- Consider a complex signal $y_c(t) = a(t)e^{j\theta(t)}$

  where $a(t)$ and $\theta(t)$ are real-valued functions.

- And also its real part $y(t) = a(t)\cos\theta(t)$.

- For AM,
  - We have $\theta(t)$ linear: $\theta(t) = \omega_c t + \theta_c$
  - And the information signal is embedded into $a(t)$:

    suppressed carier: $a(t) = x(t)$

    non-suppressed carier: $a(t) = x(t) + A$

- For "<u>Angle Modulation</u>",
  - We have $a(t)$ constant
  - The information signal is embedded into $\theta(t)$.

  - Phase Modulation or "PM": information signal embedded directly into $\theta(t)$.
  - Frequency Modulation or "FM": information signal embedded into $\frac{d}{dt}\theta(t)$.

$\Rightarrow$ often we use "FM" generically to refer to all forms of angle modulation.

- Instantaneous Frequency

- For angle modulation, we have signals $y_c(t)$ and/or $y(t)$ where $a(t)$ is a constant:

$$a(t) = A$$
$$y_c(t) = A e^{j\theta(t)}$$
$$y(t) = A \cos\theta(t)$$

- For "FM" signals like these, the derivative of the phase signal $\theta(t)$ is called the <u>instaneous</u> <u>frequency</u> $\omega_i(t)$:

$$\omega_i(t) = \frac{d}{dt}\theta(t)$$

- When the phase $\theta(t)$ is <u>linear</u>, $y(t)$ is a pure sinusoid that could be, e.g., the carrier signal in an AM system. In this case, $\omega_i(t)$ is the "frequency" {Fourier frequency} of the sinusoid (the "carrier frequency" if $y(t)$ is an AM carrier wave):

$$\theta(t) = \omega_c t + \theta_c$$
$$y(t) = A\cos[\omega_c t + \theta_c]$$
$$\omega_i(t) = \frac{d}{dt}\theta(t) = \omega_c .$$

- When the phase $\theta(t)$ is quadratic,

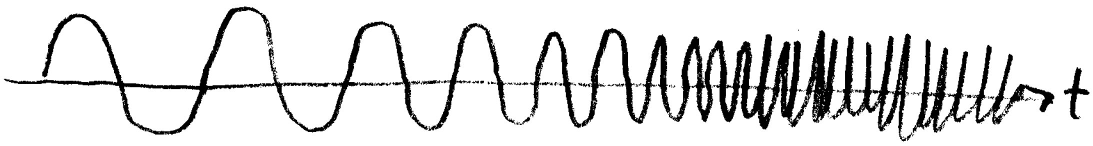
$$\theta(t) = \alpha t^2 + \beta t + \gamma$$

$$y(t) = A\cos\left[\alpha t^2 + \beta t + \gamma\right]$$

$$\omega_i(t) = 2\alpha t + \beta$$

→ instantaneous frequency is linear

→ $y(t)$ is like a "pure sinusoid" with a frequency that goes up or down linearly with time



- When you play a signal like this through a speaker, it makes a "chirp-like" sound.

- For this reason, FM signals with a quadratic phase are called "chirp signals" or just "chirps."

- Chirps are frequently used in modern radar systems.

- For AM, we had

$$y(t) = a(t)\cos[\omega_c t + \theta_c]$$

Suppressed carrier: $a(t) = x(t)$

non-Suppressed carrier: $a(t) = x(t) + A$

→ This type of $y(t)$ is modulated by multiplication of signals, which is a nonlinear operation.

→ But we could still use linear techniques to analyze AM signals, thanks to the frequency convolution property of the Fourier transform:

$$x_1(t)\,x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)\,X_2(\omega-\theta)\,d\theta$$

- For FM, we have

$$y(t) = A\cos[\theta(t)]$$

→ In this case, the nonlinearity arises through the transcendental cosine function.

→ This makes the analysis of FM signals much more difficult than AM signals.

"PM": phase modulation

- The carrier is a pure sinusoid.
- Modulation is done by altering the phase in proportion to the information signal $x(t)$.

$$y(t) = A\cos\theta(t)$$

$$\theta(t) = \omega_c t + \theta_c(t)$$

constant

$$\theta_c(t) = \theta_0 + k_p x(t)$$

$$\Rightarrow y(t) = A\cos\left[\omega_c t + \theta_0 + k_p x(t)\right].$$

"FM": frequency Modulation

- The information signal is coded directly into the instantaneous frequency... into the derivative of the phase.

$$y(t) = A\cos\theta(t)$$

constant

$$\theta'(t) = \omega_c + k_f x(t)$$

$\Rightarrow$ $y(t)$ is a "sinusoid" where the rate of oscillation (instantaneous frequency) speeds up and slows down in proportion to $x(t)$.

→ So PM and FM are related by the derivative.

→ Doing PM with $x(t)$ is the same as doing FM with $x'(t)$.

PM　　　　　　　　　　FM

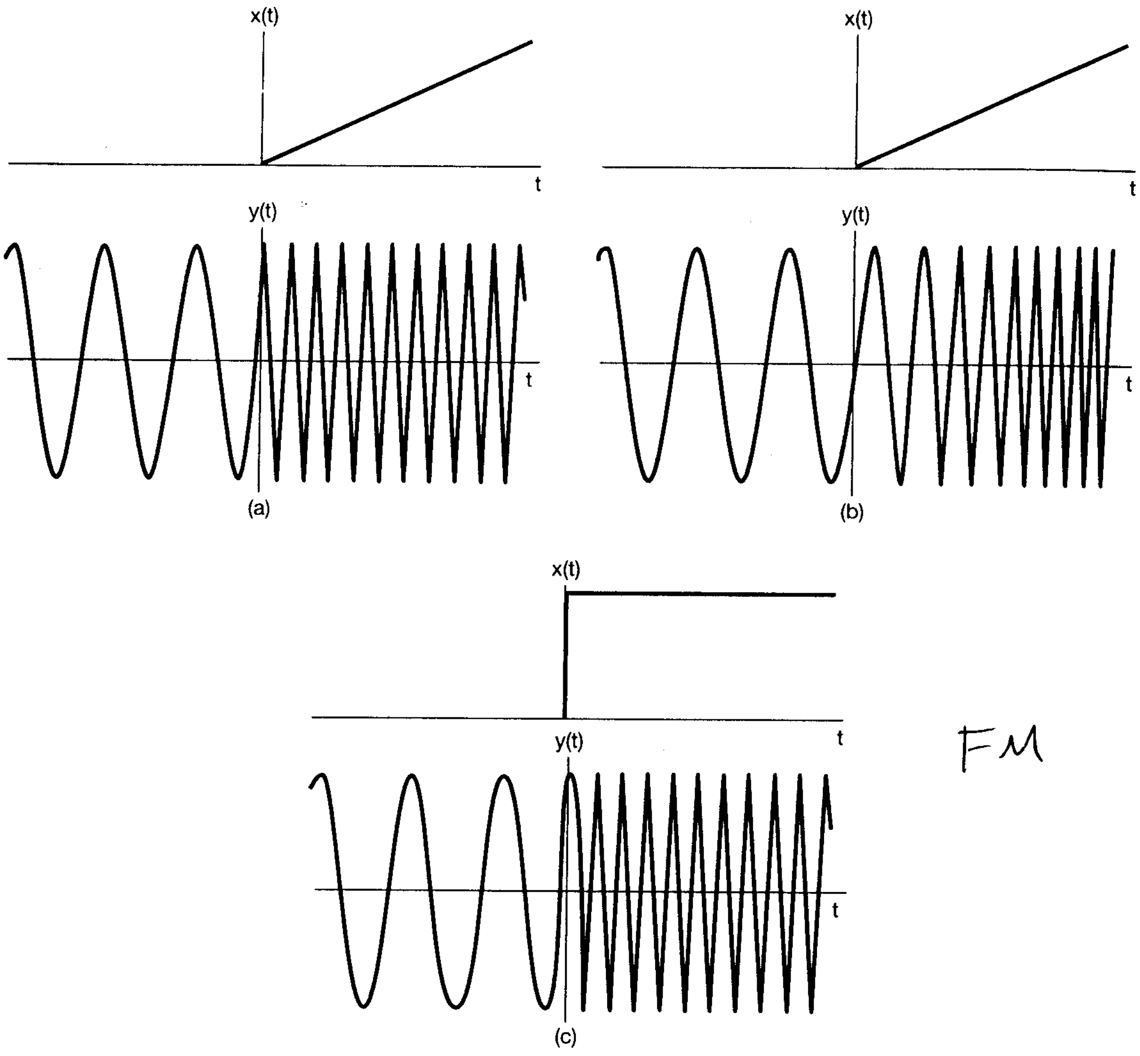


Figure 8.32 Phase modulation, frequency modulation, and their relationship: (a) phase modulation with a ramp as the modulating signal; (b) frequency modulation with a ramp as the modulating signal; (c) frequency modulation with a step (the derivative of a ramp) as the modulating signal.

Narrowband FM

- Let the information signal be
$$x(t) = a\cos\omega_m t$$
- Suppose the carrier amplitude is $A=1$.
$$y(t) = A\cos\theta(t) = \cos\theta(t)$$
- For FM, we have
$$\omega_i(t) = \theta'(t) = \omega_c + k_f a\cos\omega_m t$$
  → The inst. freq. varies sinusoidally between $\omega_c - k_f a$ and $\omega_c + k_f a$.
  → Let $\Delta\omega = k_f a$
  → Total variation about the carrier frequency is $2\Delta\omega$
- The transmitted signal is (t_0 = "on time")
$$y(t) = \cos\theta(t) = \cos\left[\int_{t_0}^{t} \omega_i(\tau)\,d\tau\right]$$
$$= \cos\left[\int_{t_0}^{t} \omega_c + \underbrace{k_f a}_{\Delta\omega}\cos\omega_m\tau\,d\tau\right]$$
$$= \cos\left[\omega_c t + \frac{\Delta\omega}{\omega_m}\sin\omega_m t + \theta_0\right]$$
→ θ_0 is an integration constant that depends on t_0
→ Assume $\theta_0 = 0$ for simplicity.

Then $y(t) = \cos\left[\omega_c t + \frac{\Delta\omega}{\omega_m}\sin\omega_m t\right]$

$m \equiv \frac{\Delta\omega}{\omega_m} \equiv$ modulation index for sinusoidal FM.

- When $\frac{\Delta\omega}{\omega_m}$ m is small $(<< \frac{\pi}{2})$, this is called "Narrowband FM". $\rightarrow k_f a$, amplitude of $x(t)$, is small compared to rate of oscillation of $x(t)$, ω_m.

- Trig identities:

$$y(t) = \cos(\omega_c t + m\sin\omega_m t)$$
$$= [\cos\omega_c t][\cos(m\sin\omega_m t)] - [\sin\omega_c t][\sin(m\sin\omega_m t)]$$

$\Rightarrow$ For small α, $\cos\alpha \approx 1$
$\sin\alpha \approx \alpha$

$\Rightarrow$ when m small, $m\sin\omega_m t$ is small,

$$\left.\begin{aligned} \cos(m\sin\omega_m t) &\approx 1 \\ \sin(m\sin\omega_m t) &\approx m\sin\omega_m t \end{aligned}\right\} \text{Narrowband FM}$$

$$y(t) \approx \cos\omega_c t - m[\sin\omega_m t][\sin\omega_c t]$$

$\Rightarrow$ This is Like AM Double sideband with carrier, but the carrier is a <u>sine</u> times -1 for 2nd term.

$$(\Delta\omega = k_f a) \quad (x(t) = a\cos\omega_m t)$$

$$\left(m = \frac{\Delta\omega}{\omega_m}\right)$$

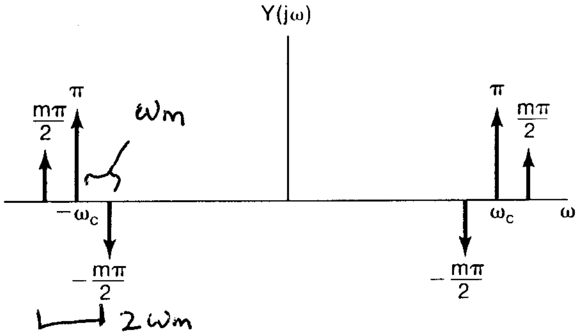
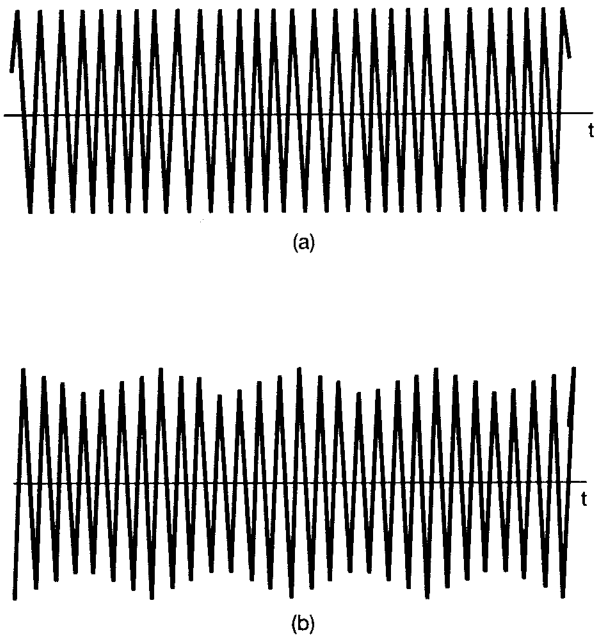


Figure 8.33 Approximate spectrum for narrowband FM.

Narrowband FM

$$y(t) \approx \cos \omega_c t - m[\sin \omega_m t][\sin \omega_c t]$$



Equivalent AM-DSB/WC

$$y(t) = [1 + x(t)] \cos \omega_c t$$
$$= [1 + m \cos \omega_m t] \cos \omega_c t$$
$$= \cos \omega_c t + m[\cos \omega_m t][\cos \omega_c t]$$

Figure 8.34 Comparison of narrowband FM and AM-DSB/WC: (a) narrowband FM; (b) AM-DSB/WC.

Wideband FM

- If $m = \frac{\Delta\omega}{\omega_m}$ is large, $k_f \overset{(\Delta\omega)}{a}$ is large amplitude of $x(t)$ compared to ω_m, the rate of oscillation of $x(t)$.

  This is called wideband FM.

- Small angle approximations no longer apply.
- we're back to

$$y(t) = [\cos\omega_c t][\cos(m\sin\omega_m t)] - [\sin\omega_c t][\sin(m\sin\omega_m t)]$$

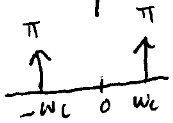
periodic w/ Fund period ω_m.

- Fourier Xforms of $\cos(m\sin\omega_m t)$ and $\sin(m\sin\omega_m t)$ are impulse trains with spacing ω_m, ~~we~~
  - impulse weights can be determined from Fourier Series, they are Bessel functions of the first kind in this case.
  - what we have is $\cos\omega_c t$ Amplitude modulated by $\cos(m\sin\omega_m t)$

    and $-\sin\omega_c t$ Amplitude modulated by $\sin(m\sin\omega_m t)$.

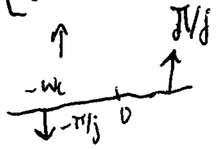
- So we get

$$Y(\omega) = \frac{1}{2\pi}\mathcal{F}[\cos\omega_c t] * \underbrace{\mathcal{F}[\cos(m\sin\omega_m t)]}_{\text{weighted impulse train}}$$



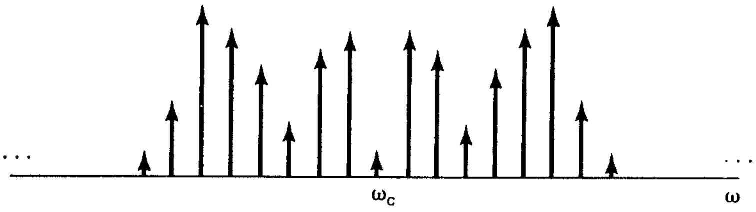
~~periodic~~

$$- \frac{1}{2\pi}\mathcal{F}[\sin\omega_c t] * \underbrace{\mathcal{F}[\sin(m\sin\omega_m t)]}_{\text{weighted impulse train}}$$



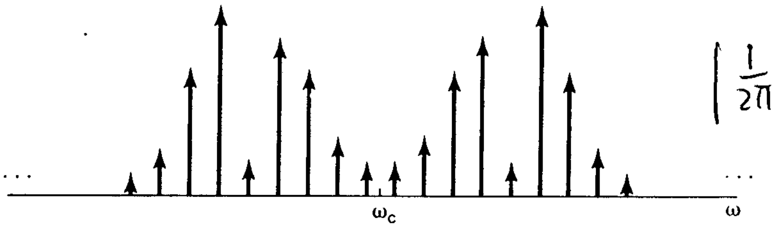
~~FM-7~~

This shows only one sideband



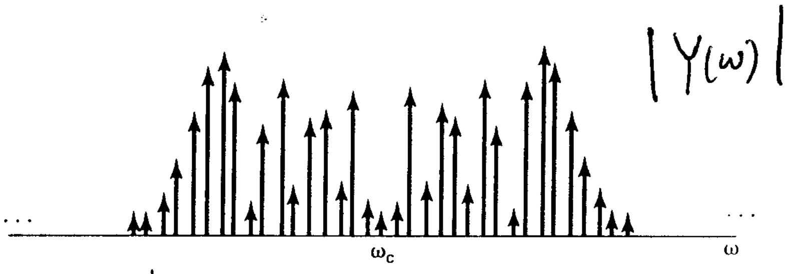
(a)

$$\left|\frac{1}{2\pi}\mathcal{F}[\cos\omega_c t] * \mathcal{F}[\cos(m\sin\omega_m t)]\right|$$



(b)

$$\left|\frac{1}{2\pi}\mathcal{F}[\sin\omega_c t] * \mathcal{F}[\sin(m\sin\omega_m t)]\right|$$



(c)

B

Figure 8.35 Magnitude of spectrum of wideband frequency modulation with $m = 12$: (a) magnitude of spectrum of $\cos \omega_c t \cos[m \sin \omega_m t]$; (b) magnitude of spectrum of $\sin \omega_c t \sin[m \sin \omega_m t]$; (c) combined spectral magnitude of $\cos[\omega_c t + m \sin \omega_m t]$.

NOTE: $y(t)$ is not bandlimited in general.

For $x(t) = a\cos\omega_m t$ in the wideband FM case (sinusoidal modulation), however, the Fourier series coefficients (Bessel functions) fall off relatively fast, so each sideband is approximately bandlimited to

$$B \approx 2k_f a = 2\underset{\uparrow}{m}\,\omega_m = 2\underset{\uparrow}{\Delta\omega}$$

$m = \frac{k_f a}{\omega_m}$ $\qquad$ $m\omega_m$, since $m = \frac{\Delta\omega}{\omega_m}$

Bandwidth of modulated signal is much larger than that of modulating signal when m is large.

8.FM.11

~~FM-8~~

EX: square-wave FM. $\omega_i(t)$ alternates between $\omega_c + \Delta\omega$ and $\omega_c - \Delta\omega$

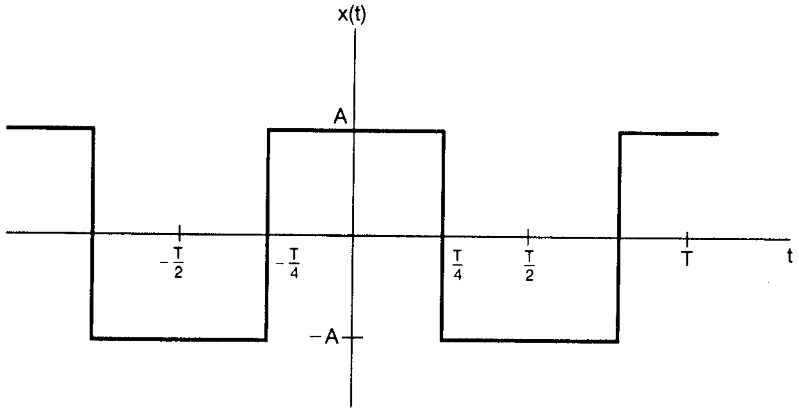


Figure 8.36 Symmetric periodic square wave.

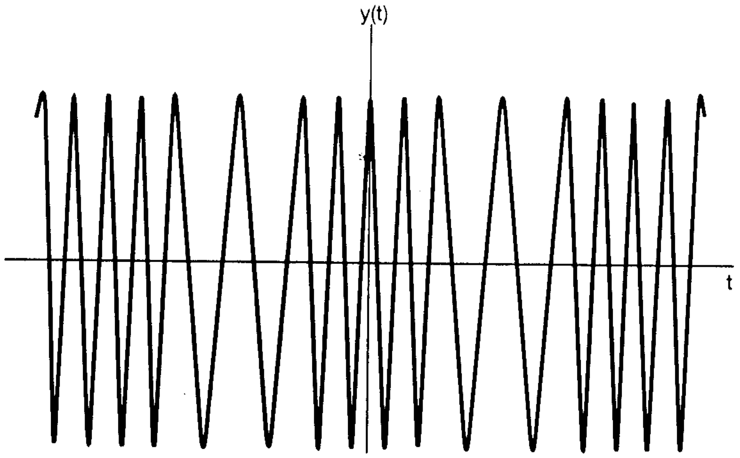


Figure 8.37 Frequency modulation with a periodic square-wave modulating signal.

Again, this shows only <u>one</u> sideband

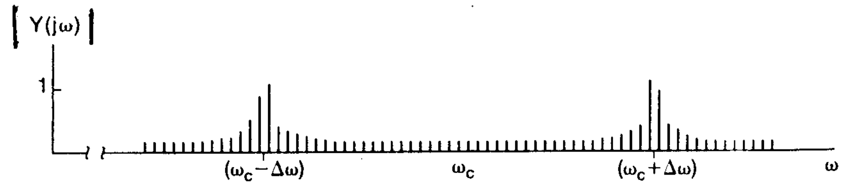


Figure 8.39 Magnitude of the spectrum for $\omega > 0$ corresponding to frequency modulation with a periodic square-wave modulating signal. Each of the vertical lines in the figure represents an impulse of area proportional to the height of the line.