EE 3793
Test 2

Friday, April 18, 1997
12:30 PM - 1:20 PM

Spring 1997 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded.

You have 50 minutes to complete the test. All work must be your own. You may use two 8.5 × 11 inch two-sided note sheets as well as the summation formula sheet given out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. **2.** **3.** **4.** **5.**

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________
5. (25) ________

TOTAL (100):

1. **25 pts.** Consider a system H whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = \frac{d}{dt}x(t).$$

(a) **2 pts.** Is the system linear? *(justify your answer)*

Let $x(t) = a_1 x_1(t) + a_2 x_2(t)$.

$$y(t) = \frac{d}{dt}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 \frac{d}{dt}x_1(t) + a_2 \frac{d}{dt}x_2(t)$$

$$= a_1 y_1(t) + a_2 y_2(t) \checkmark$$

IT IS LINEAR.

(b) **2 pts.** Is the system shift invariant? *(justify your answer)*

For input $x(t-t_0)$, the output is

$$\frac{d}{dt}x(t-t_0) \overset{\text{Chain Rule}}{=\!=} \frac{d}{dt}x(t-t_0)\,\frac{d}{dt}(t-t_0) = \frac{d}{dt}x(t-t_0) \checkmark$$

IT IS SHIFT INVARIANT.

(c) **8 pts.** What is the system frequency response?

$$y(t) = \frac{d}{dt}x(t)$$

$$\mathcal{F}[y(t)] = \mathcal{F}\left[\frac{d}{dt}x(t)\right]$$

$$Y(\omega) = j\omega X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = j\omega}$$

Problem 1, cont...

(d) **8 pts.** What is the system transfer function?

$$y(t) = \frac{d}{dt} x(t)$$

$$\mathcal{L}[y(t)] = \mathcal{L}\left[\frac{d}{dt} x(t)\right]$$

$$Y(s) = s X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = s}$$

(e) **5 pts.** What is the system impulse response?

$\boxed{\text{METHOD 1:}}$

$$h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}[s] = \mathcal{L}^{-1}[s \cdot 1]$$

So $h(t) = \frac{d}{dt} \mathcal{L}^{-1}[1] = \frac{d}{dt} \delta(t)$

$$\boxed{h(t) = \delta'(t), \text{ the unit doublet}}$$

$\boxed{\text{Method 2:}}$ $h(t)$ is the response when $\delta(t)$ is the input,

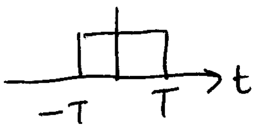
So $h(t) = \frac{d}{dt}\delta(t) = \delta'(t)$.

2. Find the signal $x(t)$ that has a Fourier transform

$$X(\omega) = \mathcal{F}[x(t)] = \begin{cases} \cos\frac{\pi\omega}{2}, & |\omega| \le 16 \\ 0 & |\omega| > 16. \end{cases}$$

Hint: Apply the frequency shifting and duality properties to the Fourier transform pair for the boxcar time function.

$$x(t) = \begin{cases} 1, & |t| \le T \\ 0, & |t| > T \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\sin\omega T}{\omega}$$



because $\cos At = \dfrac{e^{jAt} + e^{-jAt}}{2}$

Freq. shift:

$$x(t) = \begin{cases} \cos At, & |t| \le T \\ 0, & |t| > T \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin[(\omega - A)T]}{\omega - A} + \frac{\sin[(\omega + A)T]}{\omega + A}$$

Duality:

$$\frac{\sin[(t-A)T]}{t-A} + \frac{\sin[(t+A)T]}{t+A} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \begin{cases} 2\pi\cos(-A\omega), & |-\omega| \le T \\ 0, & |-\omega| > T \end{cases}$$

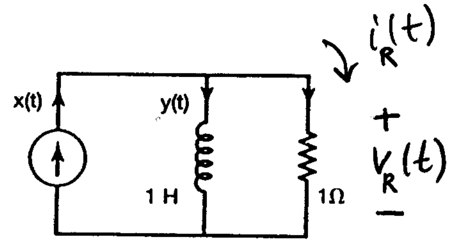
$$= \begin{cases} 2\pi\cos(A\omega), & |\omega| \le T \\ 0, & |\omega| > T. \end{cases}$$

So,

$$\frac{\sin[(t-\frac{\pi}{2})16]}{2\pi(t-\frac{\pi}{2})} + \frac{\sin[(t+\frac{\pi}{2})16]}{2\pi(t+\frac{\pi}{2})} \overset{\mathcal{F}}{\longleftrightarrow} \begin{cases} \cos\frac{\pi\omega}{2}, & |\omega| \le 16 \\ 0, & |\omega| > 16 \end{cases}$$

$$\boxed{x(t) = \frac{\sin(16t - 8\pi)}{2\pi t - \pi^2} + \frac{\sin(16t + 8\pi)}{2\pi t + \pi^2}}$$

3. **25 pts.** Consider the LSI system shown below, where current $x(t)$ is the input and current $y(t)$ is the output. Find the frequency response $H(\omega)$ of the system.



Hints: The voltage-current relationship for a resistor is $v(t) = i(t)R$. For an inductor, $v(t) = L\frac{di(t)}{dt}$. Since the inductor and resistor are in parallel, the resistor voltage must be equal to the inductor voltage. Use this fact to determine the resistor current in terms of $y(t)$. Apply KCL to the top node to establish that $x(t)$ = inductor current + resistor current. Apply Fourier transforms to both sides of this equation.

$$V_R(t) = L\frac{d}{dt}y(t) \Rightarrow i_R(t) = \frac{L}{R}\frac{d}{dt}y(t).$$

KCL: $$x(t) = y(t) + i_R(t) = y(t) + \frac{L}{R}\frac{d}{dt}y(t)$$

$$x(t) = y(t) + \frac{d}{dt}y(t)$$

$$X(\omega) = Y(\omega) + j\omega Y(\omega) = (1+j\omega)Y(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1+j\omega}}$$

More workspace for problem 3.

4. **25 pts**. The input $x(t)$ and output $y(t)$ of an LSI system are related by

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t).$$

Suppose that $y(0) = 0$, but that $\frac{d}{dt}y(0) = 1$, so that the system is **not** initially relaxed. Find the output $y(t)$ when the input is $x(t) = u(t)$.

Hint: Use the unilateral Laplace transform.

$$s^2 Y_u(s) - \cancel{s}y(0^-) - y'(0^-) + 3sY_u(s) - 3y(0^-) + 2Y_u(s) = \frac{1}{s}$$

$$s^2 Y_u(s) - 1 + 3s Y_u(s) + 2Y_u(s) = \frac{1}{s}$$

$$(s^2+3s+2)Y_u(s) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$Y_u(s) = \frac{s+1}{s(s^2+3s+2)} = \frac{s+1}{s(s+1)(s+2)} = \frac{1}{s(s+2)}$$

$$= \frac{A}{s} + \frac{B}{s+2}$$

$$A = \left.\frac{1}{s+2}\right|_{s=0} = \frac{1}{2} \quad ; \quad B = \left.\frac{1}{s}\right|_{s=-2} = -\frac{1}{2}$$

$$Y_u(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$$

$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$$

$$\boxed{y(t) = \frac{1}{2}u(t)\left[1 - e^{-2t}\right]}$$

5. **25 pts.** Suppose $x(t)$ is a cable television signal. The signal is transmitted from Atlanta to Oklahoma City via a satellite communications channel. Because of cosmic rays, solar radiation, water vapor in the lower atmosphere, and other factors, the signal is **distorted** during transmission.

Thus, the received signal $r(t)$ is not exactly equal to the transmitted signal $x(t)$. Suppose that the channel distortion can be modeled according to

$$r(t) = x(t) * d(t)$$

where

$$d(t) = \delta(t) + 5e^{-3t}u(t).$$

Design an LSI filter H to recover $x(t)$ from $r(t)$, so that in Oklahoma City we can perform the filtering operation

$$y(t) = r(t) * h(t) \equiv x(t)$$

to get the original signal from the received signal. Give the impulse response $h(t)$ of the filter H.

$$Y(\omega) = R(\omega)H(\omega) = X(\omega)D(\omega)H(\omega) \equiv X(\omega)$$

so $D(\omega)H(\omega) = \dfrac{X(\omega)}{X(\omega)} = 1$

So $H(\omega) = \dfrac{1}{D(\omega)}$

$$D(\omega) = \mathcal{F}\left[\delta(t) + 5e^{-3t}u(t)\right] = 1 + 5\frac{1}{3+j\omega}$$

$$= \frac{3+j\omega}{3+j\omega} + \frac{5}{3+j\omega} = \frac{8+j\omega}{3+j\omega}$$

$$H(\omega) = \frac{1}{D(\omega)} = \frac{3+j\omega}{8+j\omega} = \frac{3}{8+j\omega} + \frac{j\omega}{8+j\omega}$$

$\longrightarrow$

More workspace for problem 5.

$$h(t) = \mathcal{F}^{-1}\left[\frac{3}{8+j\omega}\right] + \mathcal{F}^{-1}\left[j\omega \frac{1}{8+j\omega}\right]$$

$$= 3e^{-8t}u(t) + \frac{d}{dt}\mathcal{F}^{-1}\left[\frac{1}{8+j\omega}\right]$$

$$= 3e^{-8t}u(t) + \frac{d}{dt}\left\{e^{-8t}u(t)\right\}$$

$$= 3e^{-8t}u(t) + e^{-8t}\delta(t) + (-8)e^{-8t}u(t)$$

$$= (3-8)e^{-8t}u(t) + e^{-8\cdot 0}\delta(t)$$

$$= \delta(t) - 5e^{-8t}u(t)$$

$$\boxed{h(t) = \delta(t) - 5e^{-8t}u(t)}$$