Proof by Contrapositive

- We often need to prove statements of the form "if A, then B." In math, this is written $A \to B$.
- Often, people will read "$A \to B$" as "A implies B."
- But I think the easiest way to understand it is if you read it this way: "any time you have A, you also have B."
- Now, "$A \to B$" is a *logical statement*. It has a value – either True (T) or False (F). Here's an example: "if you came to class today, then you were in Oklahoma."

  We have A="you came to class today" and B="you were in Oklahoma."

  If you came to class today, then you had to be in Dale Hall. And Dale Hall is in Oklahoma. So, in this case, any time you have A, you also have B. So the statement $A \to B$ is True.
- Here's another example: "if you are in Norman, OK, then it is raining." We have A="you are in Norman, OK" and B="it is raining." This statement is False, because there are days when it does not rain in Norman. In other words, there are times when you have A, but you do not have B.
- To *prove* the statement $A \to B$, you have to show that any time you have A, you also have B.
- To do a *direct proof*, you start by assuming that you have A. Then, you try to show from this that it must also be true that you have B.
- Now look at page 2.53 of the course notes. In the first part of the proof, we have

  $$A = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

  and

  $$B = \text{“the system is stable.”}$$

  The proof given on page 2.53 through to the top of page 2.54 is a direct proof. It starts by assuming that you have A, and then it shows that this means you must also have B.
- But there are cases where a direct proof is very difficult. Sometimes, it is easier to assume that you **don't** have B and show that this means you also **don't** have A. In other words, sometimes it is easier to prove that $\overline{B} \to \overline{A}$.
- And it turns out that the statements $A \to B$ and $\overline{B} \to \overline{A}$ are logically equivalent: if one of them is True, then the other one is True. If one of them is False, then the other one is False.
- The statement $\overline{B} \to \overline{A}$ is called the *contrapositive* of $A \to B$.
- Here is a truth table that shows that the two statements $A \to B$ and $\overline{B} \to \overline{A}$ are equivalent... i.e., they have the same truth table:

A	B	$A \rightarrow B$	$\overline{B}$	$\overline{A}$	$\overline{B} \rightarrow \overline{A}$
F	F	**T**	T	T	**T**
F	T	**T**	F	T	**T**
T	F	**F**	T	F	**F**
T	T	**T**	F	F	**T**

- If the first two lines of this table seem confusing, then try thinking about it this way: the statement $A \rightarrow B$ means "if you have A, then you also have B." It says *nothing* about what happens when you **don't** have A. So, this statement is *automatically* **True** any time that you don't have A.

- On page 2.54 of the notes, we have

$$A = \text{"the system is stable."}$$

and

$$B = \sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

In this case it turns out to be *really* hard to prove $A \rightarrow B$. It is *less* hard (but still hard) to prove $\overline{B} \rightarrow \overline{A}$; i.e., to do a proof by contrapositive. And that's what is shown in the notes on pp. 2.54-2.55.