ECE 3793

HW3 SOLUTION

HAVLICEK

(1.28) c) $y[n] = n x[n]$

(1) Memoryless: at each n, the value $y[n]$ of the output signal depends on the current value $x[n]$ of the input signal, but not on any past values of the input signal (like $x[n-1]$) and not on any future values of the input signal (like $x[n+1]$).

→ So the system is memoryless.

(2) Time Invariant: Let $n_0 \in \mathbb{Z}$ be a constant and let the input be $x_1[n]$.

Then the output is $y_1[n] = H\{x_1[n]\} = n x_1[n]$.

Then $y_1[n-n_0] = (n-n_0) x_1[n-n_0]$.

Now let the input be $x_2[n] = x_1[n-n_0]$.

Then the output is

$$
\begin{aligned}
y_2[n] = H\{x_2[n]\} &= n x_2[n] \\
&= n x_1[n-n_0] \\
&\neq y_1[n-n_0].
\end{aligned}
$$

→ So the system is not time invariant.

1.28 c)...

(3) Linear: let $x_1[n]$ and $x_2[n]$ be two arbitrary input signals and let $c_1, c_2 \in \mathbb{C}$ be two constants.

Then $y_1[n] = H\{x_1[n]\} = n x_1[n]$

and $y_2[n] = H\{x_2[n]\} = n x_2[n]$.

We have $c_1 y_1[n] + c_2 y_2[n] = c_1 n x_1[n] + c_2 n x_2[n]$.

Now let $x_3[n] = c_1 x_1[n] + c_2 x_2[n]$.

Then

$$
\begin{aligned}
y_3[n] &= H\{x_3[n]\} \\
&= n x_3[n] \\
&= n(c_1 x_1[n] + c_2 x_2[n]) \\
&= c_1 n x_1[n] + c_2 n x_2[n] \\
&= c_1 y_1[n] + c_2 y_2[n] \checkmark
\end{aligned}
$$

→ The system <u>is</u> <u>linear</u>.

(4) Causal: The system <u>is</u> <u>causal</u> because it is memoryless. In particular, for any time n, the current value of the output signal $y[n]$ depends on the current value of the input signal $x[n]$, but <u>not</u> on any <u>future</u> values of the input signal.

1.28 c)...

(5) Stable: I guess that this system is <u>not</u> stable. If I make a bounded input that's "turned on" for all the positive n's, then $nx[n]$ will make the output grow without bound. So I will have a <u>bounded</u> input that produces an <u>unbounded</u> output.

Here's a formal proof of that by contradiction:

→ The system is <u>not</u> stable.

<u>Proof:</u> let the input be $x[n] = u[n]$. Since $x[n] \leq 1 \ \forall n \in \mathbb{Z}$, this is a <u>bounded</u> input.

Then $y[n] = H\{x[n]\} = nx[n] = nu[n]$.

Suppose that this output <u>is</u> bounded.

Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|y[n]| \leq B \ \forall n \in \mathbb{Z}$.

But, when $n = \lceil B \rceil + 1$,

smallest integer $\geq B$... the "ceiling" function.

we have

$$|y[n]| = |nu[n]|_{n = \lceil B \rceil + 1} = (\lceil B \rceil + 1) \cdot 1$$

$$= \lceil B \rceil + 1 > B \quad \times\times$$

→ This is a contradiction.

→ Since the assumption that $y[n]$ is bounded by B leads us to a <u>contradiction</u>, no such B can exist.

→ So the output signal is <u>not</u> bounded.

→ The system is <u>unstable</u> because a bounded input produced an unbounded output. QED.

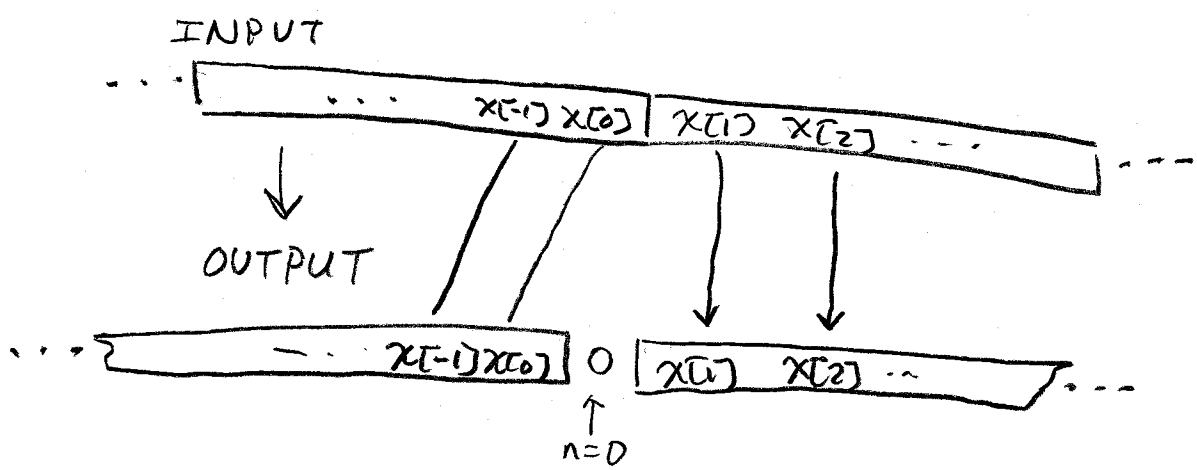
1.28e) $y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n+1] & n \leq -1 \end{cases}$

(1) MEMORYLESS: The third part of the "rule" for the system I/O relation tells us that this system is <u>not</u> memoryless:

when $n=-1$, we have $y[-1] = x[-1+1] = x[0]$. So the value of the output signal at $n=-1$ depends on the value of the input signal at $n=0$, which is a different time.

$\longrightarrow$ <u>NOT MEMORYLESS</u>.

(2) Time Invariant: the action of the system is to take the input from $-\infty < n \leq 0$ and slide it "one to the left", dropping in a zero at $n=0$:



So shifting the input will never move the zero at $y[0]=0$. Therefore, I guess that this system $\rightarrow$

1.28e) ... is <u>not</u> time invariant.

Let the input be $x_1[n] = n$. Then the output is

$$y_1[n] = \begin{cases} x_1[n], & n \geq 1 \\ 0, & n=0 \\ x_1[n+1], & n \leq -1 \end{cases} = \begin{cases} n, & n \geq 1 \\ 0, & n=0 \\ n+1, & n \leq -1 \end{cases}$$

So $y_1[n-2] = \begin{cases} n-2, & n \geq 3 \\ 0, & n=2 \\ n-1, & n \leq 1 \end{cases}$ (you can verify this by making the graph of $y_1[n]$ and then shifting it).

Now let $x_2[n] = x_1[n-2] = n-2$.

Then $$y_2[n] = \begin{cases} x_2[n], & n \geq 1 \\ 0, & n=0 \\ x_2[n+1], & n \leq -1 \end{cases} = \begin{cases} n-2, & n \geq 1 \\ 0, & n=0 \\ n-1, & n \leq -1 \end{cases}$$

n	$y_1[n-2]$	$y_2[n]$	
1	0	-1	X
0	-1	0	X

So $y_1[n-2] \neq y_2[n]$ and the system is <u>not time invariant</u>.

1.28e)... $\boxed{3}$

(3) Linear: let $x_1[n]$ and $x_2[n]$ be two arbitrary input signals and let $c_1, c_2 \in \mathbb{C}$ be two constants.

Then $y_1[n] = H\{x_1[n]\} = \begin{cases} x_1[n], & n \geq 1 \\ 0, & n=0 \\ x_1[n+1], & n \leq -1 \end{cases}$,

$$y_2[n] = H\{x_2[n]\} = \begin{cases} x_2[n], & n \geq 1 \\ 0, & n=0 \\ x_2[n+1], & n \leq -1 \end{cases},$$

and $c_1 y_1[n] + c_2 y_2[n] = \begin{cases} c_1 x_1[n] + c_2 x_2[n], & n \geq 1 \\ 0, & n=0 \\ c_1 x_1[n+1] + c_2 x_2[n+1], & n \leq -1 \end{cases}$.

Now let $x_3[n] = c_1 x_1[n] + c_2 x_2[n]$.

Then $y_3[n] = H\{x_3[n]\} = \begin{cases} x_3[n], & n \geq 1 \\ 0, & n=0 \\ x_3[n+1], & n \leq -1 \end{cases}$

$$= \begin{cases} c_1 x_1[n] + c_2 x_2[n], & n \geq 1 \\ 0, & n=0 \\ c_1 x_1[n+1] + c_2 x_2[n+1], & n \leq -1 \end{cases}$$

$$= c_1 y_1[n] + c_2 y_2[n] \checkmark$$

$\rightarrow$ So the system <u>is</u> <u>linear</u>.

1.28e)...

(4) Causal: When $n=-1$, the value of the output signal is $y[-1] = x[0]$ ... which depends on the future value of the input signal from time $n=0$.

$\rightarrow$ The system is not causal.

(5) Stable: Suppose $x[n]$ is a bounded input signal. Then $\exists B \in \mathbb{R}$, $B \geq 0$, such that $|x[n]| \leq B \quad \forall n \in \mathbb{Z}$.

Now, $$|y[n]| = \begin{cases} |x[n]|, & n \geq 1 \\ 0, & n=0 \\ |x[n+1]|, & n \leq -1 \end{cases}$$

$$\leq \begin{cases} B, & n \geq 1 \\ B, & n=0 \\ B, & n \leq -1 \end{cases} = B.$$

So $|y[n]| \leq B \quad \forall n \in \mathbb{Z}$. Then $y[n]$ is bounded and every bounded input signal produces a bounded output signal. So the system is stable.

1.28g) $y[n] = x[4n+1]$

(1) Memoryless: when $n=0$, the value of the output signal is $y[0] = x[1]$, which depends on the value of the input signal from a different time. So this system is <u>not memoryless</u>.

(2) Time Invariant: three out of every four samples of the input signal are "thrown away":

$$y[0] = x[1] \leftarrow x[2], x[3], x[4] \text{ thrown away.}$$
$$y[1] = x[5]$$
$$y[2] = x[9] \leftarrow x[6], x[7], x[8] \text{ thrown away.}$$

If I shift the input by 1, then different samples of $x[n]$ will be kept. So I guess "not time invariant."

Let $x_1[n] = \delta[n-1]$. Then $y_1[n] = \delta[n]$

and $y_1[n-1] = \delta[n-1]$.

Now let $x_2[n] = x_1[n-1] = \delta[n-2]$.

Then $y_2[n] = 0 \neq y_1[n-1]$

So this system is <u>not</u> <u>time</u> <u>invariant</u>.

1.28g)...

(3) Linear: Let $x_1[n]$ and $x_2[n]$ be arbitrary input signals and let $c_1, c_2 \in \mathbb{C}$ be two constants.

Then $y_1[n] = H\{x_1[n]\} = x_1[4n+1]$

and $y_2[n] = H\{x_2[n]\} = x_2[4n+1]$.

So $c_1 y_1[n] + c_2 y_2[n] = c_1 x_1[4n+1] + c_2 x_2[4n+1]$.

Now let $x_3[n] = c_1 x_1[n] + c_2 x_2[n]$.

Then
$$\begin{aligned} y_3[n] = H\{x_3[n]\} &= x_3[4n+1] \\ &= c_1 x_1[4n+1] + c_2 x_2[4n+1] \\ &= c_1 y_1[n] + c_2 y_2[n] \; \checkmark \end{aligned}$$

So the system <u>is</u> <u>linear</u>.

(4) Causal: when $n=0$ we have $y[0] = x[1]$ which depends on a future value of the input signal. So this system is <u>not</u> <u>causal</u>.

1.28g)...

(5) Stable: let $x[n]$ be a bounded input signal. Then $\exists B \in \mathbb{R}$, $B > 0$, s.t. $|x[n]| \leq B$ $\forall n \in \mathbb{Z}$.

We have

$$|y[n]| = |x[4n+1]| \leq B.$$

So $y[n]$ is bounded, which shows that every bounded input signal produces a bounded output signal. The system is therefore <u>Stable</u>.

1.30a) $y(t) = x(t-4)$.

The action of this system is to shift the input signal to the right by 4. This can be "undone" by a system that shifts left by 4. So the system is invertible and the I/O relation of the inverse system is

$$y(t) = x(t+4).$$

1.30 c)

$y[n] = n x[n]$

$\rightarrow$ Not invertible.

Inputs $x_1[n] = 0$ and $x_2[n] = \delta[n]$ both make the same output $y[n] = 0$.

1.30e)

$$y[n] = \begin{cases} x[n-1], & n \geq 1 \\ 0, & n=0 \\ x[n], & n \leq -1 \end{cases}$$

For $n<0$, the output equals the input.

For $n>0$, the output signal is given by the input signal shifted right by 1.

This leaves a "hole" at $n=0$, which the system "fills in" by setting $y[0]=0$.

This can be "undone" by shifting back left by 1 for $n \geq 0$. So the system is invertible and the I/O relation for the inverse system is

$$y[n] = \begin{cases} x[n+1], & n \geq 0 \\ x[n], & n<0 \end{cases}$$

2.1a) $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$

$h[n] = 2\delta[n+1] + 2\delta[n-1]$

$$y[n] = x[n] * h[n]$$

$$= x[n] * (2\delta[n+1] + 2\delta[n-1])$$

$$= 2\delta[n+1] * x[n] + 2\delta[n-1] * x[n]$$

$$= 2x[n+1] + 2x[n-1]$$

$$= 2\delta[n+1] + 4\delta[n] - 2\delta[n-2]$$
$$+ 2\delta[n-1] + 4\delta[n-2] - 2\delta[n-4]$$

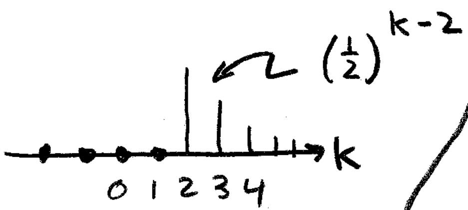
$$= 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

2.3) $x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$

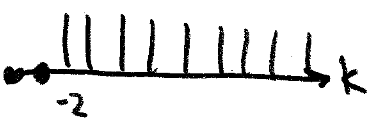
$h[n] = u[n+2]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

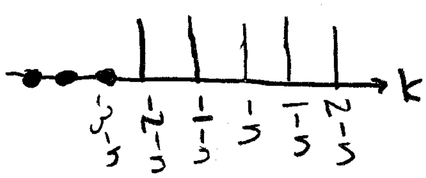
$x[k]$



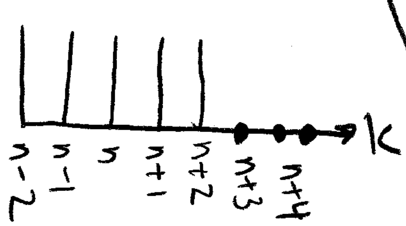
$h[k]$



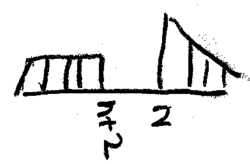
$h[k--n] = h[k+n]$



$h[n-k]$

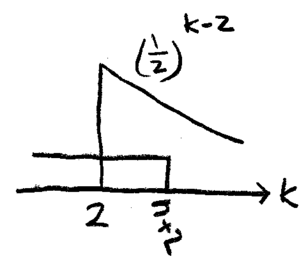


Case I) $n+2 < 2$: $n < 0$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n+2 \geq 2$: $n \geq 0$



$$y[n] = \sum_{k=2}^{n+2} 1 \cdot \left(\frac{1}{2}\right)^{k-2}$$

$$= \left(\frac{1}{2}\right)^{-2} \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k} = 4 \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k}$$

$$= 4\, \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{n+3}}{1-\frac{1}{2}} = 4\, \frac{\frac{1}{4} - \frac{1}{8}\left(\frac{1}{2}\right)^n}{1/2}$$

$$= 8\left[\frac{1}{4} - \frac{1}{8}\left(\frac{1}{2}\right)^n\right] = 2 - \left(\frac{1}{2}\right)^n$$

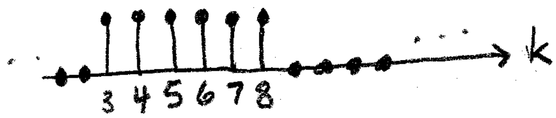
All Together:

$$y[n] = \begin{cases} 0, & n < 0 \\ 2 - \left(\frac{1}{2}\right)^n, & n \geq 0 \end{cases}$$

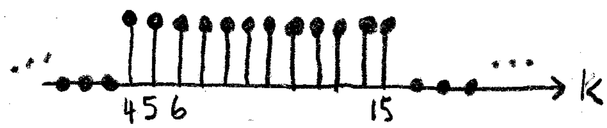
$$= \left[2 - \left(\frac{1}{2}\right)^n\right] u[n]$$

2.4)

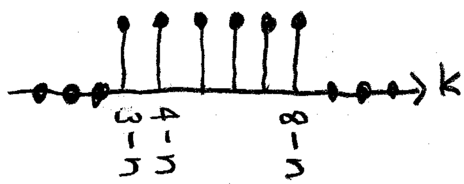
$$x[k] = \begin{cases} 1, & 3 \le k \le 8 \\ 0, & \text{other} \end{cases}$$



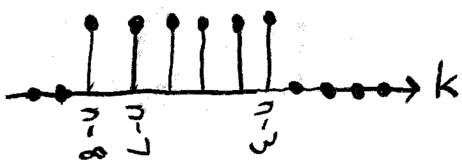
$$h[k] = \begin{cases} 1, & 4 \le k \le 15 \\ 0, & \text{other} \end{cases}$$



$$x[n+k] = x[k-(-n)]:$$

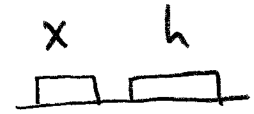


$$x[n-k] = x[-k-(-n)]:$$

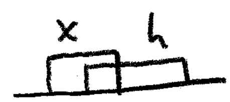


$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

case 1) $n-3 < 4 \Rightarrow n < 7$: $y[n] = 0$



case 2) $n-3 \ge 4$ and $n-8 < 4 \Rightarrow 7 \le n < 12$;

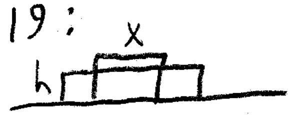


$$y[n] = \sum_{k=4}^{n-3} h[k]\,x[n-k] = \sum_{k=4}^{n-3} 1$$

$$= (n-3) - 4 + 1 = \underline{\underline{n-6}} \longrightarrow$$

2.4)...

case 3) $n-8 \geq 4$ and $n-3 < 16 \Rightarrow 12 \leq n < 19$:

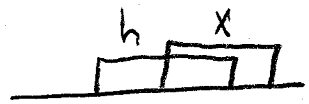


$$y[n] = \sum_{k=n-8}^{n-3} h[k]x[n-k]$$

$$= \sum_{k=n-8}^{n-3} 1 = (n-3)-(n-8)+1$$

$$= n-3-n+8+1 = \underline{\underline{6}}$$

case 4) $n-3 \geq 16$ and $n-8 < 16 \Rightarrow 19 \leq n < 24$:



$$y[n] = \sum_{k=n-8}^{15} h[k]x[n-k]$$

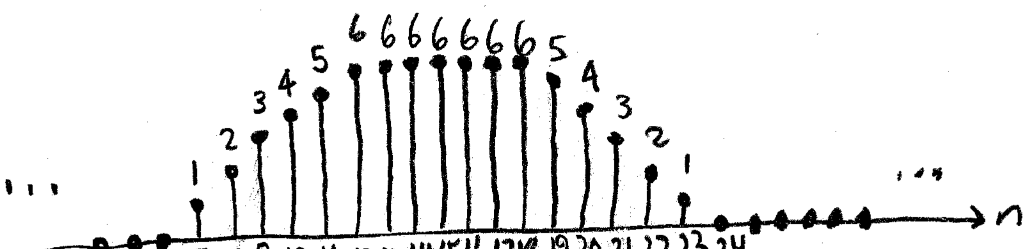
$$= \sum_{k=n-8}^{15} 1 = 15-(n-8)+1$$

$$= 15+8-n+1 = \underline{\underline{24-n}}$$

case 5) $n \geq 24$: $y[n] = 0$.

All Together:

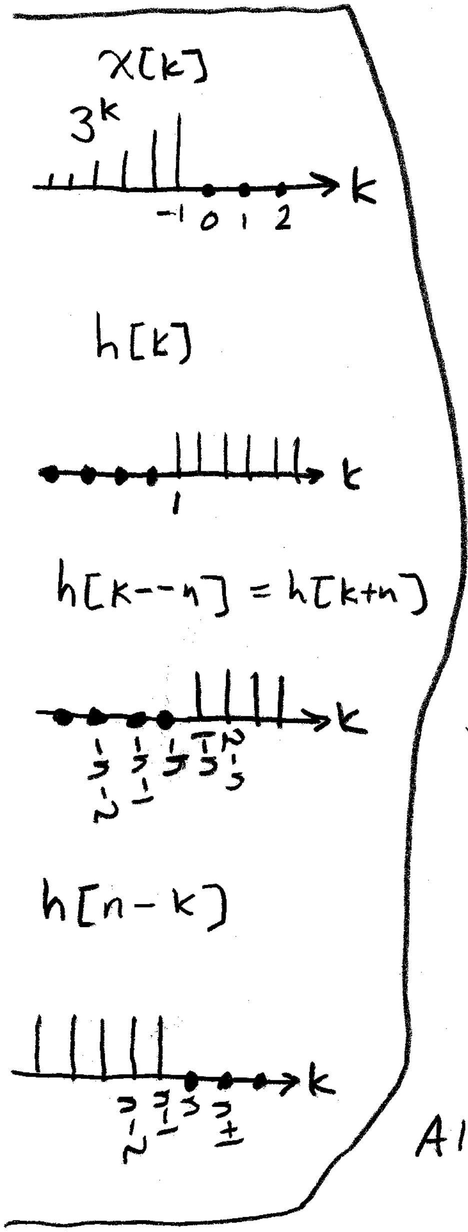
$$y[n] = \begin{cases} 0, & n<7 \\ n-6, & 7 \leq n < 12 \\ 6, & 12 \leq n < 19 \\ 24-n, & 19 \leq n < 24 \\ 0, & n \geq 24 \end{cases}$$

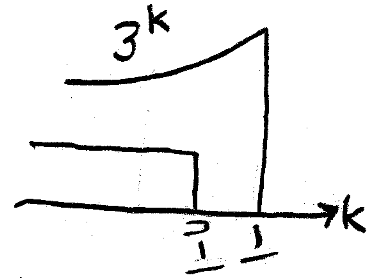


2.6) $x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1] = 3^n u[-n-1]$

$h[n] = u[n-1]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

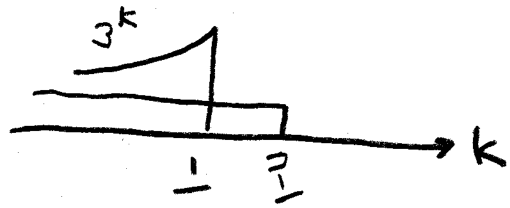




case I) $n-1 \le -1$: $n \le 0$

$$y[n] = \sum_{k=-\infty}^{n-1} 1 \cdot 3^k = \lim_{A\to\infty} \sum_{k=-A}^{n-1} 3^k$$

$$= \lim_{A\to\infty} \frac{\cancelto{0}{3^{-A}} - 3^n}{1-3} = \frac{-3^n}{-2} = \frac{1}{2}(3)^n$$



case II) $n-1 > -1$
$n > 0$

$$y[n] = \sum_{k=-\infty}^{-1} 3^k$$

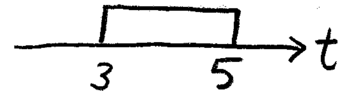
$$= \lim_{A\to\infty} \sum_{k=-A}^{-1} 3^k = \lim_{A\to\infty} \frac{\cancelto{0}{3^{-A}} - 3^0}{1-3}$$

$$= \frac{-1}{-2} = \frac{1}{2}$$

All Together :

$$y[n] = \begin{cases} \frac{3^n}{2}, & n \le 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$

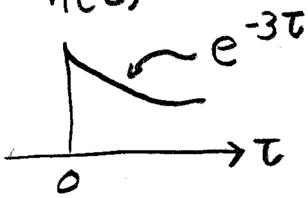
2.11a) $x(t) = u(t-3) - u(t-5) =$



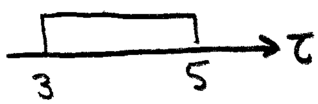
$h(t) = e^{-3t}u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

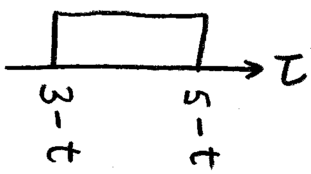
$h(\tau)$



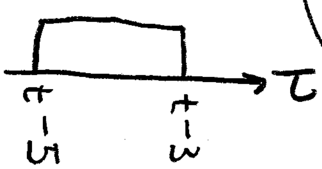
$x(\tau)$



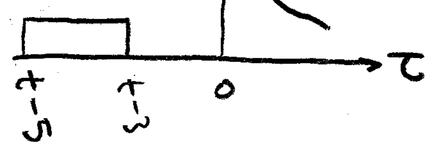
$x(\tau - -t) = x(t+\tau)$



$x(t-\tau)$

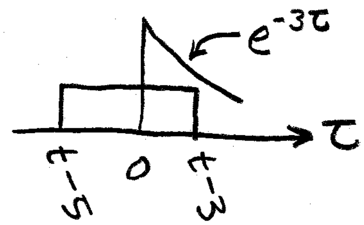


Case I) $t-3<0$: $t<3$



$$y(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0$$

Case II) $t-3 \geq 0$ and $t-5<0$: $3 \leq t < 5$

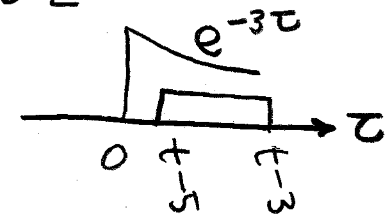


$$y(t) = \int_0^{t-3} e^{-3\tau}\,d\tau$$

$$= -\frac{1}{3}\left[e^{-3\tau}\right]_{\tau=0}^{t-3}$$

$$= -\frac{1}{3}\left[e^{9-3t} - 1\right] = \frac{1}{3}\left[1 - e^{-3(t-3)}\right]$$

Case III) $t-5 \geq 0$: $t \geq 5$



$$y(t) = \int_{t-5}^{t-3} e^{-3\tau}\,d\tau$$

$$= -\frac{1}{3}\left[e^{-3\tau}\right]_{\tau=t-5}^{t-3} = -\frac{1}{3}\left[e^{-3(t-3)} - e^{-3(t-5)}\right]$$

$$= \frac{1}{3}\left[e^{-3t}e^{15} - e^{-3t}e^{9}\right] = \frac{1}{3}\left[e^{6}e^{-3t}e^{9} - e^{-3t}e^{9}\right]$$

$$= \frac{1}{3}(e^6-1)e^{-3t}e^{9} = \frac{1}{3}(e^6-1)e^{-3(t-3)}$$

All Together: $$y(t) = \begin{cases} 0, & t<3 \\ \frac{1}{3}\left[1 - e^{-3(t-3)}\right], & 3 \leq t < 5 \\ \frac{1}{3}(e^6-1)e^{-3(t-3)}, & t \geq 5 \end{cases}$$

2.20b) The integrand is

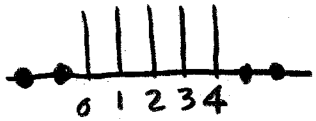
$$\sin(2\pi t)\,\delta(t+3)$$

which is "turned on" at $t=-3$.

But $t=-3$ is <u>not</u> included in the limits of integration.

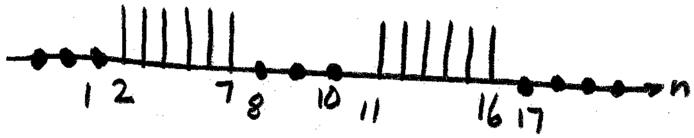
So
$$\int_0^5 \sin(2\pi t)\,\delta(t+3)\,dt$$
$$= \int_0^5 0\,dt = \underline{\underline{0}}.$$

2.21 d) $x[n] =$



$= \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$.

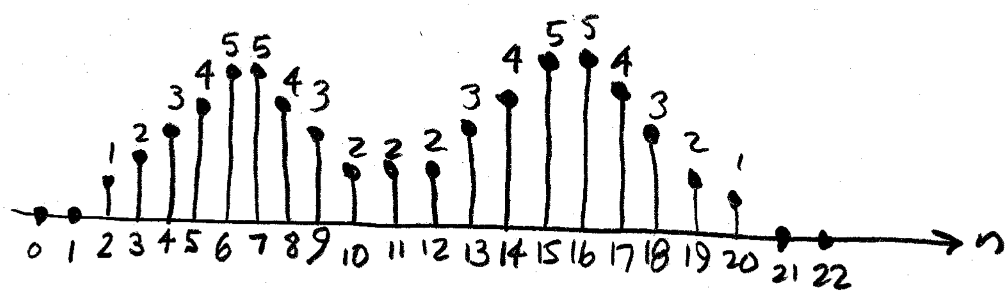
$h[n] =$



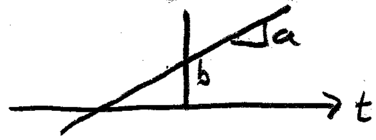
$$y[n] = x[n] * h[n] = (\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]) * h[n]$$

$$= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]$$

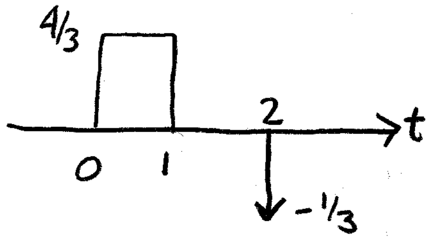
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$h[n]$	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0	0
$h[n-1]$	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0
$h[n-2]$	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0
$h[n-3]$	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0
$h[n-4]$	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0
$y[n]$	0	0	1	2	3	4	5	5	4	3	2	2	2	3	4	5	5	4	3	2	1	0



$$y[n] = \delta[n-2] + 2\delta[n-3] + 3\delta[n-4] + 4\delta[n-5] + 5\delta[n-6] + 5\delta[n-7] + 4\delta[n-8]$$
$$+ 3\delta[n-9] + 2\delta[n-10] + 2\delta[n-11] + 2\delta[n-12] + 3\delta[n-13]$$
$$+ 4\delta[n-14] + 5\delta[n-15] + 5\delta[n-16] + 4\delta[n-17] + 3\delta[n-18]$$
$$+ 2\delta[n-19] + \delta[n-20].$$

2.22d) $x(t) =$ 

$= at + b$

$h(t) =$ 

$= \frac{4}{3}\left[u(t) - u(t-1)\right] - \frac{1}{3}\delta(t-2).$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{4}{3}\left[u(\tau) - u(\tau-1)\right] x(t-\tau) - \frac{1}{3}\delta(\tau-2)\, x(t-\tau)\, d\tau$$

$$= \frac{4}{3}\int_{0}^{1} x(t-\tau)\, d\tau - \frac{1}{3}\int_{-\infty}^{\infty} \delta(\tau-2)\, x(t-\tau)\, d\tau$$

$$= \frac{4}{3}\int_{0}^{1} a(t-\tau) + b\, d\tau - \frac{1}{3}\int_{-\infty}^{\infty} \delta(\tau-2)\left[a(t-\tau) + b\right] d\tau$$

$$= \frac{4}{3}\int_{0}^{1} at - a\tau + b\, d\tau - \frac{1}{3}\left[a(t-\tau) + b\right]_{\tau=2}$$

$$= \frac{4}{3}\left\{\int_{0}^{1} at\, d\tau - \int_{0}^{1} a\tau\, d\tau + \int_{0}^{1} b\, d\tau\right\} - \frac{1}{3}\left[a(t-2) + b\right]$$

$$= \frac{4}{3}\left\{at\int_{0}^{1} d\tau - a\int_{0}^{1} \tau\, d\tau + b\int_{0}^{1} d\tau\right\} - \frac{1}{3}at + \frac{2}{3}a - \frac{1}{3}b$$

$$= \frac{4}{3}\left\{at\left[\tau\right]_{0}^{1} - \frac{1}{2}a\tau^{2}\Big|_{0}^{1} + b\tau\Big|_{0}^{1}\right\} - \frac{1}{3}at + \frac{2}{3}a - \frac{1}{3}b$$

2.22d)...

$$y(t) = \frac{4}{3}\left\{at - \frac{1}{2}a + b\right\} - \frac{1}{3}at + \frac{2}{3}a - \frac{1}{3}b$$

$$= \frac{4}{3}at - \frac{2}{3}a + \frac{4}{3}b - \frac{1}{3}at + \frac{2}{3}a - \frac{1}{3}b$$

$$= at + b \qquad = x(t) \text{ in this case.}$$

In other words, this particular input signal is not changed by the system.