ECE 3793
Test 2

Wednesday, April 21, 1999
7:00 PM - 9:30 PM

SP99
~~Fall 1999~~

Name: SOLUTION

Dr. Havlicek

Student Num:

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded. You have 150 minutes to complete the test. All work must be your own. You may use two 8.5 × 11 inch two-sided note sheets as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

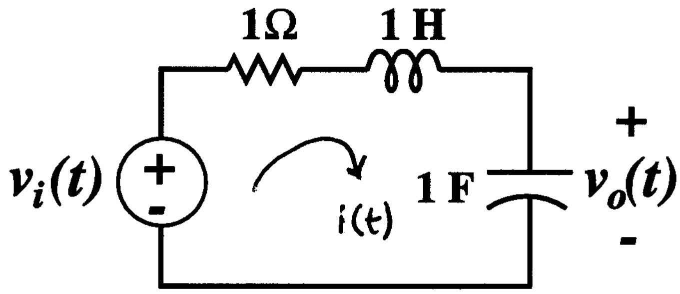
1. 2. 3. 4. 5.

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________
5. (25) ________

TOTAL (100):

1. **25 pts**. Consider the RLC circuit shown below:



This circuit is a continuous-time LSI system. Source voltage $v_i(t)$ is the system input. Capacitor voltage $v_o(t)$ is the system output.

(a) **7 pts**. Find the differential equation relating $v_i(t)$ and $v_o(t)$.

Capacitor: $i(t) = C\frac{d}{dt}V_o(t) = V_o'(t)$

Inductor: $V_I(t) = L\frac{d}{dt}i(t) = i'(t) = V_o''(t)$

Resistor: $V_R(t) = Ri(t) = i(t) = V_o'(t)$

KVL: $V_R(t) + V_I(t) + V_o(t) = V_i(t)$

$V_o'(t) + V_o''(t) + V_o(t) = V_i(t)$

$$\underline{\underline{V_o''(t) + V_o'(t) + V_o(t) = V_i(t)}}$$

(b) **8 pts**. Find the system frequency response $H(\omega)$.

$(j\omega)^2 V_o(\omega) + j\omega V_o(\omega) + V_o(\omega) = V_i(\omega)$

$(-\omega^2 + j\omega + 1)\, V_o(\omega) = V_i(\omega)$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \underline{\underline{\frac{1}{1 + j\omega - \omega^2}}}$$

Problem 1, cont...

(c) **10 pts.** Find the system impulse response $h(t)$.

denominator of $H(\omega) = (j\omega)^2 + j\omega + 1$

Quadratic Formula: $a = b = c = 1$

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$H(\omega) = \frac{1}{\left[j\omega + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right]\left[j\omega + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right]} = \frac{A}{\left(j\omega + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)} + \frac{B}{\left(j\omega + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}$$

$$A = \left.\frac{1}{j\omega + \frac{1}{2} + j\frac{\sqrt{3}}{2}}\right|_{j\omega = -\frac{1}{2} + j\frac{\sqrt{3}}{2}} = \frac{1}{-\frac{1}{2} + j\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}} = \frac{1}{j\sqrt{3}}$$

$$B = \left.\frac{1}{j\omega + \frac{1}{2} - j\frac{\sqrt{3}}{2}}\right|_{j\omega = -\frac{1}{2} - j\frac{\sqrt{3}}{2}} = \frac{1}{-\frac{1}{2} - j\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}} = \frac{-1}{j\sqrt{3}}$$

$$H(\omega) = \frac{1}{j\sqrt{3}}\left[\frac{1}{\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + j\omega} - \frac{1}{\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + j\omega}\right]$$

Table: $h(t) = \frac{1}{j\sqrt{3}}\left[e^{-\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)t} u(t) - e^{-\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)t} u(t)\right]$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t}\left[\frac{e^{j\frac{\sqrt{3}}{2}t} - e^{-j\frac{\sqrt{3}}{2}t}}{2j}\right] u(t)$$

$$h(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) u(t)$$

2. **25 pts.** Consider the input signal $x(t) = \cos t$.

(a) **15 pts.** H_1 is an LSI system with impulse response $h_1(t) = u(t)$. Find the system output when $x(t)$ is the input.

$$X(\omega) = \pi\left[\delta(\omega-1) + \delta(\omega+1)\right]$$

$$H_1(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$Y_1(\omega) = H_1(\omega)X(\omega) = \frac{\pi}{j\omega}\left[\delta(\omega-1) + \delta(\omega+1)\right]$$

$$= \frac{\pi}{j}\left[\frac{1}{1}\delta(\omega-1) + \frac{1}{-1}\delta(\omega+1)\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega-1) - \delta(\omega+1)\right]$$

$$y_1(t) = \sin t$$

Problem 2, cont...

(b) **10 pts.** H_2 is an LSI system with impulse response $h_2(t) = 2te^{-t}u(t)$. Find the system output when $x(t)$ is the input.

$$H_2(\omega) = \frac{2}{(1+j\omega)^2}$$

$$Y_2(\omega) = H_2(\omega)\,X(\omega) = \frac{2\pi}{(1+j\omega)^2}\left[\delta(\omega-1) + \delta(\omega+1)\right]$$

$$= \frac{2\pi}{(1+j)^2}\delta(\omega-1) + \frac{2\pi}{(1-j)^2}\delta(\omega+1)$$

$$= \frac{2\pi}{1+2j-1}\delta(\omega-1) + \frac{2\pi}{1-2j-1}\delta(\omega+1)$$

$$= \frac{2\pi}{2j}\delta(\omega-1) - \frac{2\pi}{2j}\delta(\omega+1)$$

$$= \frac{\pi}{j}\left[\delta(\omega-1) - \delta(\omega+1)\right]$$

$$\underline{\underline{Y_3(t) = \sin t}}$$

3. **25 pts.** The signal $x[n] = (\frac{4}{5})^n u[n]$ is input to a causal and stable discrete-time LSI system H. The system output is observed to be $y[n] = n(\frac{4}{5})^n u[n]$.

(a) **15 pts.** Find the system frequency response $H(e^{j\omega})$.

$$X(e^{j\omega}) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}$$

$$y[n] = n x[n] \xleftrightarrow{\mathcal{F}} j\frac{d}{d\omega}X(e^{j\omega}) = j\frac{d}{d\omega}\left(1 - \frac{4}{5}e^{-j\omega}\right)^{-1}$$

$$= -j\left(1 - \frac{4}{5}e^{-j\omega}\right)^{-2}\left(\frac{4}{5}j e^{-j\omega}\right)$$

$$= \frac{\frac{4}{5}e^{-j\omega}}{\left(1 - \frac{4}{5}e^{-j\omega}\right)^2}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{4}{5}e^{-j\omega}}{1 - \frac{4}{5}e^{-j\omega}}$$

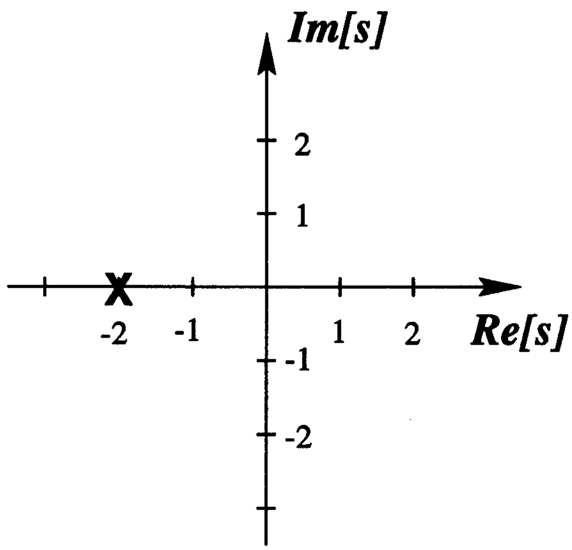
(b) **10 pts.** For an arbitrary input signal $x[n]$, find the difference equation that relates the input and output signals of the system.

$$\left(1 - \frac{4}{5}e^{-j\omega}\right)Y(e^{j\omega}) = \frac{4}{5}e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega}) - \frac{4}{5}e^{-j\omega}Y(e^{j\omega}) = \frac{4}{5}e^{-j\omega}X(e^{j\omega})$$

$$y[n] - \frac{4}{5}y[n-1] = \frac{4}{5}x[n-1]$$

4. **25 pts.** Consider a continuous-time BIBO stable, causal LSI system with rational transfer function $H(s)$. The pole-zero plot of $H(s)$ is shown in the figure below.



The maximum value of the system impulse response is $\max_t h(t) = 2$.

(a) **10 pts.** Specify the region of convergence (ROC) of $H(s)$.

Since the system is causal, $h(t)$ is right-sided.
$\Rightarrow$ The ROC is a right half-plane

$$\underline{\underline{\text{ROC} = Re[s] > -2.}}$$

(b) **15 pts** Find $h(t)$.

$$H(s) = \frac{K}{s+2}, \quad Re[s] > -2, \quad K \text{ constant.}$$

$$h(t) = Ke^{-2t}u(t)$$

$$\max_t h(t) = h(0) = K = 2$$

$$\underline{\underline{h(t) = 2e^{-2t}u(t)}}$$

5. **25 pts**. Consider two *different* discrete-time LSI systems H_1 and H_2. Both systems have the *same* transfer function $H(z)$, which is given by

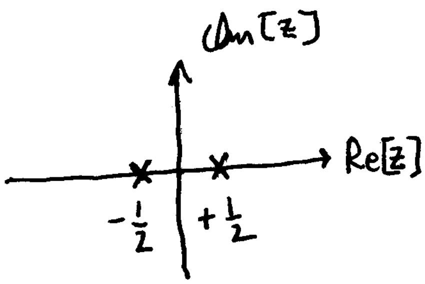
$$H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4}z^{-2}}.$$

Find the impulse responses $h_1[n]$ and $h_2[n]$.

$$H(z) = \frac{1-z^{-1}}{1-\frac{1}{4}z^{-2}} = \frac{1-z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=-\frac{1}{2}} = \frac{1+2}{1+1} = \frac{3}{2}$$

$$B = \left.\frac{1-z^{-1}}{1+\frac{1}{2}z^{-1}}\right|_{z=\frac{1}{2}} = \frac{1-2}{1+1} = -\frac{1}{2}$$



pole-zero plot

$$H(z) = \frac{3/2}{1+\frac{1}{2}z^{-1}} - \frac{1/2}{1-\frac{1}{2}z^{-1}}$$

There are two possible ROC's:

$ROC_1 = |z| > \frac{1}{2}$ $\qquad$ $ROC_2 = |z| < \frac{1}{2}$

For ROC_1, $h_1[n] = \frac{3}{2}\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$

For ROC_2, $h_2[n] = -\frac{3}{2}\left(-\frac{1}{2}\right)^n u[-n-1] + \frac{1}{2}\left(\frac{1}{2}\right)^n u[-n-1]$