Consider a complex signal \( y_c(t) = a(t)e^{j\theta(t)} \)

where \( a(t) \) and \( \theta(t) \) are real-valued functions.

And also its real part \( y(t) = a(t)\cos \theta(t) \).

For AM,
- We have \( \theta(t) \) linear: \( \theta(t) = \omega_c t + \theta_c \)
- And the information signal is embedded into \( a(t) \):
  - Suppressed Carrier: \( a(t) = X(t) \)
  - non-suppressed Carrier: \( a(t) = X(t) + A \)

For "Angle Modulation",
- We have \( a(t) \) constant
- The information signal is embedded into \( \theta(t) \).

- Phase Modulation or "PM":
  information signal embedded directly into \( \theta(t) \).

- Frequency Modulation or "FM":
  information signal embedded into \( \frac{d}{dt} \theta(t) \).

\( \Rightarrow \) often we use "FM" generically to refer to all forms of angle modulation.
- Instantaneous Frequency

- For angle modulation, we have signals $y_c(t)$ and/or $y(t)$ where $a(t)$ is a constant:
  
  $$a(t) = A$$
  $$y_c(t) = A e^{j \theta(t)}$$
  $$y(t) = A \cos \theta(t)$$

- For "FM" signals like these, the derivative of the phase signal $\theta(t)$ is called the **instantaneous frequency** $\omega_i(t)$:
  
  $$\omega_i(t) = \frac{d}{dt} \theta(t)$$

- When the phase $\theta(t)$ is **linear**, $y(t)$ is a pure sinusoid that could be, e.g., the carrier signal in an AM system. In this case, $\omega_i(t)$ is the "frequency" of the sinusoid (the "carrier frequency" if $y(t)$ is an AM carrier wave):
  
  $$\theta(t) = \omega_c t + \theta_c$$
  $$y(t) = A \cos [\omega_c t + \theta_c]$$
  $$\omega_i(t) = \frac{d}{dt} \theta(t) = \omega_c$$.

---

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When the phase $\Theta(t)$ is quadratic,

$$\Theta(t) = \alpha t^2 + \beta t + \gamma$$
$$y(t) = A \cos[\alpha t^2 + \beta t + \gamma]$$
$$\omega_i(t) = 2\alpha t + \beta$$

$\rightarrow$ instantaneous frequency is linear

$\rightarrow$ $y(t)$ is like a "pure sinusoid" with a frequency that goes up or down linearly with time

\[\text{graph}\]

- When you play a signal like this through a speaker, it makes a "chirp-like" sound.

- For this reason, FM signals with a quadratic phase are called "chirp signals" or just "chirps." Chirps are frequently used in modern radar systems.
- For AM, we had
\[ y(t) = a(t) \cos[\omega_c t + \Theta_c] \]

  Suppressed carrier: \( a(t) = X(t) \)
  Non-Suppressed carrier: \( a(t) = X(t) + A \)

  This type of \( y(t) \) is modulated by multiplication of signals, which is a nonlinear operation.

  But we could still use linear techniques to analyze AM signals, thanks to the frequency convolution property of the Fourier transform:
  \[ x_1(t) x_2(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\theta) X_2(\omega-\theta) d\theta \]

- For FM, we have
\[ y(t) = A \cos[\Theta(t)] \]

  In this case, the nonlinearity arises through the transcendental cosine function.

  This makes the analysis of FM signals much more difficult than AM signals.
"PM": phase modulation

- The carrier is a pure sinusoid.
- Modulation is done by altering the phase in proportion to the information signal $X(t)$.

$$y(t) = A \cos \Theta(t)$$

$$\Theta(t) = \omega_c t + \Theta_c(t)$$

$$\Theta_c(t) = \Theta_0 + k_p X(t)$$

$$\Rightarrow y(t) = A \cos \left[ \omega_c t + \Theta_0 + k_p X(t) \right].$$

"FM": frequency Modulation

- The information signal is coded directly into the instantaneous frequency... into the derivative of the phase.

$$y(t) = A \cos \Theta(t)$$

$$\Theta'(t) = \omega_c + k_f X(t)$$

$$\Rightarrow y(t)$$ is a "sinusoid" where the rate of oscillation (instantaneous frequency) speeds up and slows down in proportion to $X(t)$.

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So PM and FM are related by the derivative.

Doing PM with $X(t)$ is the same as doing FM with $X'(t)$.

\[ \text{PM} \quad \text{FM} \]

\[ (a) \quad (b) \quad (c) \]

Figure 8.32 Phase modulation, frequency modulation, and their relationship: (a) phase modulation with a ramp as the modulating signal; (b) frequency modulation with a ramp as the modulating signal; (c) frequency modulation with a step (the derivative of a ramp) as the modulating signal.
Narrowband FM

- Let the information signal be
  \[ X(t) = A \cos \omega_m t \]

- Suppose the carrier amplitude is \( A = 1 \).
  \[ y(t) = A \cos \theta(t) = \cos \theta(t) \]

- For FM, we have
  \[ \omega_i(t) = \theta'(t) = \omega_c + k_f A \cos \omega_m t \]

  \( \rightarrow \) The inst. freq. varies sinusoidally
  between \( \omega_c - k_f A \) and \( \omega_c + k_f A \).

  \( \rightarrow \) Let \( \Delta \omega = k_f A \)

  \( \rightarrow \) Total variation about the carrier
  frequency is \( 2 \Delta \omega \)

- The transmitted signal is \((t_0 = "a\,time")\)
  \[ y(t) = \cos \theta(t) = \cos \left[ \int_{t_0}^{t} \omega_i(\tau) \, d\tau \right] \]
  \[ = \cos \left[ \int_{t_0}^{t} \omega_c + \frac{k_f A \cos \omega_m \tau}{\Delta \omega} \, d\tau \right] \]
  \[ = \cos \left[ \omega_c t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t + \theta_0 \right] \]

  \( \rightarrow \theta_0 \) is an integration constant that depends on \( t_0 \)

  \( \rightarrow \) Assume \( \theta_0 = 0 \) for simplicity...
Then
\[ y(t) = \cos \left( w_c t + \frac{\Delta w}{\omega_m} \sin \omega_m t \right) \]

\[ m = \frac{\Delta w}{\omega_m} = \text{modulation index for sinusoidal FM}. \]

- When \( \left( \frac{\Delta w}{\omega_m} \right)^m \) is small \( (\ll \frac{\pi}{2}) \), this is called "Narrowband FM". \( \Rightarrow K_{\phi A} \), amplitude of \( x(t) \), is small compared to rate of oscillation of \( x(t), \omega_m \).

- Trig identities:

\[ y(t) = \cos \left( w_c t + m \sin \omega_m t \right) \]

\[ = \left[ \cos w_c t \right] \left[ \cos (m \sin \omega_m t) \right] - \left[ \sin w_c t \right] \left[ \sin (m \sin \omega_m t) \right] \]

\[ \Rightarrow \text{For small } \alpha, \quad \cos \alpha \approx 1 \]
\[ \quad \sin \alpha \approx \alpha \]

\[ \Rightarrow \text{when } m \text{ small, } m \sin \omega_m t \text{ is small,} \]
\[ \cos (m \sin \omega_m t) \approx 1 \]
\[ \sin (m \sin \omega_m t) \approx m \sin \omega_m t \]

\[ y(t) \approx \cos w_c t - m \left[ \sin \omega_m t \right] \left[ \sin w_c t \right] \]

\[ \Rightarrow \text{This is like AM double sideband with carrier, but the carrier is a sine times -1 for } 2^{nd} \text{ term.} \]

\[ (\Delta w = K_{\phi A}) \quad (x(t) = A \cos \omega_m t) \]

\[ (m = \frac{\Delta w}{\omega_m}) \]

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Figure 8.33 Approximate spectrum for narrowband FM.

\[ y(t) \approx \cos \omega t - m \left[ \sin \omega t \right] \left[ \sin \omega t \right] \]

Narrowband FM

Equivalent AM-DSB/SC

\[ y(t) = \left[ 1 + x(t) \right] \cos \omega t \]
\[ = \left[ 1 + m \cos \omega t \right] \cos \omega t \]
\[ = \cos \omega t + m \left[ \cos \omega t \right] \left[ \cos \omega t \right] \]

Figure 8.34 Comparison of narrowband FM and AM-DSB/SC:
(a) narrowband FM; (b) AM-DSB/SC.
Wideband FM

- If $m = \frac{\Delta \omega}{\omega_m}$ is large, $k_{cA}$ is large amplitude of $x(t)$ compared to $\omega_m$, the rate of oscillation of $X(t)$.

  This is called wideband FM.

- Small angle approximations no longer apply.

- We're back to

  $$y(t) = \left[ \cos(\omega t) \right] \left[ \cos(m \sin(\omega_m t)) \right] - \left[ \sin(\omega t) \right] \left[ \sin(m \sin(\omega_m t)) \right]$$

- Fourier Xforms of $\cos(m \sin(\omega_m t))$ and $\sin(m \sin(\omega_m t))$ are impulse trains with spacing $\omega_m$.

- Impulse weights can be determined from Fourier Series, they are Bessel functions of the first kind in this case.

- What we have is $\cos(\omega t)$ Amplitude modulated by $\cos(m \sin(\omega_m t))$

  and $-\sin(\omega t)$ Amplitude modulated by $\sin(m \sin(\omega_m t))$.

- So we get $Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \cos(\omega t) \right] \ast \left[ \cos(m \sin(\omega_m t)) \right]$ weighted impulse train

  $- \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \sin(\omega t) \right] \ast \left[ \sin(m \sin(\omega_m t)) \right]$ weighted impulse train

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This shows only one sideband

\[ \frac{1}{2\pi} \int [\cos \omega t] \cos (m \sin \omega_m t) \, dt \]

\[ \frac{1}{2\pi} \int [\sin \omega t] \sin (m \sin \omega_m t) \, dt \]

Figure 8.35 Magnitude of spectrum of wideband frequency modulation with \( m = 12 \):
(a) magnitude of spectrum of \( \cos \omega t \cos (m \sin \omega_m t) \);
(b) magnitude of spectrum of \( \sin \omega t \sin (m \sin \omega_m t) \);
(c) combined spectral magnitude of \( \cos(\omega t + m \sin \omega_m t) \).

**Note:** \( y(t) \) is not bandlimited in general.

For \( x(t) = A \cos \omega_m t \) in the wideband FM case
(sinusoidal modulation), however, the Fourier series coefficients (Bessel functions) fall off relatively fast,
so each sideband is approximately bandlimited to

\[ B \approx 2K A = \frac{2m \omega_m}{\Delta \omega} \]

\[ \text{Bandwidth of modulated signal is much lower than that of modulating signal when } m \text{ is large.} \]
EX: square-wave FM. $\omega_i(t)$ alternates between $\omega_c + \Delta \omega$ and $\omega_c - \Delta \omega$

![Figure 8.36 Symmetric periodic square wave.]

Again, this shows only one sideband $Y$

![Figure 8.37 Frequency modulation with a periodic square-wave modulating signal.]

| $|Y(\omega)|$ |
| --- |
| ![Magnitude of the spectrum for $\omega > 0$ corresponding to frequency modulation with a periodic square-wave modulating signal. Each of the vertical lines in the figure represents an impulse of area proportional to the height of the line.](image) |

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