S-Domain Characterization of Continuous-Time LSI Systems

- We have seen that the Fourier transform is useful for analyzing and designing LSI systems.

  → with the basis $\{e^{j\omega t}\}_{\omega \in \mathbb{R}}$, the system output is the product of the input and the system frequency response:

$$Y(\omega) = X(\omega) H(\omega)$$

  → This happens because the signal $e^{j\omega t}$, for any real ω, is an eigenfunction of any LSI system.

- If $x(t)$ is square integrable

$$\left[\int_{-\infty}^{\infty} |x(t)|^2 dt \right]^{1/2} < \infty$$

or absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

Then $x(t)$ has a Fourier transform.

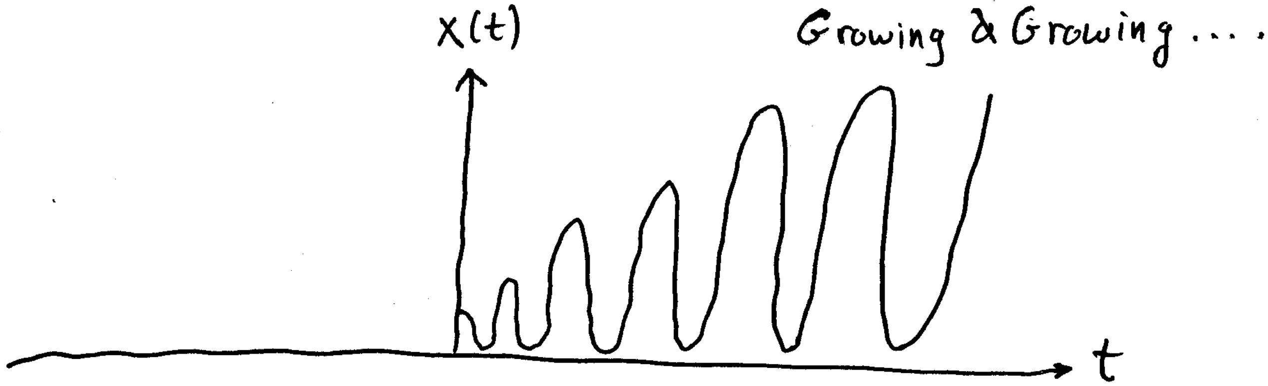
- As we have seen, some signals that are <u>not</u> square integrable and are <u>not</u> absolutely integrable also have Fourier transforms.

EX: $x(t) = u(t)$

$x(t) = \cos\omega_0 t$

- But what about "bad" signals that do <u>not</u> have a Fourier transform?

Ex: $x(t) = [1 + \cos\omega_0 t]e^{2t}u(t)$



- This signal does not have a Fourier transform.

- Can we find a way to do frequency domain analysis on signals like this?

- The answer is yes, but we have to "fix up" the signal first.
- This "fixing up" will give us the Laplace transform.

Fourier, (Jean Baptiste) Joseph, Baron [fooryay] (1768–1830) French mathematician: discovered Fourier series and the Fourier Integral Theorem.

Fourier, the son of a tailor, was orphaned at age 8. He had a mixed education, at military school, an abbey and later (after a narrow escape from the guillotine during the French Revolution in 1794) at the École Normale. He joined the staff of the École Normale, newly-formed to train senior teachers, and the École Polytechnique in Paris. When Napoleon invaded Egypt in 1798 Fourier accompanied him, but it is unlikely that he became governor of Lower Egypt as is often stated. He became the prefect of the *département* of Grenoble for 14 years but resigned during Napoleon's Hundred Days campaign. Fourier died in 1830 of a disease contracted while in Egypt.

Fourier established linear partial differential equations as a powerful tool in mathematical physics, particularly in boundary value problems. For example, to find the conduction of heat through a body of a given shape when its boundaries are at particular temperatures, the heat diffusion equation can be solved as a sum of simpler trigonometric components (Fourier series). This way of solving the linear differential equations that often occur in physics has led to much use of the method on many new problems to the present day. Importantly, any arbitrary repeating function may be represented by a Fourier series; for instance, a complex musical waveform can always be represented as the sum of many individual frequencies. At a different level, the understanding of Fourier series and integrals has contributed greatly to the development of pure analysis, particularly of functional analysis. On the problem of heat conduction through a uniform solid, Fourier's Law states that the heat flux is given by the product of the thermal conductivity and the temperature gradient.

Laplace, Pierre Simon, marquis de [lahplas] (1749–1827) French mathematician, astronomer and mathematical physicist: developed celestial mechanics; suggested hypothesis for origin of the solar system.

Although from a poor family, Laplace's talent led him to become an assistant to LAVOISIER in thermochemistry. Later he moved to astronomy, and became a minister and senator, skilfully contriving to hold a state office despite violent political changes. Laplace's most important work was on celestial mechanics. In 1773 he showed that gravitational perturbations of one planet by another would not lead to instabilities in their orbits (NEWTON had believed that such small irregularities would, without divine intervention, eventually lead to the end of the world). He later proved two theorems involving the mean distances and eccentricities of the planetary orbits and showed that the solar system has long-term stability. In 1796 Laplace proposed in a note that the Sun and planets were formed from a rotating disk of gas; he did not know that KANT had made a similar suggestion; modified forms of this nebular hypothesis are still accepted. Between 1799 and 1825 he published his five-volume opus *Mécanique céleste* (Celestial Mechanics), which incorporated developments in celestial mechanics since Newton as well as his own important contributions. (The book has its oddity: frequently the phrase 'it is obvious that' occurs, in mathematical equations, when it is far from obvious. And Napoleon is said to have remarked, critically, that it made no mention of God.) Laplace is also remembered for putting probability theory on a firm foundation, and for developing the concept of a 'potential' and its description by the Laplace equation. In the fields in which Laplace worked and where Newton had worked previously, he is seen as second only to Newton in his talent.

- For the "bad" signal $x(t) = [1+\cos\omega_0 t]e^{2t}u(t)$, we can "fix things up" by multiplying $x(t)$ by the "fixer upper" signal e^{-2t}.

- The "fixed up" signal is then

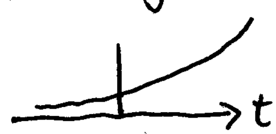
$$x(t)e^{-2t} = [1+\cos\omega_0 t]u(t),$$

$$\frac{1}{j\omega} + \pi\delta(\omega) + [\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]] \otimes [\tfrac{1}{j\omega} + \pi\delta(\omega)]$$

which has a Fourier transform (what is it?).

<u>NOTE:</u> The "fixer upper" signal e^{-3t} will also work in this case.

<u>NOTE:</u> The "fixer upper" signal e^{3t} will <u>not</u> work in this case: it just makes things <u>worse</u>.



<u>NOTE:</u> The "fixer upper" signal e^{-3t} works for $x(t)$ above. But, for the signal

$$x(t) = [1+\cos\omega_0 t]e^{-2t}u(-t),$$

this "fixer upper" only makes things <u>worse</u>.

- In general, we will "fix up" bad signals by multiplying them times $e^{-\sigma t}$, $\sigma \in \mathbb{R}$.

- The fixed up signal $x(t)e^{-\sigma t}$ will then have a Fourier transform for <u>certain choices</u> of σ, but <u>not for others</u>.

- The Fourier transform of the "fixed up" signal is

$$\mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

→ Think of this transform as having <u>two</u> independent variables:

→ σ is the "fixer upper" variable. The Fourier transform exists for some σ and not for others.

→ For each σ, ω is the usual frequency variable.

- It is convenient to write the two variables σ and ω together as a complex variable s:

$$s = \sigma + j\omega.$$

- Then, the Fourier transform of the "fixed up" signal becomes:

$$\mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \qquad (*)$$

- This is called the <u>BILATERAL LAPLACE TRANSFORM</u> of $x(t)$:

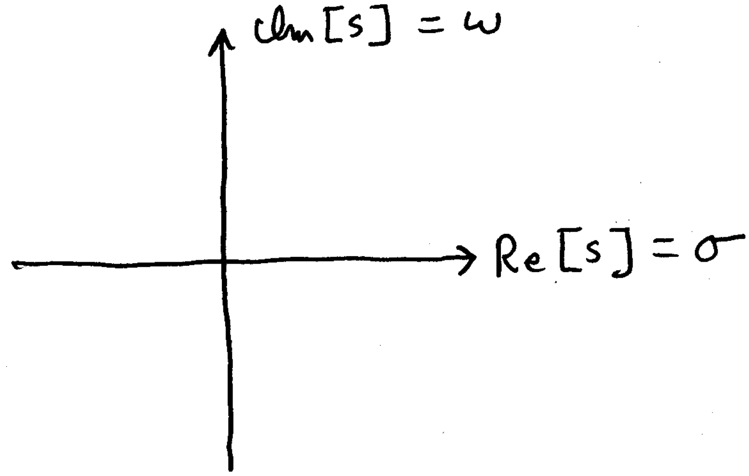
$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

- we write

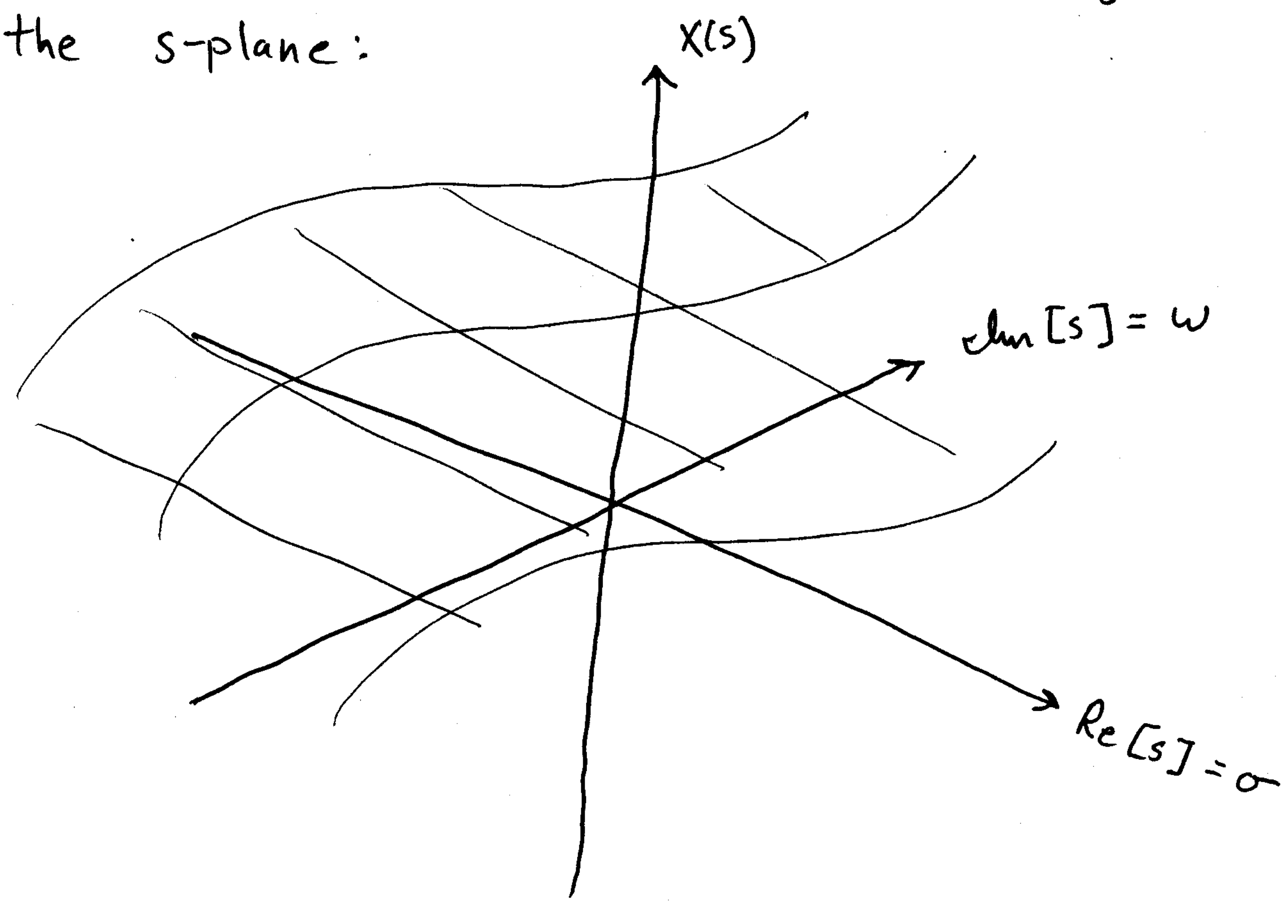
$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$$

to indicate that $x(t)$ and $X(s)$ are a Laplace transform pair.

- $X(s)$ is a function of the complex variable s, which takes values in the complex "s-plane":



- The graph of $X(s)$ is a "sheet" lying above the s-plane:



- For certain values of s, this sheet will generally be equal to ∞.

Ex: A "fixer upper" signal $e^{-\sigma t}$ that makes things <u>worse</u> will cause $X(s)$ to diverge to ∞ for all s with $Re[s]=\sigma$.

<u>NOTE</u>: The set of s with $Re[s]=\sigma=0$ is a <u>line</u> in the s-plane. It is, in fact, the imaginary or "$j\omega$" axis of the s-plane.

→ If we evaluate $X(s)$ along this line, we get

$$\mathcal{L}[x(t)]\Big|_{\sigma=0} = X(s)\Big|_{s=j\omega}$$

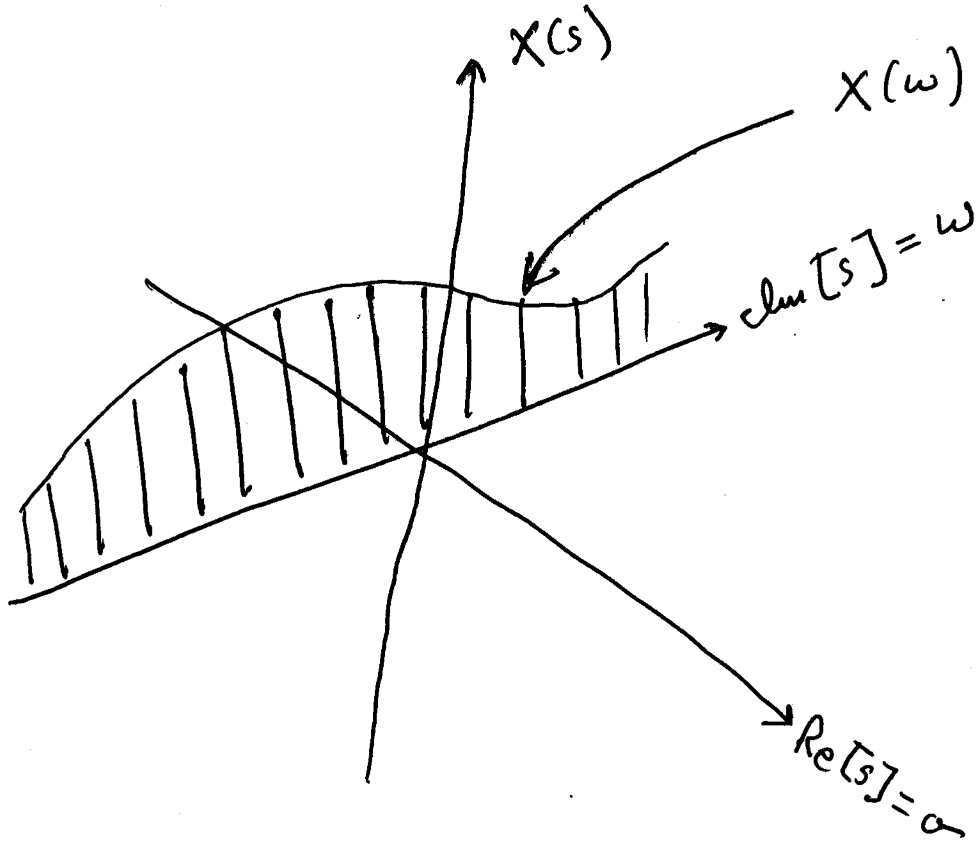
$$= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt\Big|_{\sigma=0}$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= X(\omega) = \mathcal{F}[x(t)].$$

- Thus, $X(s)\Big|_{s=j\omega}$ <u>is</u> the Fourier transform of $x(t)$, which might or might not exist in general.

→ $X(\omega)$ is like a <u>slice</u> through the sheet $X(s)$, with $\sigma=0$:



NOTE: Read Section 3.2 of the book.

- Suppose that H is a continuous-time LSI system with impulse response $h(t)$.

- For some arbitrary fixed value of the complex variable s, let the system input be

$$x(t) = e^{st}$$

- Then the output is

$$y(t) = e^{st} * h(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= \underset{\text{the input}}{e^{st}} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}_{\text{A complex number for any fixed } s.}$$

- Thus e^{st}, for any fixed complex value of s, is an eigenfunction of any LSI system.

- The eigenvalue is $\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$.

- We will write $H(s)$ for the collection of eigenvalues for all the different complex values of s.

- $H(s)$ is called the "transfer function" of the system H.

- NOTE: $H(s)$ is the Laplace transform of $h(t)$:

$$H(s) = \mathcal{L}[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

- Recall that $s = \sigma + j\omega$,

and $X(s) = \mathcal{L}[x(t)]$ is the Fourier transform of $x(t)$ times the "fixer upper" signal $e^{-\sigma t}$

→ In general, the Laplace transform integral

$$X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

will converge for some values of s and diverge for others.

⇒ The set of s for which $X(s)$ converges is called the "Region of Convergence" of $X(s)$, or the "ROC".

EX: $x(t) = e^{-at}u(t)$, $a \in \mathbb{R}$.

Find $X(s)$.

- Recall the Fourier transform pair

$$e^{-\alpha t}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{\alpha + j\omega}, \quad Re[\alpha] > 0.$$

- we have

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-(\sigma+j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(a+\sigma)t}\, u(t)\, e^{-j\omega t}\, dt$$

$$= \mathcal{F}\left[e^{-(a+\sigma)t}u(t)\right]$$

$$= \frac{1}{(\sigma+a)+j\omega}, \quad \sigma + a > 0$$

$$= \frac{1}{a+(\sigma+j\omega)}, \quad \sigma > -a$$

$$= \frac{1}{s+a}, \quad Re[s] > -a.$$

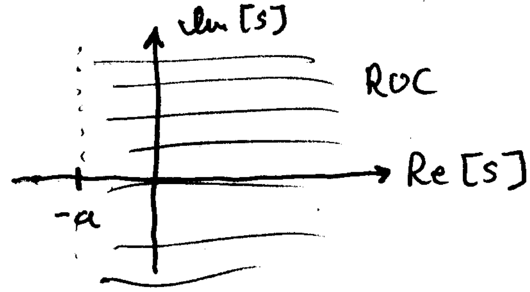
So,

$$x(t) = e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad Re[s] > -a.$$

$\Rightarrow$ The ROC of $X(s)$ is the half-plane $Re[s] > -a$.

NOTE: The Laplace transform in the preceeding example converges if $Re[s] > -a$.

→ If $a > 0$, then the ROC includes the $j\omega$-axis of the s-plane:



→ So we can evaluate $X(s)$ when $\sigma = 0$ to get the Fourier transform of $x(t)$:

$$X(\omega) = \mathcal{F}[x(t)] = X(s)\Big|_{Re[s]=0} = \frac{1}{s+a}\Big|_{Re[s]=0}$$

$$= \frac{1}{(\sigma + j\omega) + a}\Big|_{\sigma=0} = \frac{1}{a+j\omega} \checkmark$$

⇒ But, if $a < 0$, the ROC does not include the $j\omega$ axis.

→ In this case, $X(\omega)$ diverges. However, $X(s)$ converges for all s satisfying $Re[s] > -a$.

EX: $x(t) = -e^{-at}u(-t)$. Find $X(s)$.

→ This time we will work it out directly without using the Fourier transform.

$$X(s) = -\int_{-\infty}^{\infty} e^{-at} u(-t) e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$= \frac{1}{s+a}\left[e^{-(s+a)t}\right]_{-\infty}^{0}$$

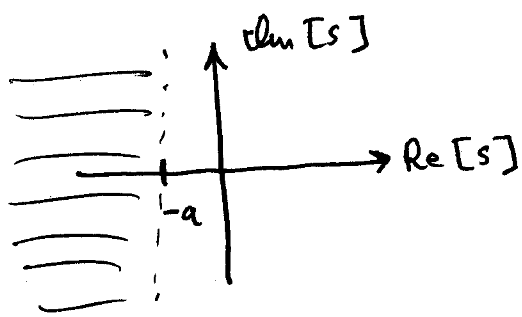
$$= \frac{1}{s+a}\left[1 - \lim_{A\to\infty} e^{(s+a)A}\right]$$

→ if $Re[s+a] \geq 0$, or $Re[s] \geq -a$, then the integral diverges.

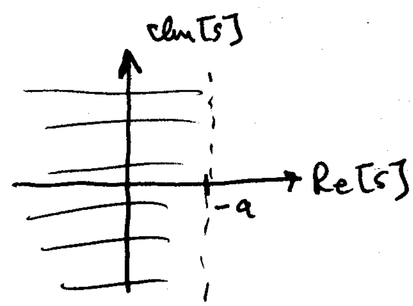
→ if $Re[s+a] < 0$, or $Re[s] < -a$, then we have

$$X(s) = \frac{1}{s+a}[1-0] = \frac{1}{s+a}, \quad Re[s] < -a.$$

⇒ The ROC is the left half plane $Re[s] < -a$:



$a>0$



$a<0$

- In the last two examples, we showed that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad Re[s] > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad Re[s] < -a.$$

⇒ These two different signals have the <u>same</u> functional form for their Laplace transforms.

⇒ Only the ROC's are different.

⇒ Thus, we see that the ROC is very important. Given $X(s) = \frac{1}{s+a}$, we can't tell which signal $x(t)$ is unless the ROC is specified.

<u>EX</u>: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} \left[3e^{-2t} - 2e^{-t}\right] u(t) e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} 3e^{-2t}u(t)e^{-st}\, dt - \int_{-\infty}^{\infty} 2e^{-t}u(t)e^{-st}\, dt$$

$$= 3\mathcal{L}\left[e^{-2t}u(t)\right] - 2\mathcal{L}\left[e^{-t}u(t)\right]$$

$$= \underbrace{\frac{3}{s+2}}_{\text{ROC: } Re[s] > -2} - \underbrace{\frac{2}{s+1}}_{\text{ROC: } Re[s] > -1} \longrightarrow$$

→ For $X(s)$ to converge, both $\mathcal{L}[e^{-2t}u(t)]$ and $\mathcal{L}[e^{-t}u(t)]$ must converge.

→ So, the ROC is the intersection of the two individual ROC's :

$$ROC = \{Re[s] > -2\} \cap \{Re[s] > -1\}$$
$$= Re[s] > -1.$$

⇒ Look at example 9.4 on page 659 of the book. ☆☆

Poles and Zeros

- For the examples we have seen so far, the Laplace transforms have been rational functions of s.

  ⇒ A "rational" function is a ratio of two polynomials.

- If $X(s)$ is a rational function of s, then

$$X(s) = \frac{N(s)}{D(s)},$$

where $N(s)$ and $D(s)$ are polynomials.

→ The roots of $D(s)$ are poles of $X(s)$.

→ The roots of $N(s)$ are zeros of $X(s)$.

- Suppose $N(s)$ is a polynomial of order N, and $D(s)$ is a polynomial of order M.

- If $M=N$, then the roots of $N(s)$ and $D(s)$ are <u>all</u> of the poles and zeros of $X(s)$.

- If $M<N$, then, as $s \to \infty$, the numerator of $X(s)$ grows faster than the denominator.

  $\Rightarrow$ So $X(s)$ has an additional pole at $s=\infty$ of order $N-M$.

- If $M>N$, then the denominator of $X(s)$ grows faster than the numerator as $s \to \infty$.

  $\Rightarrow$ So $X(s)$ has an additional zero at $s=\infty$ of order $M-N$.

<u>EX</u>: in the last example, we had $X(s) = \frac{3}{s+2} - \frac{2}{s+1}$

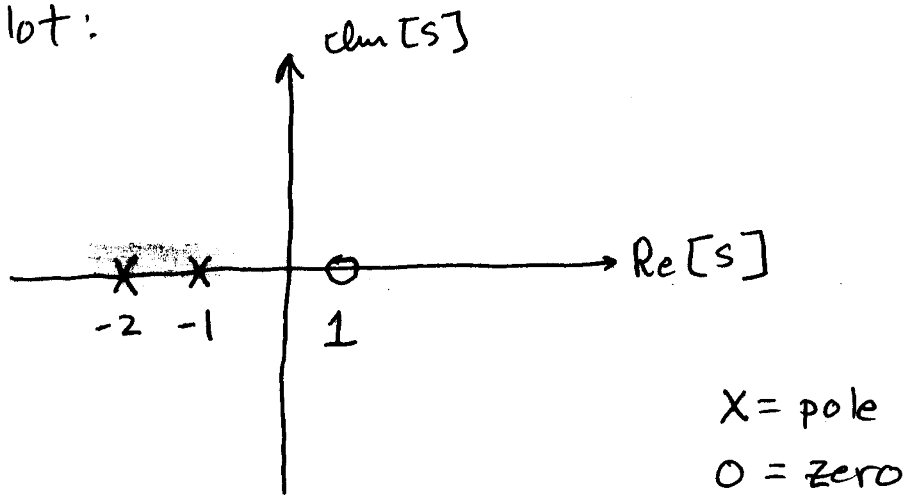
$$\text{So } X(s) = \frac{3(s+1) - 2(s+2)}{(s+2)(s+1)} = \frac{s-1}{s^2+3s+2}$$

$N(s) = s-1$: One finite zero at $s=1$.

$D(s) = s^2+3s+2 = (s+2)(s+1)$: Two finite poles at $s=-2$ and $s=-1$.

- Since $D(s)$ has order 2 and $N(s)$ has order 1, there is an additional zero at $s=\infty$.

Pole-zero plot:



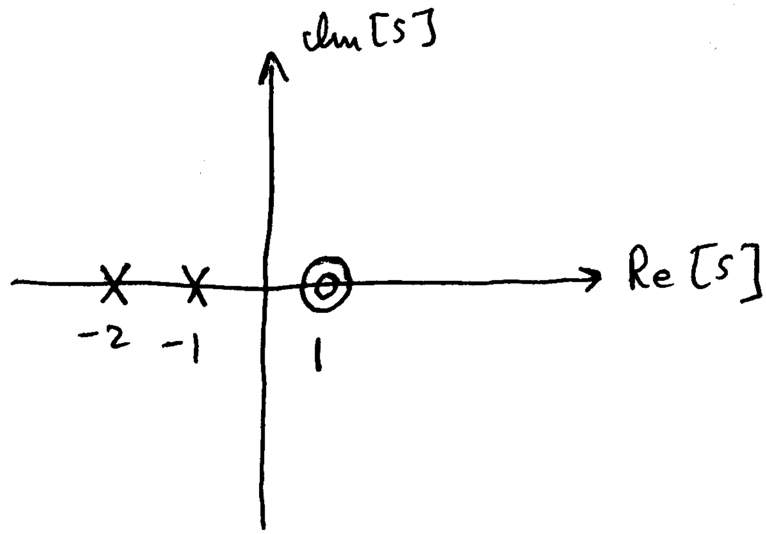
→ Only the finite poles and zeros are shown in the pole-zero plot.

EX: $X(s) = \dfrac{(s-1)^2}{(s+2)(s+1)} = \dfrac{(s-1)^2}{s^2+3s+2} = \dfrac{s^2-2s+1}{s^2+3s+2}$

→ Since numerator and denominator have the same order, all of the poles and zeros are finite.

$N(s) = (s-1)(s-1)$: 2^{nd} order zero at $s=1$

$D(s) = (s+2)(s+1)$: pole at $s=-2$ and at $s=-1$



Note: For an LSI system with input and output related by a constant-coefficients differential equation, the transfer function $H(s)$ will always be rational.

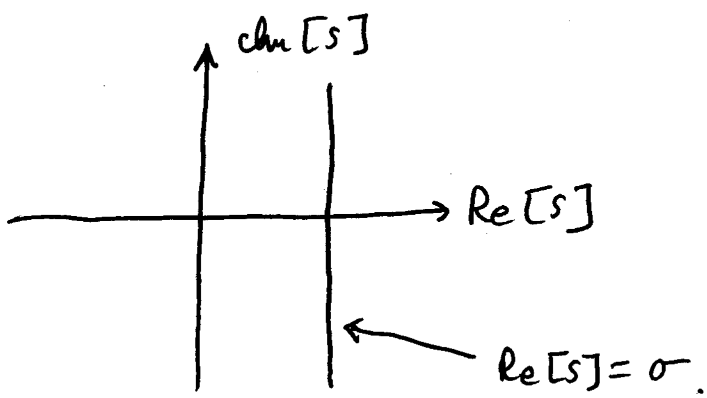
More on ROC

- The ROC of $X(s)$ is made up of strips in the s-plane running parallel to the $j\omega$ axis.

→ why? $\sigma = Re[s]$. For each σ,

$$X(s) = \mathcal{F}\left[x(t)e^{-\sigma t}\right].$$

- If this Fourier transform exists, then $X(s)$ converges for all s with $Re[s] = \sigma$.
- So, the vertical line $Re[s] = \sigma$ is in the ROC of $X(s)$:



- Since a pole of $X(s)$ is a point where $X(s)$ diverges, the ROC of $X(s)$ <u>cannot</u> contain any poles. (provided that $X(s)$ is rational).

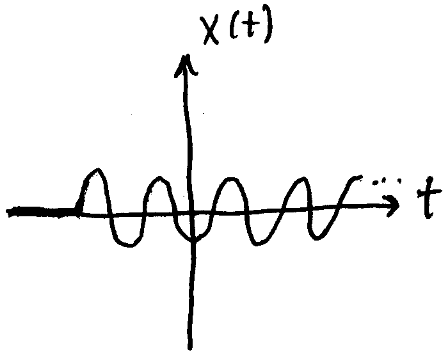
<u>FACT</u>: if $x(t) = 0$ outside a finite interval and $x(t)$ is absolutely integrable, then the ROC of $X(s)$ is the entire s-plane.

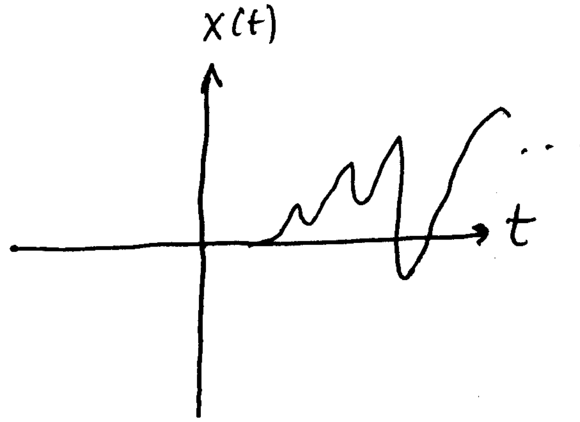
- why?
  - → $x(t)$ is absolutely integrable, so $\mathcal{F}[x(t)]$ exists.
  - → So $X(s)$ has a ROC that includes $Re[s] = 0$.
  - → Since $x(t)$ is zero outside a finite interval, $x(t)e^{-\sigma t}$ is absolutely integrable for any $\sigma \in \mathbb{R}$.
  - → So $x(t)e^{-\sigma t}$ has a Fourier transform that converges for any $\sigma \in \mathbb{R}$.
  - → So $X(s)$ converges for all $s = \sigma + j\omega$.

<u>DEF</u>: if there is a time t_0 such that $x(t)$ is zero $\forall\ t < t_0$, then $x(t)$ is called a "right sided signal."

<u>Note</u>: t_0 could be positive or negative.

- Right sided signals:





$\Rightarrow$ If $x(t)$ is right sided and the line $Re[s] = \sigma_0$ is in the ROC of $X(s)$, then all s with $Re[s] > \sigma_0$ are also in the ROC.

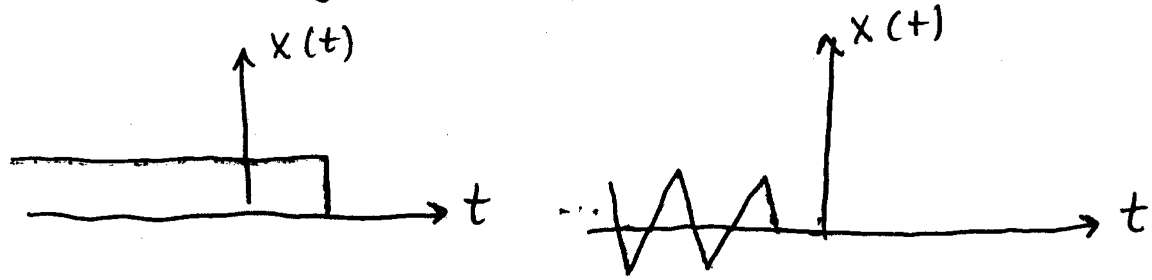
$\Rightarrow$ This implies that the ROC of $X(s)$ for a right sided signal is a right half plane.

$\rightarrow$ why? $Re[s] = \sigma_0$ is in the ROC by hypothesis.

- So $x(t)e^{-\sigma_0 t}$ has a Fourier transform.
- if $\sigma > \sigma_0$, then the "fixer upper" signal $e^{-\sigma t}$ fixes things up even more than $e^{-\sigma_0 t}$.
- So $x(t)e^{-\sigma t}$ has a Fourier transform
- So the line $Re[s] = \sigma$ is in the ROC of $X(s)$.

DEF: if $\exists\, t_0 \in \mathbb{R}$ such that $x(t) = 0 \;\; \forall\, t > t_0$, then $x(t)$ is called a "left sided signal".

- Left sided signals:



$\Rightarrow$ If $x(t)$ is left sided and the line $Re[s] = \sigma_0$ is in the ROC of $X(s)$, then all s with $Re[s] < \sigma_0$ are also in the ROC.

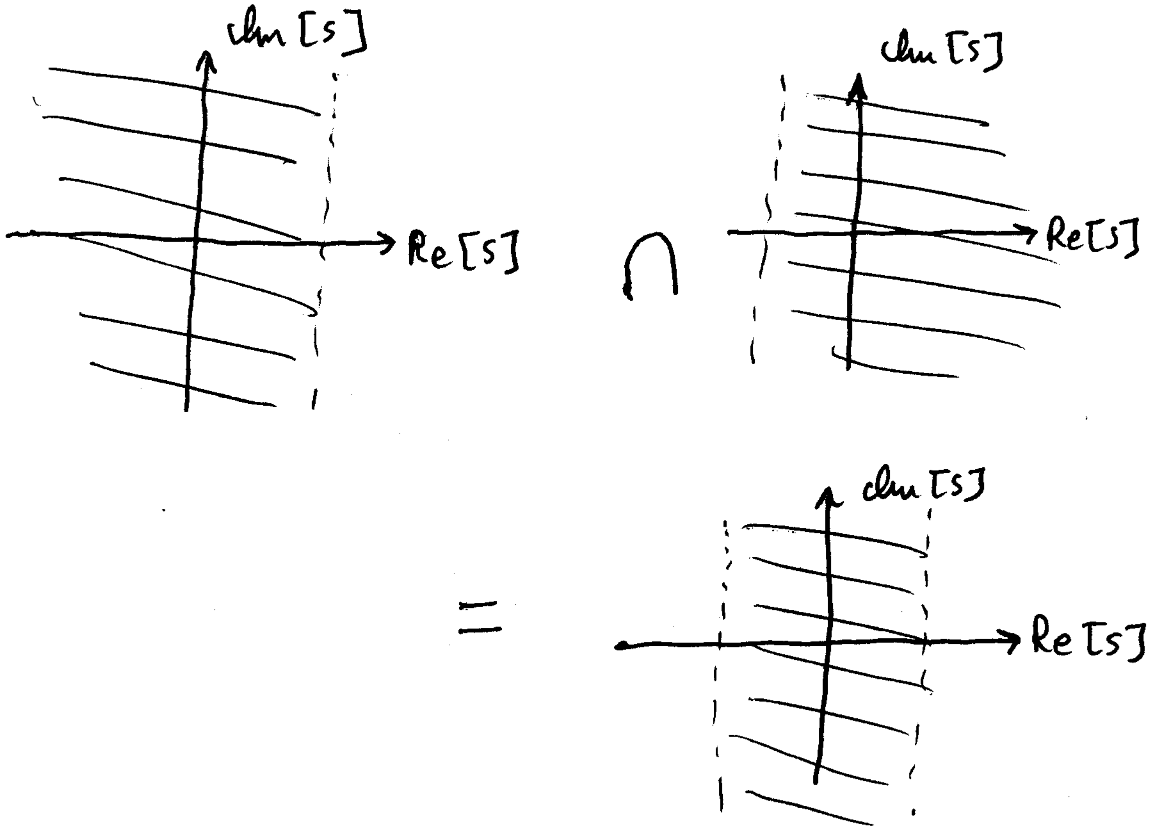
$\Rightarrow$ This implies that, for a left sided signal, the ROC of $X(s)$ is a left half plane.

- If $x(t)$ is two sided, then it can be broken into the sum of a left sided signal and a right sided signal.
  - $\rightarrow$ The ROC of the Laplace transform of the right sided part of $x(t)$ will be a right half plane.
  - $\rightarrow$ The ROC of the Laplace transform of the left sided part of $x(t)$ will be a left half plane

$\longrightarrow$

→ The ROC of X(s) will be the intersection of these two half planes... since both parts have to converge.

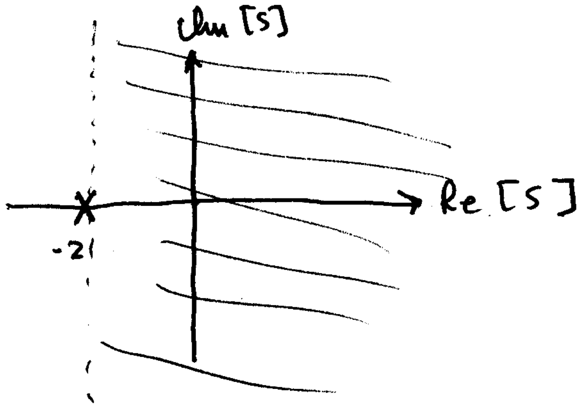
→ The intersection of two half planes, one right sided and one left sided, is either a <u>strip</u> or the empty set.



– So, for a two sided signal, the ROC of X(s) is a strip.

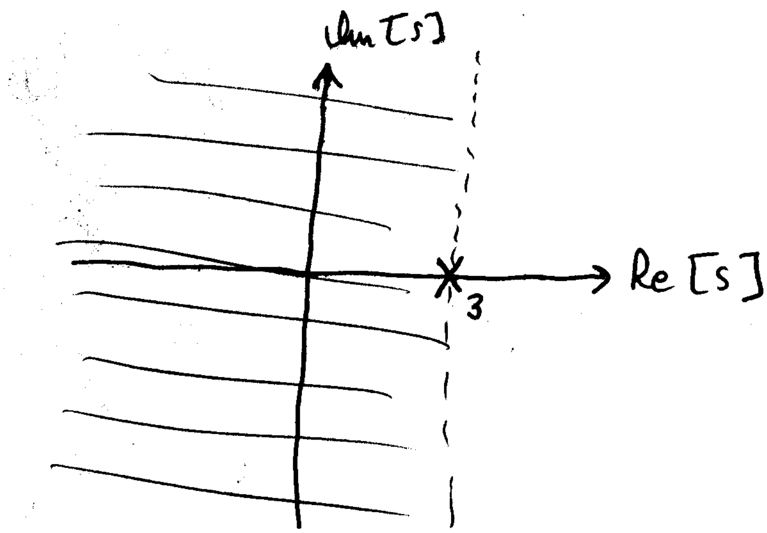
- If $X(s)$ is rational and $x(t)$ is right sided, then the ROC is the right half plane to the right of the rightmost pole.

EX: $e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}$, $Re[s] > -2$

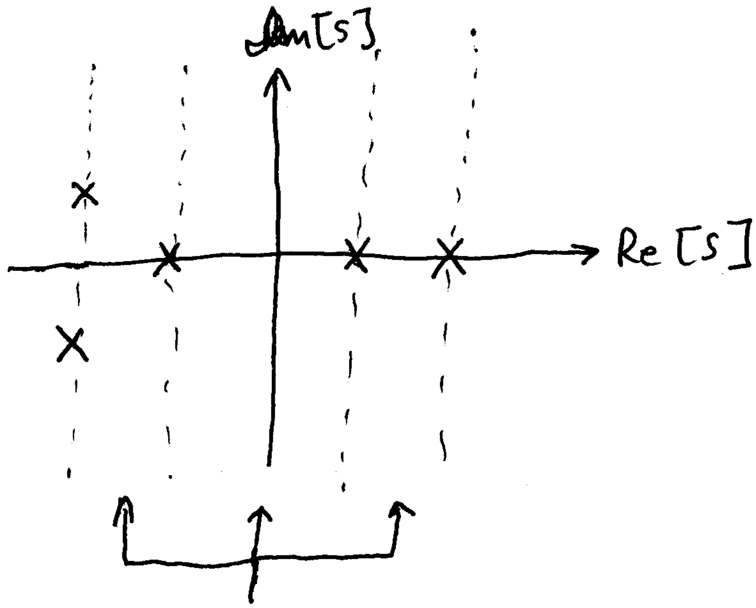


- If $X(s)$ is rational and $x(t)$ is left sided, then the ROC is the left half plane to the left of the leftmost pole.

EX: $-e^{+3t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-3}$, $Re[s] < 3$



- if $X(s)$ is rational and $x(t)$ is two sided, the ROC is a strip between two of the poles.



could be any one of these strips.

- if $X(s)$ is rational and $x(t)$ is zero outside a finite interval, then the ROC is the entire s-plane.

- if $x(t)$ is zero outside a finite interval and $x(t)$ is absolutely integrable, then the ROC of $X(s)$ is the entire s-plane (as we saw on page 9.20).

Inversion

- Suppose $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ and that the line $Re[s] = \sigma_0$ is in the ROC of $X(s)$.

- Then $X(s)\Big|_{Re[s]=\sigma_0} = \mathcal{F}\left[x(t)e^{-\sigma_0 t}\right]$.

- So
$$e^{-\sigma_0 t}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)\Big|_{Re[s]=\sigma_0} e^{j\omega t}\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_0 + j\omega)\, e^{j\omega t}\,d\omega$$

- Multiply both sides by $e^{\sigma_0 t}$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma_0 + j\omega)\, e^{(\sigma_0 + j\omega)t}\,d\omega$$

Let $s = \sigma_0 + j\omega$ as $\omega \to -\infty$, $s \to \sigma_0 - j\infty$

$ds = j\,d\omega$ as $\omega \to \infty$, $s \to \sigma_0 + j\infty$

$d\omega = \frac{ds}{j}$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} X(s)\, e^{st}\,ds$$

- The only thing we have assumed about σ_0 is that the line $\text{Re}[s] = \sigma_0$ is in the ROC of $X(s)$.

- Thus, for <u>any</u> σ such that the line $\text{Re}[s] = \sigma$ is in the ROC of $X(s)$,

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \, ds$$

→ This is the inverse Laplace transform.

- In practice, we rarely evaluate inverse Laplace transforms using this definition.
- Instead, we use tables, properties, and partial fractions.

Laplace Transform Properties

Linearity:

if $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, and a_1 and a_2 are constants, then

$$a_1 x_1(t) + a_2 x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a_1 X_1(s) + a_2 X_2(s)$$

with region of convergence the ROC of $X_1(s)$ intersected with the ROC of $X_2(s)$.

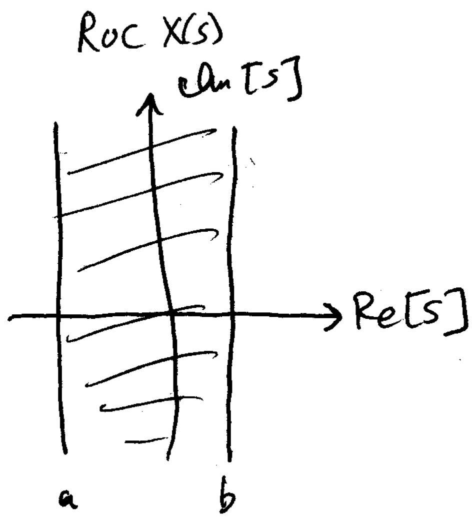
<u>Time Shifting</u>: if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R,

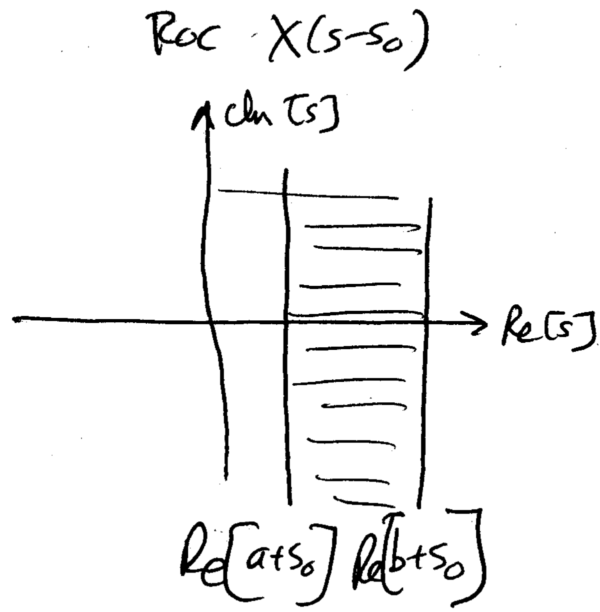
then $x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s)$ with ROC R.

<u>s-Domain Shifting</u>: if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R,

then $e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)$, with ROC $R + Re\{s_0\}$.

EX:





- In other words, if $s \in$ ROC $X(s)$, then $s + Re[s_0] \in$ ROC $X(s-s_0)$.

Time Scaling:

- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R,

then $x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$

if $s \in R$, the ROC of $X(s)$,

then $\overset{as}{\cancel{\frac{s}{a}}}$ is in the ROC of $\frac{1}{|a|} X\left(\frac{s}{a}\right)$.

this error is in some editions of the book.

Conjugation:

- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R,

then $x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$ with ROC R.

Convolution:

if $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$ with ROC R_1

and $x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$ with ROC R_2,

then $x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s) X_2(s)$

with ROC at least $R_1 \cap R_2$.

Time Differentiation:

– if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R

then $\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s)$ with ROC at least R.

s-Domain Differentiation:

– if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R,

then $-t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s)$

with ROC R.

Time domain Integration:

– if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with ROC R,

then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)$ with ROC

containing $R \cap \{s : \sigma > 0\}$

- Initial and Final Value Theorems:

- if $x(t)$ is causal... $x(t) = 0 \;\; \forall \; t < 0$, and $x(t)$ does not contain a singularity at $t = 0$, then

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} s X(s)$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s).$$

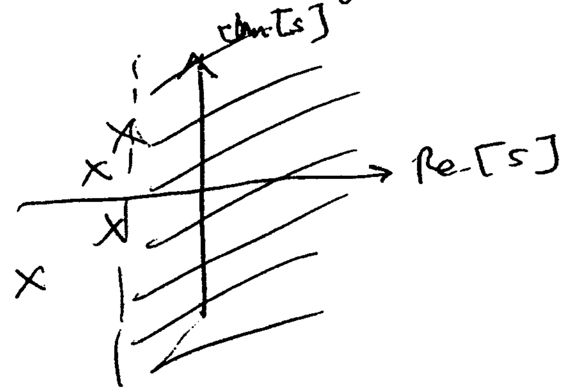
☆ READ O.W.N. Sections 9.5, 9.6

S-Domain LSI System Analysis

- READ O.W.N. SECTIONS 9.7 AND 9.8.
  SKIP SECTION 9.7.5.

- If H is an LSI system with impulse response $h(t)$ that is causal, then $h(t) = 0 \ \forall \ t < 0$.
  - $\rightarrow$ Then $h(t)$ is a right sided signal
  - $\rightarrow$ Then the ROC of $H(s) = \mathcal{L}[h(t)]$ is a right-handed half-plane. (*)

- $\rightarrow$ IF $H(s)$ is <u>RATIONAL</u>, then then the system is causal <u>if</u> <u>and</u> <u>only</u> <u>if</u> the ROC of $H(s)$ is the right half-plane to the right of the rightmost pole of $H(s)$.



- $\rightarrow$ IF $H(s)$ is <u>not</u> rational, then the <u>if</u> <u>and</u> <u>only</u> <u>if</u> might not hold. (*) above is still true, but the converse of (*) might be false.

⇒ STUDY EXAMPLES 9.17, 9.18, and 9.19 CAREFULLY.

– **Lemma:** If H is a BIBO stable LSI system with impulse response $h(t)$, then the ROC of the transfer function $H(s) = \mathcal{L}[h(t)]$ includes the imaginary axis $Re[s]=0$.

Proof: Recall that H is stable iff $\|h(t)\|_{L^1} < \infty$, i.e., if and only if $h(t)$ is absolutely integrable. So, by hypothesis we have

$$\|h(t)\|_{L^1} = \int_{\mathbb{R}} |h(t)|\, dt < \infty.$$

Now,

$$H(s)\Big|_{Re[s]=0} = \mathcal{L}\big[h(t)\big]\Big|_{Re[s]=0} = \int_{\mathbb{R}} h(t) e^{-st} dt \Big|_{Re[s]=0}$$

$$= \int_{\mathbb{R}} h(t) e^{-j\omega t} dt.$$

So,

$$\big|H(s)\big|\Big|_{Re[s]=0} \underset{\substack{\uparrow \\ \text{triangle} \\ \text{Inequality}}}{\leq} \int_{\mathbb{R}} |h(t)| \, \big|\cancelto{1}{e^{-j\omega t}}\big| \, dt = \int_{\mathbb{R}} |h(t)|\, dt$$

$$= \|h(t)\|_{L^1} < \infty$$

Therefore, $H(s) = \int_{\mathbb{R}} h(t) e^{-st} dt$ converges if $Re[s]=0$.

Q.E.D.

Theorem: If H is an LSI system with impulse response $h(t)$ and transfer function $H(s) = \mathcal{L}\{h(t)\}$, then H is BIBO stable <u>if</u> <u>and</u> <u>only</u> <u>if</u> the ROC of $H(s)$ includes the imaginary axis $Re[s] = 0$.

Proof: No time! (aren't you glad?).

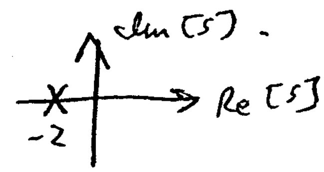
Corollary:

If H is a <u>CAUSAL</u> LSI system with rational transfer function $H(s)$, then H is BIBO stable <u>if</u> <u>and</u> <u>only</u> <u>if</u> all of the poles of $H(s)$ are in the left half-plane. That is, if and only if $Re[s] < 0$ for each pole of $H(s)$.

<u>EX</u>: Let H be an LSI system with impulse response $h(t) = e^{-2t} u(t)$.

$$\mathcal{L}[h(t)] \stackrel{\text{table}}{=} \frac{1}{s+2}, \quad \text{One pole at } s_1 = -2.$$

$$Re[s_1] = Re[-2] = -2.$$

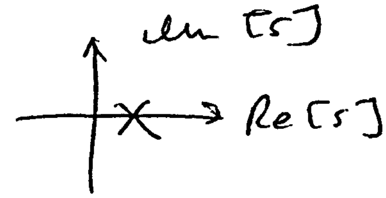


$\Rightarrow$ The system is BIBO stable.

EX: Let H be an LSI system with impulse response $h(t) = e^{2t} u(t)$.

$$\mathcal{L}[h(t)] \overset{\text{Table}}{=} \frac{1}{s-2}. \qquad \text{One pole @ } s_1 = 2$$

$$Re[s_1] = Re[2] = 2$$



$\Rightarrow$ The system is NOT BIBO stable because $H(s)$ has a right half-plane pole.

★ READ O.W.N. SECTION 4.7 ★★

- several of the Laplace transform properties we have seen have been true for ~~systems~~ LSI systems H having rational transfer functions $H(s)$.

$\Rightarrow$ There is an important class of systems for which the transfer function is always rational.

- These are the systems for which the input and output signals are related by a linear differential equation with constant coefficients.

EX suppose we have a LSI system where the input/output relationship is

$$\frac{d}{dt} y(t) + 3y(t) = x(t).$$

Taking the Laplace transform of both sides yields

$$sY(s) + 3Y(s) = X(s)$$

$$(s+3)Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

Note: There is more than one LSI system with input/output described by this differential equation. To make the solution unique, we would also need boundary conditions on $y(t)$. Thus, there is more than one system that has the transfer function $H(s) = \frac{1}{s+3}$.

However, the ROC of $H(s)$ is <u>different</u> for the different systems.

EX: $H(s) = \frac{1}{s+3}$, $\text{ROC} = \{s : \text{Re}[s] > -3\}$: $h(t) = e^{-3t}u(t)$.

$H(s) = \frac{1}{s+3}$, $\text{ROC} = \{s : \text{Re}[s] < -3\}$: $h(t) = -e^{-3t}u(-t)$.

- more generally, if the input and output signals of an LSI system are related by the differential equation

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \dots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$
$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \dots + b_0 x(t)$$

then
$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{l=0}^{M} b_l \frac{d^l}{dt^l} x(t)$$

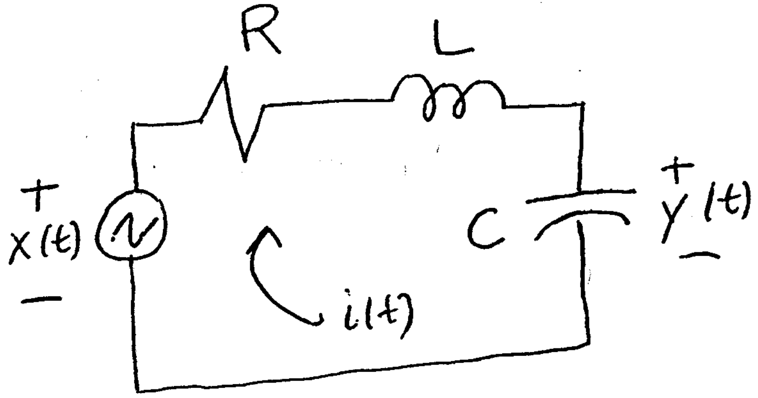
Taking the Laplace transform of both sides, we get
$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{l=0}^{M} b_l s^l X(s)$$

or
$$Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{l=0}^{M} b^l s^l .$$

So
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{l=0}^{M} b^l s^l}{\sum_{k=0}^{N} a_k s^k} .$$

- Since the numerator and denominator are polynomials in s, $H(s)$ is by definition a rational transfer function.

EX: 9.27: Consider the RLC circuit



Input: $x(t)$
output: $y(t)$.

within the "linear region" of operation (i.e. NO FIRES), this circuit is an LSI system, and CAUSAL.

- The constituent eqn. for the resistor is $V_R(t) = R\,i(t)$.
- The constituent eqn. for the inductor is $V_L(t) = L \frac{d}{dt} i(t)$.
- The constituent eqn. for the capacitor is $i(t) = C \frac{d}{dt} y(t)$.

→ Writing KVL for drops around the mesh gives:

$$-x(t) + i(t)R + L \frac{d}{dt} i(t) + y(t) = 0.$$

→ subbing in $i(t) = C \frac{d}{dt} y(t)$ then gives

$$-x(t) + RC \frac{d}{dt} y(t) + LC \frac{d^2}{dt^2} y(t) + y(t) = 0,$$

OR,

$$RC \frac{d}{dt} y(t) + LC \frac{d^2}{dt^2} y(t) + y(t) = x(t).$$

– Taking the Laplace transform of both sides, we have

$$RC\,s\,Y(s) + LC s^2 Y(s) + Y(s) = X(s)$$

$$(LC s^2 + RC s + 1)\,Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{LC s^2 + RC s + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

<u>FACT</u>: if $L > 0$, $R > 0$, and $C > 0$, then the poles of $H(s)$ are all in the left-half plane, and the system is BIBO stable.

<u>NOTE</u>: the above implies that the system frequency response is

$$H(\omega) = \frac{1/LC}{\left(\frac{1}{LC} - \omega^2\right) + j\frac{R}{L}\omega}$$

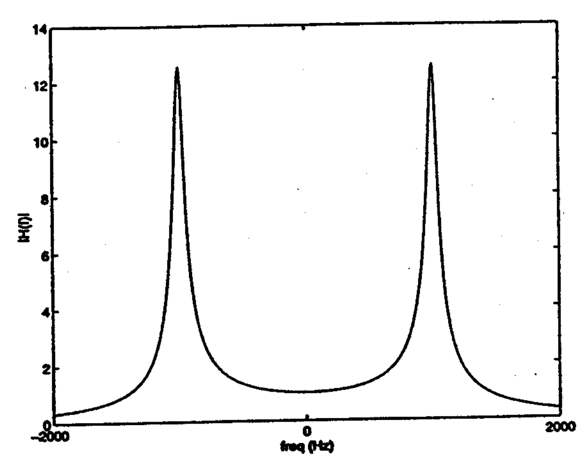
if $R = \frac{1}{2}\,\Omega$

$L = 1\,mH$

$C = 25\frac{1}{3}\,\mu F$,

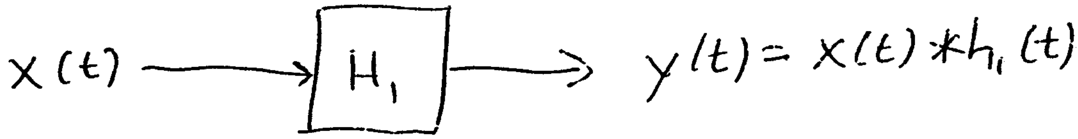
Then H is a bandpass filter. The center of the passband is

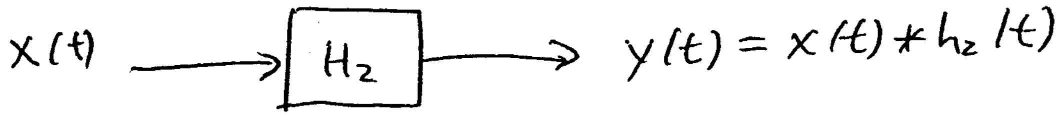
$f_0 = \pm 1000$ Hz



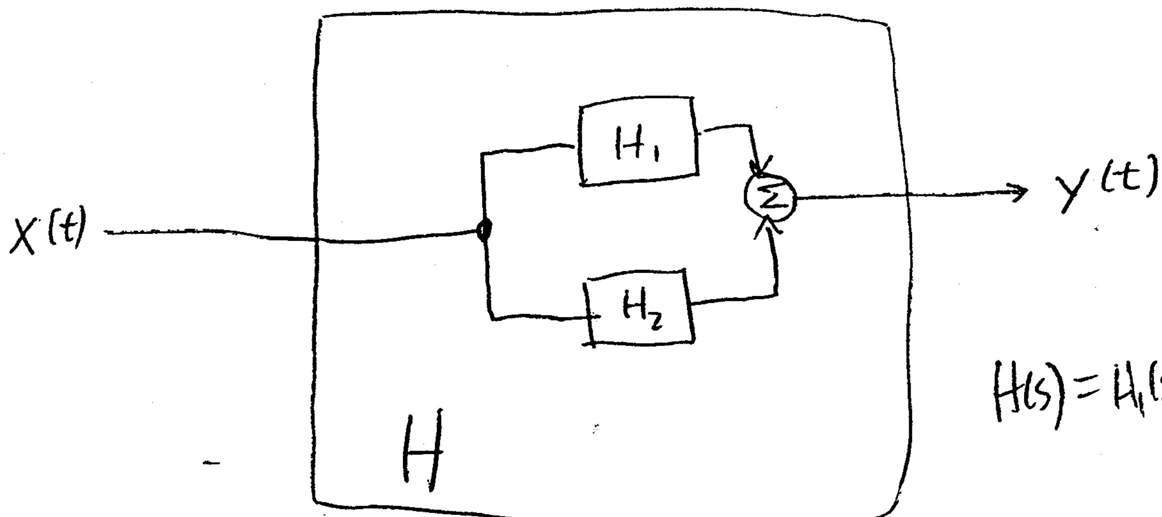
10.5 System Interconnections and Block Diagrams

- Suppose H_1 and H_2 are LSI systems with impulse responses $h_1(t)$ and $h_2(t)$ and transfer functions $H_1(s)$ and $H_2(s)$:





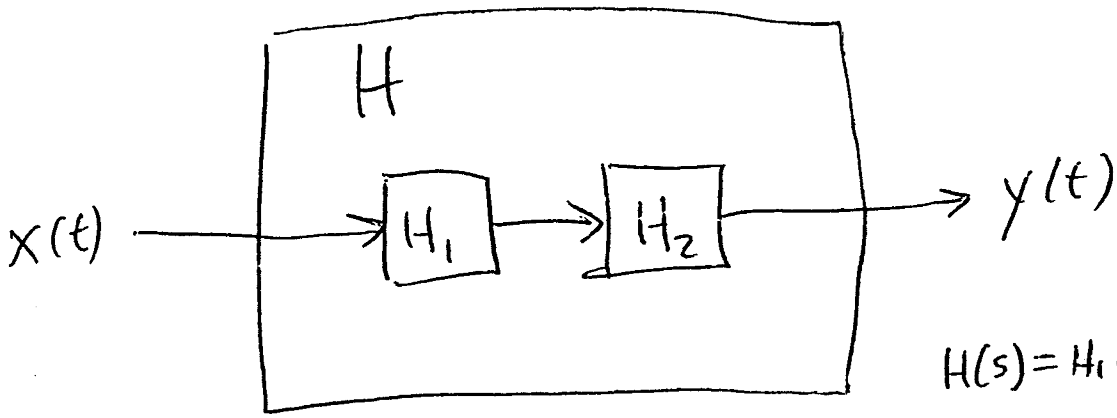
- The parallel connection of H_1 and H_2 is an LSI system H with impulse response $h_1(t) + h_2(t)$ and transfer function $H(s) = H_1(s) + H_2(s)$:



$$H(s) = H_1(s) + H_2(s)$$

$$h(t) = h_1(t) + h_2(t)$$

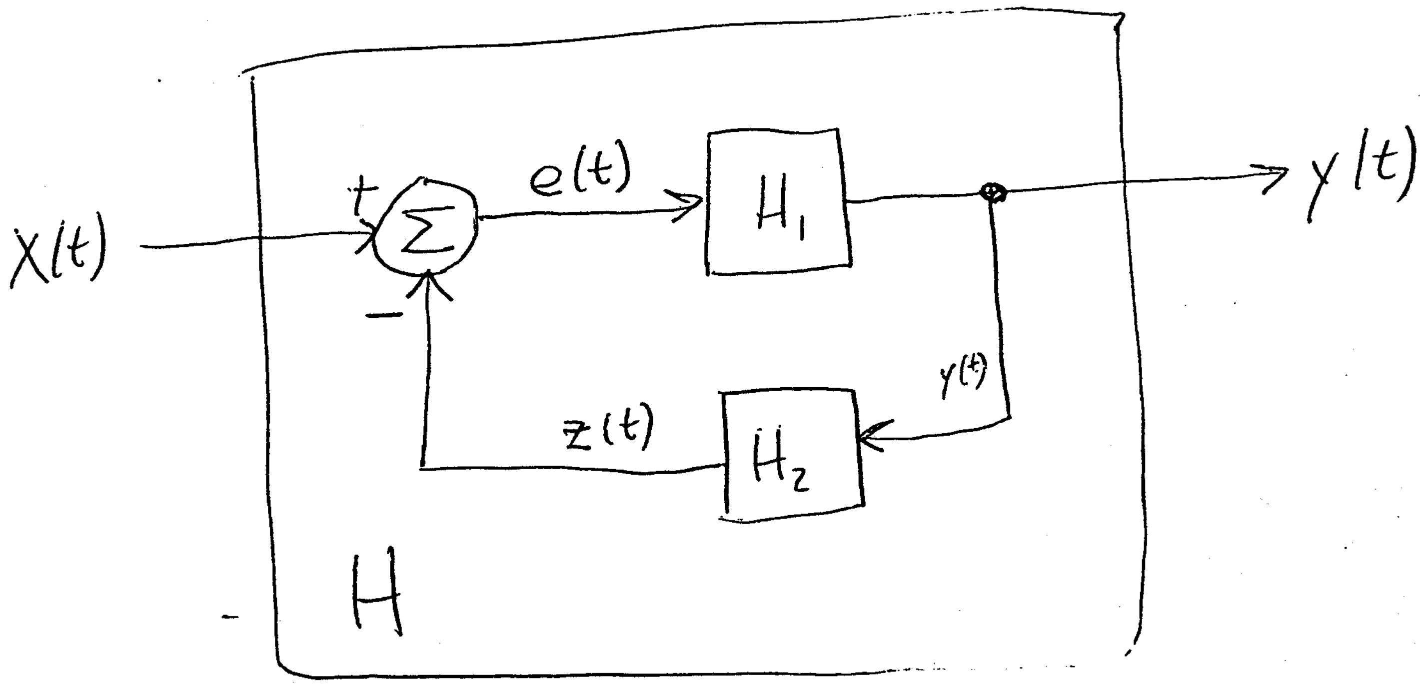
- The series connection of H_1 and H_2 is an LSI system H with impulse response $h(t) = h_1(t) * h_2(t)$ and transfer function $H(s) = H_1(s) H_2(s)$:



$$H(s) = H_1(s) H_2(s)$$

$$h(t) = h_1(t) * h_2(t).$$

- Feedback Connection:



$$H(s) = \frac{H_1(s)}{1 + H_1(s) H_2(s)}$$

- Derivation of feedback transfer function:

Clearly, $Y(s) = E(s)\,H_1(s)$.

But $E(s) = X(s) - Z(s)$,

So $Y(s) = \left[X(s) - Z(s)\right] H_1(s)$.

Now, $Z(s) = Y(s)\,H_2(s)$,

So $Y(s) = \left[X(s) - Y(s)\,H_2(s)\right] H_1(s)$

$= X(s)H_1(s) - Y(s)\,H_1(s)\,H_2(s)$

So $Y(s)\left[1 + H_1(s)H_2(s)\right] = X(s)\,H_1(s)$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \quad \star$$

10-6 The Unilateral Laplace Transform

- The unilateral Laplace transform of a signal $x(t)$ is defined by

$$X_u(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt.$$

$$= \lim_{\varepsilon \to 0^-} \int_{\varepsilon}^{\infty} x(t) e^{-st} dt = \mathcal{UL}\{x(t)\}$$

⇒ The meaning of 0^- in the lower limit of integration is that singularity functions at $t=0$ are to be included in the integration.

- Note that, for any signal $x(t)$,

$$\mathcal{UL}\{x(t)\} = X_u(s) = \mathcal{L}\{x(t)u(t)\}.$$

- So, if $x(t) = 0 \ \forall \ t \leq 0$, then

$$\mathcal{UL}\{x(t)\} = \mathcal{L}[x(t)].$$

→ Review examples 9.32 – 9.36 in the book carefully.

→ The ROC of any unilateral laplace transform is a right hand half plane.

→ If $H_u(s)$ is rational, then the ROC is the right hand half plane to the right of the right most pole of $H(s)$.

→ The following properties of the unilateral Laplace transform are the same as their bilateral counterparts:

Linearity	s-domain shifting
time scaling	conjugation
s-domain differentiation	Initial/Final value theorems

Unilateral Laplace Transform Convolution Property:

if $x_1(t) = 0 \ \forall\, t<0$ and $x_2(t) = 0 \ \forall\, t<0$

and $x_1(t) \xleftrightarrow{\mathcal{UL}} X_{1u}(s)$ and $x_2(t) \xleftrightarrow{\mathcal{UL}} X_{2u}(s)$,

then $x_1(t) * x_2(t) \xleftrightarrow{\mathcal{UL}} X_{1u}(s)\, X_{2u}(s)$.

- Thus, the unilateral Laplace transform is primarily of interest for treating systems where $h(t) = 0 \ \forall t < 0$ (causal system) and where the inputs do not become nonzero prior to time $t = 0$.

- Unilateral Laplace Transform Integration Property:

  - if $x(t) \overset{\mathcal{UL}}{\longleftrightarrow} X_u(s)$ <u>and</u> $x(t) = 0 \ \forall t < 0$, then

$$\int_{0^-}^{t} x(\tau)\,d\tau \overset{\mathcal{UL}}{\longleftrightarrow} \frac{1}{s} X_u(s).$$

→ Note: if $x(t)$ is right sided, this is the same as the bilateral Laplace transform integral property.

- Unilateral Laplace transform Time Shifting Property:

★★★ (This one is <u>not</u> in the book). ★★★

- if $x(t) \overset{\mathcal{UL}}{\longleftrightarrow} X_u(s)$ and $a > 0$, then

$$\mathcal{UL}\{x(t-a)\,u(t-a)\} = e^{-as} X_u(s)$$

Does not work for shifts the other way

proof:

$$\mathcal{UL}\{x(t-a)u(t-a)\} = \int_{0^-}^{\infty} x(t-a)u(t-a)e^{-st}\,dt$$

$$= \int_{a^-}^{\infty} x(t-a)e^{-st}\,dt$$

let $v = t-a$ $\qquad$ $t = v+a$

$dv = dt$ $\qquad$ $dt = dv$

when $t=\infty$, $v=\infty$

when $t=a^-$, $v=0^-$

$$= \int_{0^-}^{\infty} x(v)e^{-s(v+a)}\,dv$$

$$= e^{-sa}\int_{0^-}^{\infty} x(v)e^{-sv}\,dv$$

$$= e^{-as}X_u(s).$$

QED.

- unilateral Laplace transform time differentiation property:

- This is <u>the</u> <u>most</u> <u>important</u> difference between the unilateral and bilateral Laplace transforms.

- if $x(t) \overset{\mathcal{UL}}{\longleftrightarrow} X_u(s)$, then

$$\frac{d}{dt}x(t) \overset{\mathcal{UL}}{\longleftrightarrow} sX_u(s) - x(0^-).$$

NOTE: The derivative property of the unilateral Laplace transform involves an initial condition on $x(t)$.

→ This is different from the bilateral Laplace transform.

⇒ It makes the unilateral Laplace transform VERY useful for solving initial value problems

RECALL: An initial value problem is a differential equation (or a system of differential equations) together with initial conditions that constrain the solution.

- Because of their derivative properties, the Fourier, Laplace, and unilateral Laplace transforms all change a system of differential equations into a system of algebraic equations.

⇒ But only the unilateral Laplace transform gives algebraic equations that also involve initial conditions on the time domain signal.

Proof of $\mathcal{UL}$ time differention property:

$$\mathcal{UL}\left\{\frac{d}{dt}x(t)\right\} = \int_{0^-}^{\infty} \frac{d}{dt}x(t)\, e^{-st}\, dt \qquad (*)$$

→ Integrate by parts:

$$u = e^{-st} \qquad dv = \frac{d}{dt}x(t)\,dt$$

$$du = -se^{-st}\,dt \qquad v = x(t)$$

→

$$(*) = \int_{0^-}^{\infty} u\,dv = uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\,du$$

$$= e^{-st}x(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)(-s)e^{-st}\,dt$$

$$= \lim_{A\to\infty} \cancelto{0}{e^{-sA}} x(A) - \lim_{\varepsilon\to 0^-} \cancelto{1}{e^{-s\varepsilon}} x(\varepsilon) + \underbrace{s\int_{0^-}^{\infty} x(t)e^{-st}\,dt}_{\mathcal{UL}\{x(t)\} = \mathcal{X}_u(s)}$$

$$= s\mathcal{X}_u(s) - x(0^-). \quad \underline{\text{QED}}$$

- Applying the $\mathcal{UL}$ time differentiation property repeatedly, we get:

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{UL}} s^n\mathcal{X}_u(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - s^{n-3}x''(0^-) - \dots - x^{(n-1)}(0^-).$$

In words:

for the n'th derivative of $x(t)$, you get:

$s^n\mathcal{X}_u(s)$ __minus__ terms that are a power of s times a derivative of $x(t)$ evaluated at $t = 0^-$.

- The powers of s go __down__ until they reach zero.
- The orders of the derivatives go __up__ until they reach $n-1$.

☆☆ __NOTE__: This property is __not__ __in__ __the__ __book__.

BILATERAL LAPLACE TRANSFORM EXAMPLE:

Partial Fractions.

- A continuous-time LTI system has impulse response

$$h(t) = e^{-2t}u(t).$$

- The input is $x(t) = e^{-3t}u(t)$.
- Find the output signal $y(t)$.

Table: $H(s) = \frac{1}{s+2}$, $Re[s] > -2$

Table: $X(s) = \frac{1}{s+3}$, $Re[s] > -3$

$$Y(s) = X(s)H(s) = \frac{1}{(s+2)(s+3)}$$

ROC: $\{Re[s] > -3\} \cap \{Re[s] > -2\}$
$= Re[s] > -2.$

$$Y(s) = \frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = \left.\frac{1}{s+3}\right|_{s=-2} = \frac{1}{3-2} = 1$$

$$B = \left.\frac{1}{s+2}\right|_{s=-3} = \frac{1}{2-3} = -1$$

$$Y(s) = \underbrace{\frac{1}{s+2}}_{Re[s]>-2} - \underbrace{\frac{1}{s+3}}_{Re[s]>-3}$$

Table: $y(t) = e^{-2t}u(t) - e^{-3t}u(t)$

Another Example: Improper Fraction

- A causal continuous-time LTI system has input

$$x(t) = \tfrac{1}{2}e^{-t}u(t) - \tfrac{1}{2}e^{-3t}u(t)$$

- The output is given by $y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)$
- Find the impulse response $h(t)$.

Table: $$X(s) = \underbrace{\frac{1/2}{s+1}}_{Re[s]>-1} - \underbrace{\frac{1/2}{s+3}}_{Re[s]>-3} = \frac{\frac{1}{2}(s+3) - \frac{1}{2}(s+1)}{(s+1)(s+3)}$$

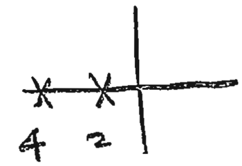
$$= \frac{\frac{1}{2}(s+3-s-1)}{(s+1)(s+3)} = \frac{\frac{1}{2}(2)}{(s+1)(s+3)}$$

$$= \frac{1}{(s+1)(s+3)}, \quad Re[s] > -1$$

Table: $$Y(s) = \underbrace{\frac{1/2}{s+2}}_{Re[s]>-2} - \underbrace{\frac{1/2}{s+4}}_{Re[s]>-4} = \frac{\frac{1}{2}(s+4) - \frac{1}{2}(s+2)}{(s+2)(s+4)}$$

$$= \frac{\frac{1}{2}(s+4-s-2)}{(s+2)(s+4)} = \frac{\frac{1}{2}(2)}{(s+2)(s+4)} = \frac{1}{(s+2)(s+4)}, \quad Re[s] > -2$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+1)(s+3)}{(s+2)(s+4)} = \frac{s^2+4s+3}{s^2+6s+8}$$



Since H is given to be <u>causal</u>, the ROC must be everything to the right of the rightmost pole.

ROC: $Re[s] > -2$.

$\Rightarrow$ Since $H(s)$ is <u>not</u> a strictly proper fraction, we cannot apply partial fractions directly.

- We must first use long division to clear the improper fraction.

$$\begin{array}{r|l} & \quad 1 \\ \hline s^2+6s+8 & s^2+4s+3 \\ & s^2+6s+8 \\ \hline & \quad -2s-5 \end{array}$$

$$\Rightarrow \quad H(s) = 1 - \frac{2s+5}{s^2+6s+8} = 1 - \frac{2s+5}{(s+2)(s+4)}, \quad Re[s] > -2$$

Now, do PFE on the remainder, which is now a strictly proper fraction:

$$\frac{2s+5}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = \left.\frac{2s+5}{s+4}\right|_{s=-2} = \frac{-4+5}{-2+4} = \frac{1}{2}$$

$$B = \left.\frac{2s+5}{s+2}\right|_{s=-4} = \frac{-8+5}{-4+2} = \frac{-3}{-2} = \frac{3}{2}$$

$$\frac{2s+5}{(s+2)(s+4)} = \frac{1/2}{s+2} + \frac{3/2}{s+4}$$

$$H(s) = \underbrace{1}_{\text{all } s} - \underbrace{\frac{1/2}{s+2}}_{Re[s]>-2} - \underbrace{\frac{3/2}{s+4}}_{Re[s]>-4}$$

Table: $h(t) = \delta(t) - \frac{1}{2}e^{-2t}u(t) - \frac{3}{2}e^{-4t}u(t)$

USING FEEDBACK TO STABILIZE AN UNSTABLE SYSTEM

- Suppose F is a causal continuous-time LTI system with impulse response $f(t) = e^{2t}u(t)$

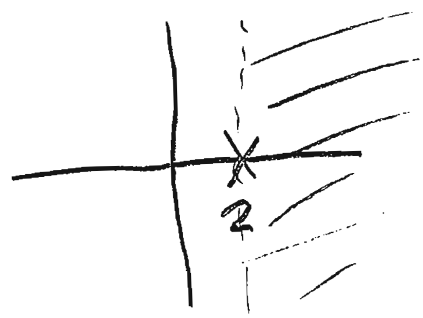
$$x(t) \longrightarrow \boxed{F} \longrightarrow y(t)$$

- We can see that F is unstable because

$$\int_{-\infty}^{\infty} |f(t)|\,dt \longrightarrow \infty .$$

- The transfer function is (Table):

$$H(s) = \frac{1}{s-2}, \quad \mathrm{Re}[s] > 2$$



- The ROC does not include the $j\omega$-axis

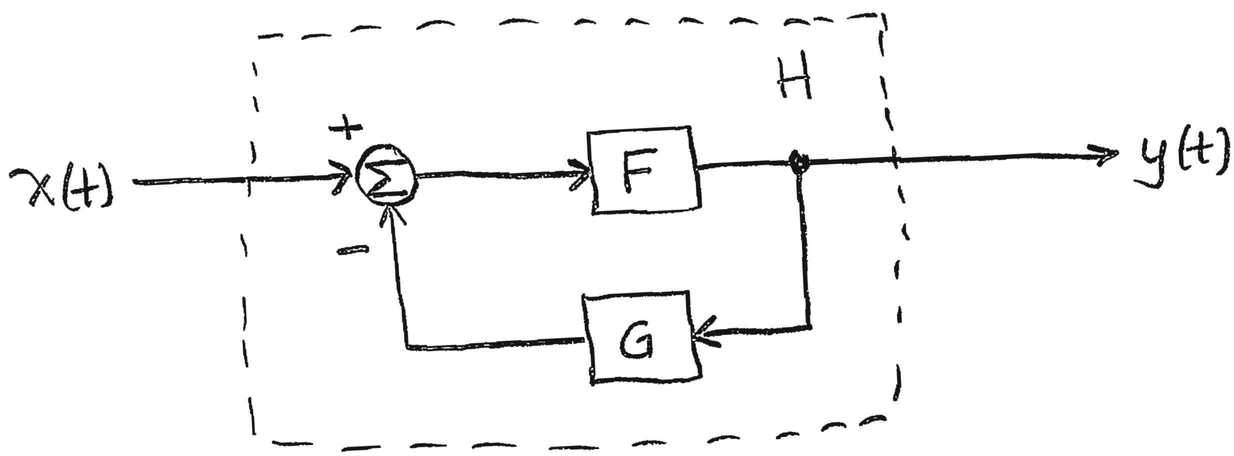
- There is a pole in the right half-plane.

$\longrightarrow$ Not Stable

- We can stabilize the system F by adding a second LTI system G in a feedback connection.
- We will call the new system H.
- For a first-order system like F, it is sufficient to make G a pure linear amplifier:

$$g(t) = K\,\delta(t) \quad , \quad K \in \mathbb{R}$$

Table: $G(s) = K$, all s.



$$H(s) = \frac{F(s)}{1+F(s)G(s)} = \frac{\frac{1}{s-2}}{1+\frac{K}{s-2}} \cdot \underbrace{\frac{s-2}{s-2}}_{\text{one}}$$

$$= \frac{1}{s-2+K} = \frac{1}{s+(K-2)}$$

- The new system H has one real pole at $s = 2-K$.
- If $K > 2$, then this pole is in the left half-plane and the system H is <u>stable</u>.
- For example, if $K = 3$, then

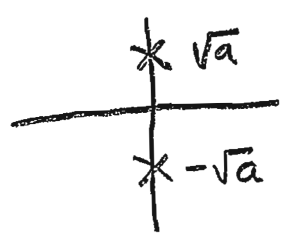
$$H(s) = \frac{1}{s+1}$$

<u>EX</u>: Stabilization of a 2^{nd}-order causal system.

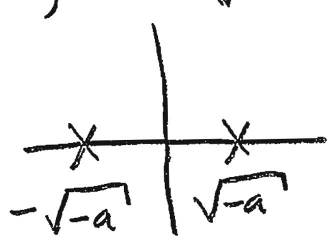
- Suppose $F(s) = \dfrac{b}{s^2+a}$, $a, b \in \mathbb{R}$.

$$x(t) \longrightarrow \boxed{F} \longrightarrow y(t)$$

- If $a > 0$, then $F(s) = \dfrac{b}{(s+j\sqrt{a})(s-j\sqrt{a})}$



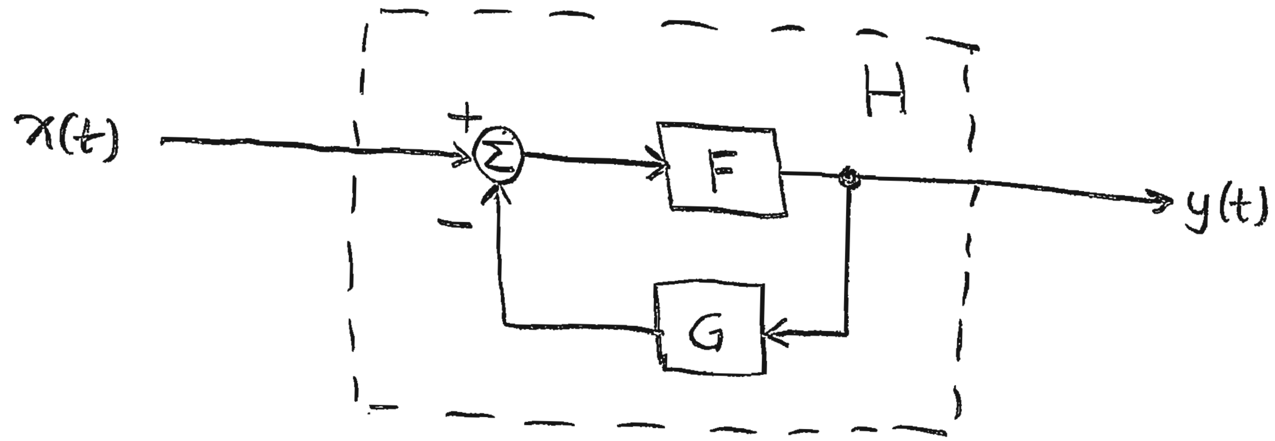
- If $a < 0$, then $\sqrt{-a}$ is real and $F(s) = \dfrac{b}{(s+\sqrt{-a})(s-\sqrt{-a})}$



$\Rightarrow$ The causal system F is unstable in either case.

- As before, we will add a second LTI system G in a feedback connection.
- Let $g(t) = K_1 \delta(t) + K_2 \delta'(t)$

Table: $G(s) = K_1 + K_2 s$, all s.



$$H(s) = \frac{F(s)}{1+F(s)G(s)} = \frac{\frac{b}{s^2+a}}{1+\frac{b}{s^2+a}(K_1+K_2 s)} \cdot \underbrace{\frac{s^2+a}{s^2+a}}_{\text{one}}$$

$$= \frac{b}{s^2+a+bK_1+bK_2 s}$$

$$= \frac{b}{s^2+bK_2 s+(a+bK_1)} = \frac{b}{\alpha s^2+\beta s+\gamma}$$

where $\alpha = 1$

$\beta = bK_2$

$\gamma = a+bK_1$

- Applying the quadratic formula to the denominator to solve for the poles of $H(s)$, we obtain

$$\text{Poles:} \quad s = \frac{-\beta}{2\alpha} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} = -\tfrac{1}{2}\beta \pm \tfrac{1}{2}\sqrt{\beta^2 - 4\gamma}$$

$$s = -\tfrac{1}{2} b K_2 \pm \tfrac{1}{2}\sqrt{b^2 K_2^2 - 4(a + bK_1)}$$

- For stability, the poles must have real parts that are negative.
- This is <u>guaranteed</u> if:

$$bK_2 > 0$$

$$\text{and} \quad a + bK_1 > 0 .$$

- we are done with Chapter 9.