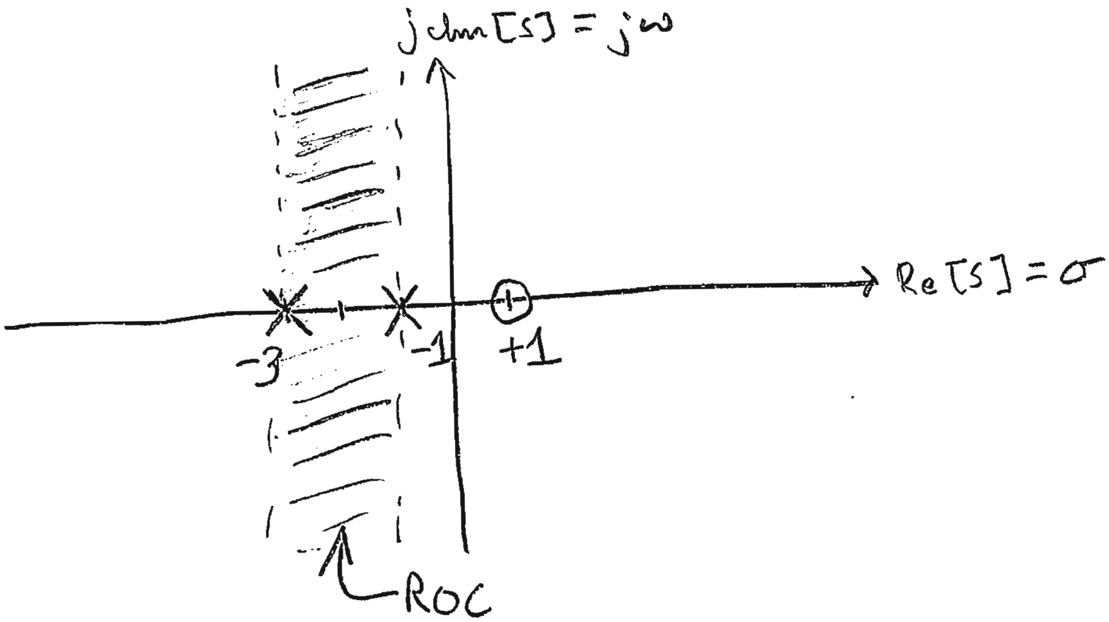
$$= \frac{s-1}{(s+1)(s+2)(s+3)}, \quad -2 < Re[s] < -1$$

$$H(s) = \frac{Y(s)}{X(s)} = Y(s)\frac{1}{X(s)} = \frac{(s-1)(s+2)}{(s+1)(s+2)(s+3)}$$

$$\boxed{H(s) = \frac{s-1}{(s+1)(s+3)}, \quad -3 < Re[s] < -1}$$

Problem 3, cont...

(b) **3 pts.** Give a pole-zero plot for $H(s)$. Be sure to indicate the ROC.



(c) **3 pts.** Is the system H BIBO stable? *Justify your answer.*

The ROC of H(s) does not include the jω-axis of the s-plane.

⇒ Not BIBO stable

(d) **3 pts.** Is the system H causal? *Justify your answer.*

The ROC of H(s) is a strip in the s-plane.

→ h(t) is two-sided

→ h(t) cannot be zero ∀ t < 0.

→ The system is NOT CAUSAL

Problem 3, cont...

(e) **8 pts.** Give the system input-output equation (differential equation) that relates $x(t)$ and $y(t)$.

$$H(s) = \frac{s-1}{(s+1)(s+3)} = \frac{s-1}{s^2+4s+3} = \frac{Y(s)}{X(s)}$$

$$[s^2+4s+3]\,Y(s) = [s-1]\,X(s)$$

$$s^2Y(s) + 4sY(s) + 3Y(s) = sX(s) - X(s)$$

$$\mathcal{L}^{-1}: \quad y''(t) + 4y'(t) + 3y(t) = x'(t) - x(t)$$

4. **25 pts.** The input $x(t)$ and output $y(t)$ of a causal LTI system H are related by

$$y''(t) + 5y'(t) + 6y(t) = x''(t) + 6x'(t) + 9x(t).$$

The system input is given by $x(t) = te^{-3t}u(t)$.
You are given the following initial conditions on $x(t)$ and $y(t)$:

$$x(0^-) = 5, \quad x'(0^-) = -10,$$
$$y(0^-) = 1, \quad y'(0^-) = 10.$$

Use the unilateral Laplace transform to find $y(t)$ for $t \geq 0$.

Table: $\mathcal{X}_u(s) = \frac{1}{(s+3)^2}$ $\quad \{n=2\}$

UL: $s^2 \mathcal{Y}_u(s) - sy(0^-) - y'(0^-) + 5[s\mathcal{Y}_u(s) - y(0^-)] + 6\mathcal{Y}_u(s)$

$$= s^2\mathcal{X}_u(s) - sx(0^-) - x'(0^-) + 6[s\mathcal{X}_u(s) - x(0^-)] + 9\mathcal{X}_u(s)$$

$$s^2\mathcal{Y}_u(s) - s(1) - 10 + 5[s\mathcal{Y}_u(s) - 1] + 6\mathcal{Y}_u(s)$$
$$= s^2\mathcal{X}_u(s) - s(5) - (-10) + 6[s\mathcal{X}_u(s) - 5] + 9\mathcal{X}_u(s)$$

$$s^2\mathcal{Y}_u(s) + 5s\mathcal{Y}_u(s) + 6\mathcal{Y}_u(s) - s - 10 - 5$$
$$= s^2\mathcal{X}_u(s) + 6s\mathcal{X}_u(s) + 9\mathcal{X}_u(s) - 5s + 10 - 30$$

$$[s^2 + 5s + 6]\mathcal{Y}_u(s) - (s+15) = [s^2 + 6s + 9]\mathcal{X}_u(s) - 5s - 20$$

$$(s+3)(s+2)\mathcal{Y}_u(s) = (s+3)(s+3)\frac{1}{(s+3)^2} - 5s - 20 + s + 15$$

$$(s+3)(s+2)\mathcal{Y}_u(s) = \frac{(s+3)^2}{(s+3)^2} - 4s - 5 = 1 - 4s - 5 = -4s - 4$$

$$\mathcal{Y}_u(s) = \frac{-4s-4}{(s+2)(s+3)} = \frac{-4(s+1)}{(s+2)(s+3)}$$

→

More Work Space for Problem 4...

$$\mathcal{Y}_u(s) = \frac{-4(s+1)}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = \left.\frac{-4(s+1)}{s+3}\right|_{s=-2} = \frac{-4(-1)}{1} = 4$$

$$B = \left.\frac{-4(s+1)}{s+2}\right|_{s=-3} = \frac{-4(-2)}{-1} = -8$$

$$\mathcal{Y}_u(s) = \underbrace{\frac{4}{s+2}}_{Re[s]>-2} - \underbrace{\frac{8}{s+3}}_{Re[s]>-3}$$

Table: $\boxed{y(t) = 4e^{-2t}u(t) - 8e^{-3t}u(t)}$