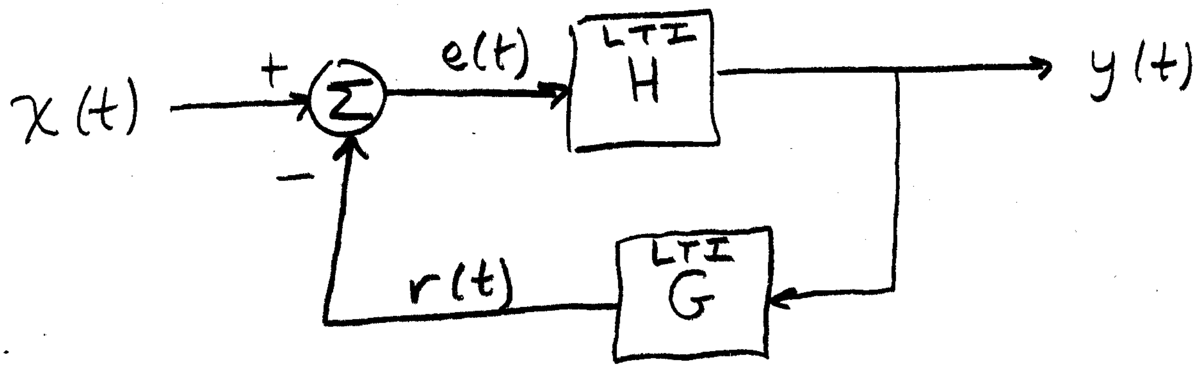
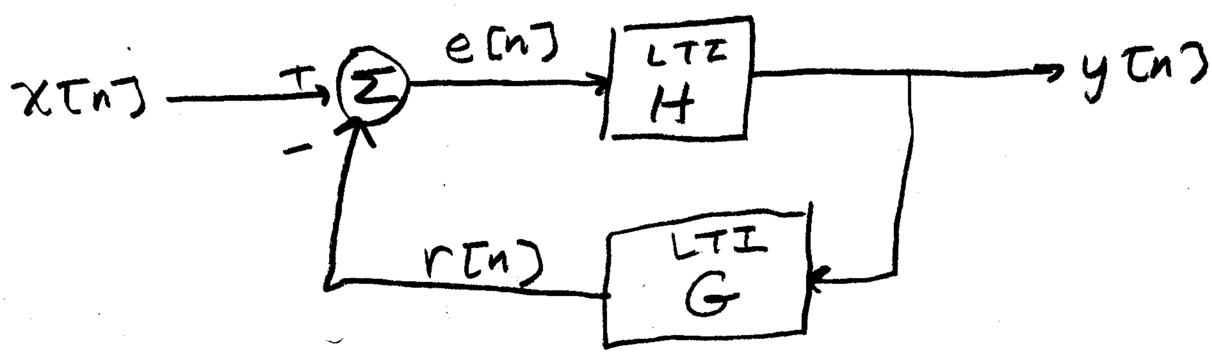
Linear Feedback



$$Q(\omega) = \frac{H(\omega)}{1+H(\omega)G(\omega)}$$

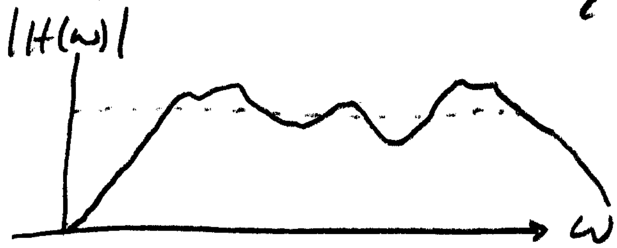
$$Q(s) = \frac{H(s)}{1+H(s)G(s)}$$



$$Q(e^{j\omega}) = \frac{H(e^{j\omega})}{1+H(e^{j\omega})G(e^{j\omega})}$$

$$Q(z) = \frac{H(z)}{1+H(z)G(z)}$$

- Generally assumed that H and G are causal LTI systems.
  - → Then the overall system Q is also causal
  - → ROC of $Q(s)$ is the half-plane to the right of the rightmost pole.
  - → ROC of $Q(z)$ is the exterior of a circle going through the largest pole.
- Systems of this type were first studied seriously in the 1920s
  - → The goal was to produce high quality audio amplifiers having a very uniform gain across their usable frequency range.
  - → High gain components $H(s)$ were available, but with poor uniformity:



  - → The idea of using <u>negative</u> feedback in a power amp design was ridiculed at first.

→ But suppose that:

- $G(\omega) = K$ is constant with $K << 1$.
- $H(\omega) >> K$, so that $H(\omega)G(\omega) >> 1$.

→ Then

$$Q(\omega) = \frac{H(\omega)}{1 + KH(\omega)} \approx \frac{H(\omega)}{KH(\omega)} = \frac{1}{K} >> 1.$$

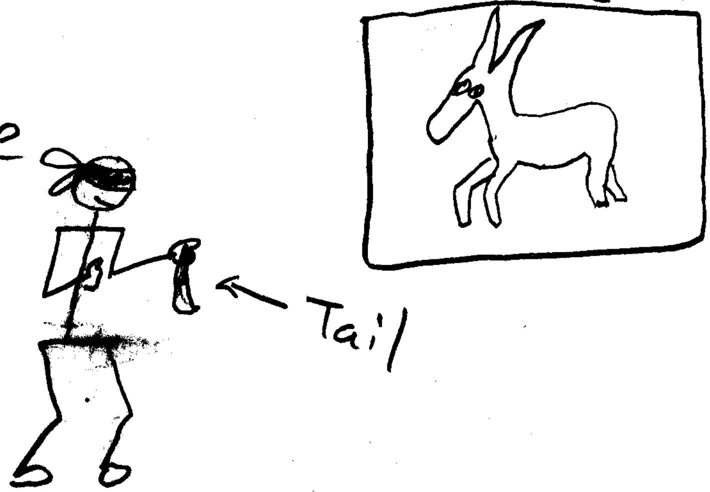
→ This can result in a high power amp with a very uniform frequency response.

→ However, since the overall "closed loop" gain is $\frac{1}{K}$ and the "open loop" gain $H(\omega)$ must be $>> K$, a lot of power is wasted in obtaining the uniformity.

The notion of feedback for control

- Consider the game "pin the tail on the donkey".

  - There is a picture of a donkey on the wall. This donkey has no tail.

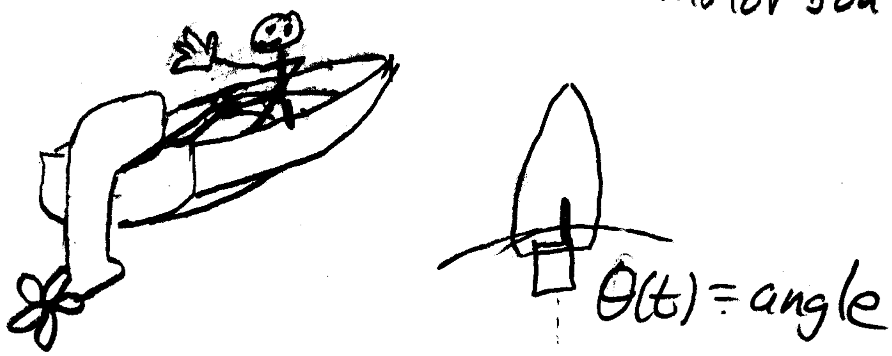


- The objective is for the child to pin a tail in the correct position on the hindquarters of the beast.

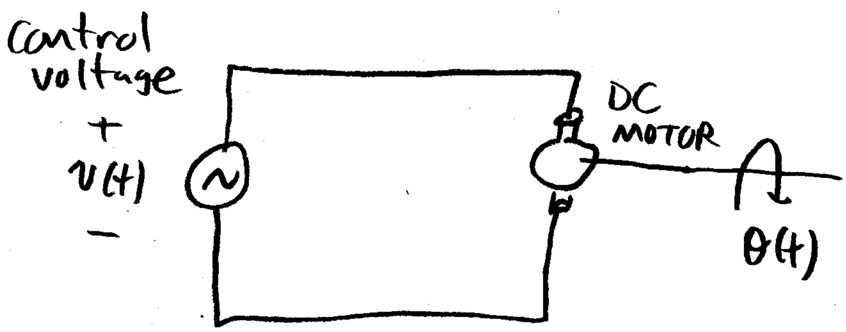
- In the "open loop" system the child is blinfolded so that visual information about the position error can not be fed back.
  - → The expected performance is poor.

- when the child cheats by removing the blindfold, this becomes a "closed loop" system with feedback.
  - → The feedback dramatically improves the system performance.

- Another example: we want to make a steering system for an outboard motorboat.



- An electric stepper motor will be used to position the out board motor.



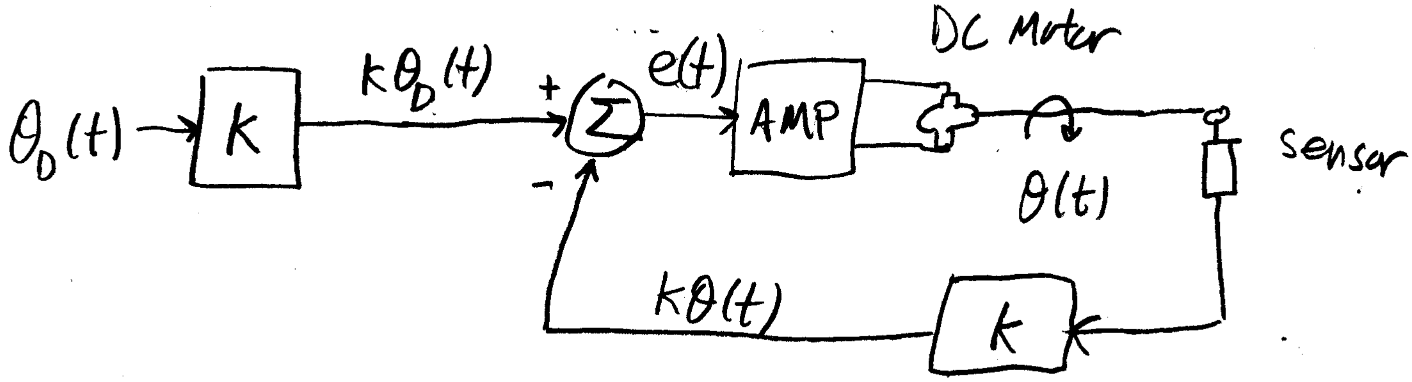
Open Loop System

$$v(t) \longrightarrow \boxed{\text{DC Motor}} \longrightarrow \theta(t)$$

- The closed loop system does not need frequent calibration, provides smooth control, and is insensitive to drifting parameter values:

  $\theta_D(t)$: control signal : the desired angle.

  $\theta(t)$: actual angle measured by a rotary position sensor.



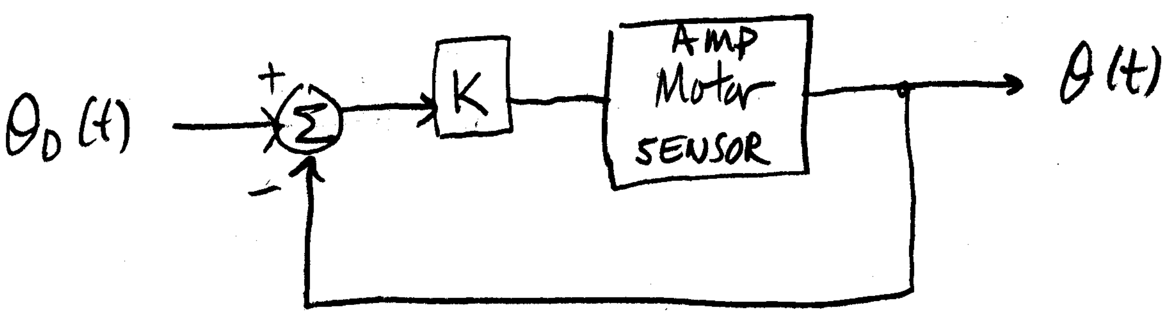
- The motor is driven by the error signal

$$e(t) = K[\theta_D(t) - \theta(t)].$$

→ When the error is large, the DC motor is driven hard.

→ when the error is small, the DC motor moves gently.

System Model:

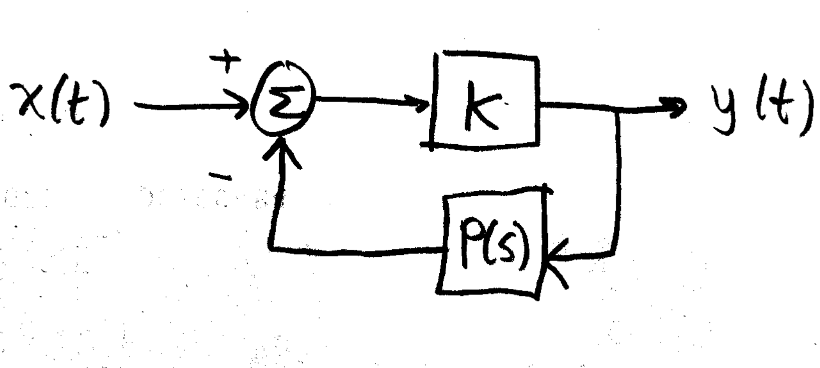


Using Feedback to Implement Inverse Systems

- You have an LTI system $P(s)$ and you want to build the inverse system $\frac{1}{P(s)}$.
  - The details of $P(s)$ may be unknown or slowly drifting.
  - EX: $P(s)$ might be a capacitor where the ambient temperature is causing the capacitance to slowly drift.

$$v(t) \longrightarrow \boxed{C\frac{d}{dt}} \longrightarrow i(t)$$

- To construct the inverse system, we let the reverse path be $P(s)$ and the forward path be a constant K such that $KP(s) >> 1$:

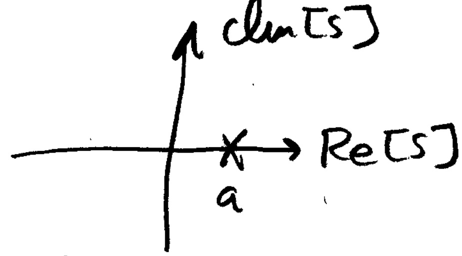


$$Q(s) = \frac{K}{1+KP(s)} \approx \frac{K}{KP(s)} = \frac{1}{P(s)}$$

Using Feedback for Stabilization

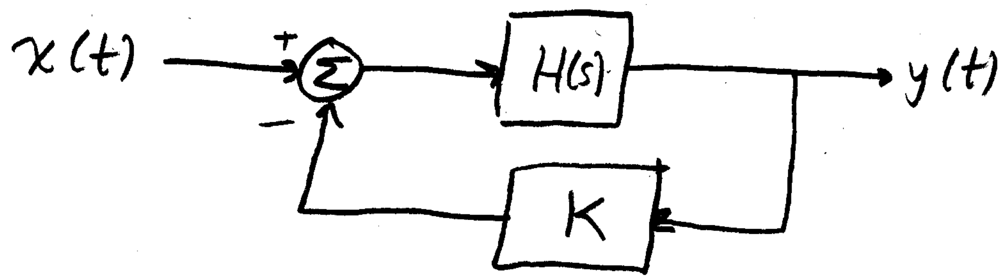
- The addition of negative feedback moves the poles of an open loop system.
- This can be used to make a stable closed loop system out of an unstable open loop system.

EX: $H(s) = \dfrac{b}{s-a}, \quad a > 0$



For a causal system, the pole at $s=a$ implies instability.

Apply feedback with a constant gain reverse path $G(s) = K$:



- The closed loop gain is

$$Q(s) = \frac{H(s)}{1 + KH(s)} = \frac{\frac{b}{s-a}}{1 + \frac{Kb}{s-a}} \cdot \frac{s-a}{s-a}$$

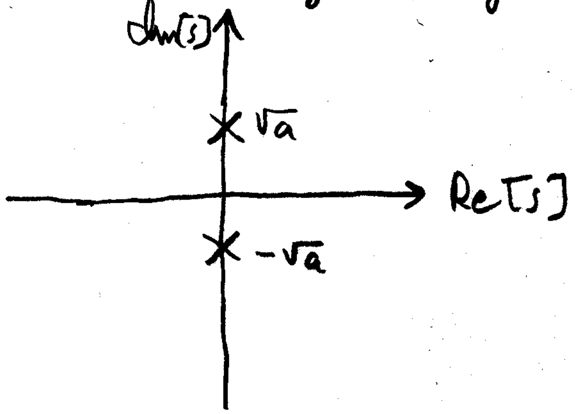
$$= \frac{b}{s-a+Kb} = \frac{b}{s-(a-Kb)}$$

- The pole of the closed loop system has been moved from $s = a$ to $s = a - kb$.
  - → if $kb > a$, the new pole is in the left half-plane and the closed loop system is stable.

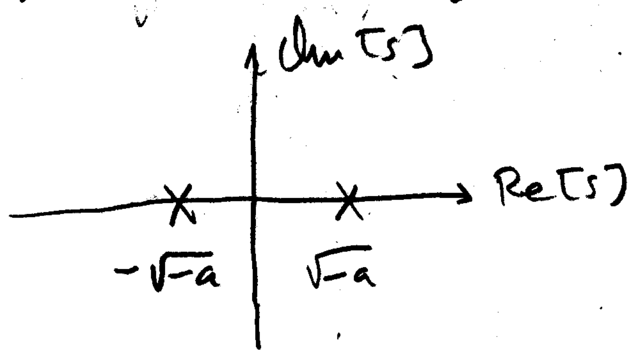
- Now consider a 2^{nd}-order causal system

$$x(t) \longrightarrow \boxed{H(s) = \frac{b}{s^2 + a}} \longrightarrow y(t) \qquad (a \in \mathbb{R})$$

→ if $a > 0$, $\quad H(s) = \dfrac{b}{(s + j\sqrt{a})(s - j\sqrt{a})}$



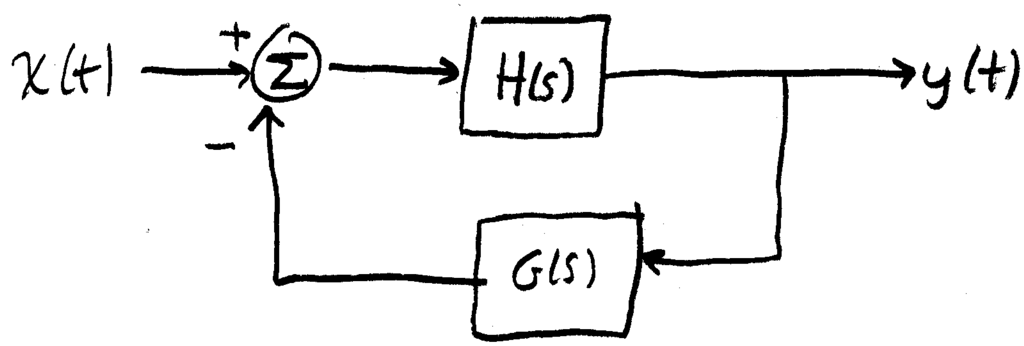
- if $a < 0$, then $\sqrt{-a}$ is real and $H(s) = \dfrac{b}{(s + \sqrt{-a})(s - \sqrt{-a})}$



- The causal system H is unstable in either case.

- Constant gain "proportional" control cannot stabilize this system.

- Instead, we take $G(s) = K_1 + K_2 s$



- We have

$$Q(s) = \frac{H(s)}{1 + H(s)G(s)} = \frac{\frac{b}{s^2+a}}{1 + \frac{b}{s^2+a}(K_1 + K_2 s)}$$

$$\times\left(\frac{s^2+a}{s^2+a}\right) = \frac{b}{s^2 + a + bK_1 + bK_2 s}$$

$$= \frac{b}{s^2 + bK_2 s + (a + bK_1)}$$

$$Q(s) = \frac{b}{1s^2 + bK_2 s + (a+bK_1)} = \frac{b}{\alpha s^2 + \beta s + \gamma}$$

where

$$\alpha = 1$$
$$\beta = bK_2$$
$$\gamma = a + bK_1$$

The poles are

$$s = \frac{-\beta}{2\alpha} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$= -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\gamma}$$

$$= -\frac{1}{2} bK_2 \pm \frac{1}{2}\sqrt{b^2K_2^2 - 4(a+bK_1)}$$

- For stability, we need the poles to have real parts that are negative.

- This is guaranteed if

$$bK_2 > 0 \quad \text{and} \quad a + K_1 b > 0.$$

- Discrete Example

- The open loop system is described by the difference equation

$$y[n] = 2y[n-1] + x[n]$$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

- This could describe, e.g., the growth in the population of a certain organism that tends to double its population each time step.

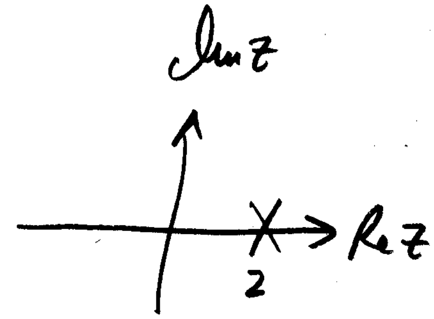
- $x[n]$ represents deaths and other modifications to the population (removal or insertion of individuals).

- We have:

$$y[n] - 2y[n-1] = x[n]$$

$$Y(z)\left[1 - 2z^{-1}\right] = X(z)$$

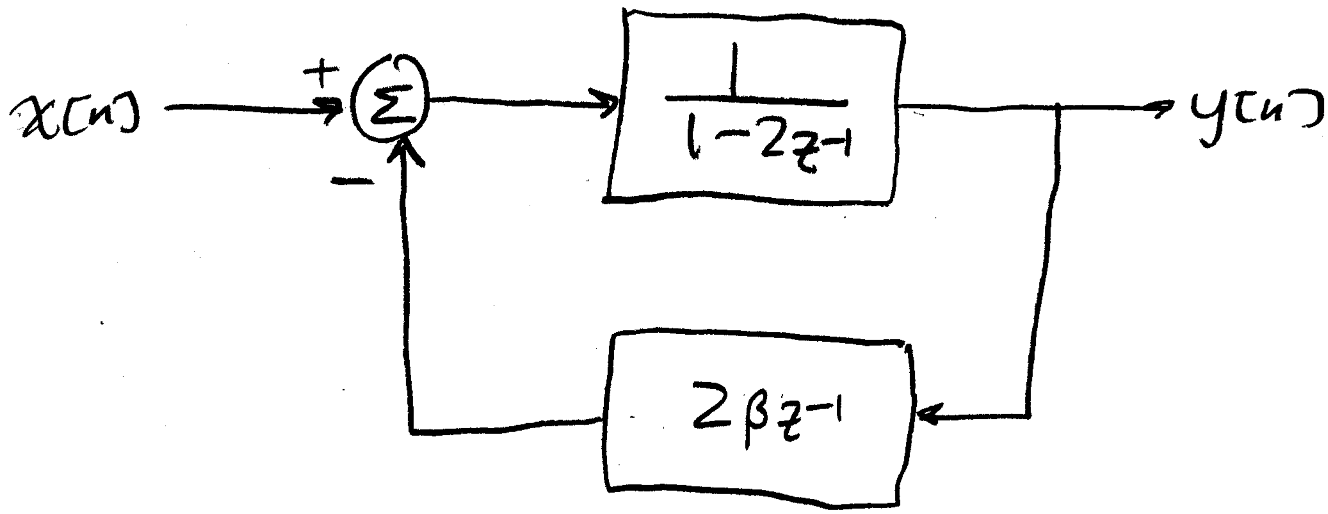
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-2z^{-1}}$$



→ The causal system is unstable because the pole at $z=2$ is outside the unit circle.

- Now we introduce negative feedback with gain $G(z) = 2\beta z^{-1}$.

  → For the population dynamics model, this could represent a fraction β of the population at each time step being killed off by predators or some other outside influence.



- Now we have

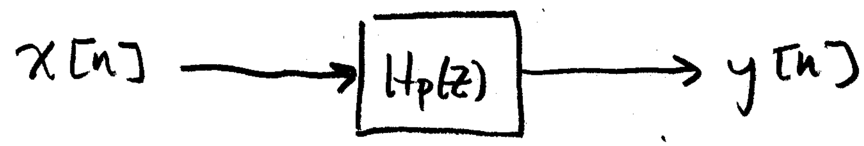
$$Q(z) = \frac{H(z)}{1+H(z)G(z)} = \frac{\frac{1}{1-2z^{-1}}}{1+\frac{1}{1-2z^{-1}}2\beta z^{-1}} \cdot \frac{1-2z^{-1}}{1-2z^{-1}}$$

$$= \frac{1}{1-2z^{-1}+2\beta z^{-1}} = \frac{1}{1-2(1-\beta)z^{-1}}$$

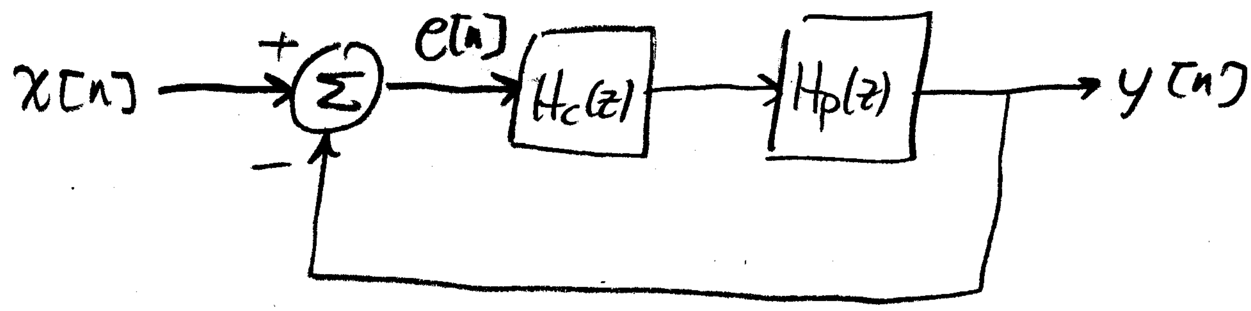
- The new pole is at $z = 2(1-\beta)$.
- This is inside the unit circle if $\frac{1}{2} < \beta < \frac{3}{2}$.

Tracking and Disturbance (using discrete as example)

Open loop system:



Ideal Closed loop system:



$H_p(z)$: the system to be controlled

$H_c(z)$: the compensator to be designed

$x[n]$: the control input

$y[n]$: system output

$e[n]$: error signal; desire it to be close to zero.

- Let $H(z) = H_c(z) H_p(z)$.

- Then $$Q(z) = \frac{H(z)}{1 + H(z)}$$

- So $$Y(z) = \frac{H(z)}{1+H(z)} X(z)$$

- but $Y(z) = H(z) E(z)$, so

$$H(z) E(z) = \frac{H(z)}{1+H(z)} X(z)$$

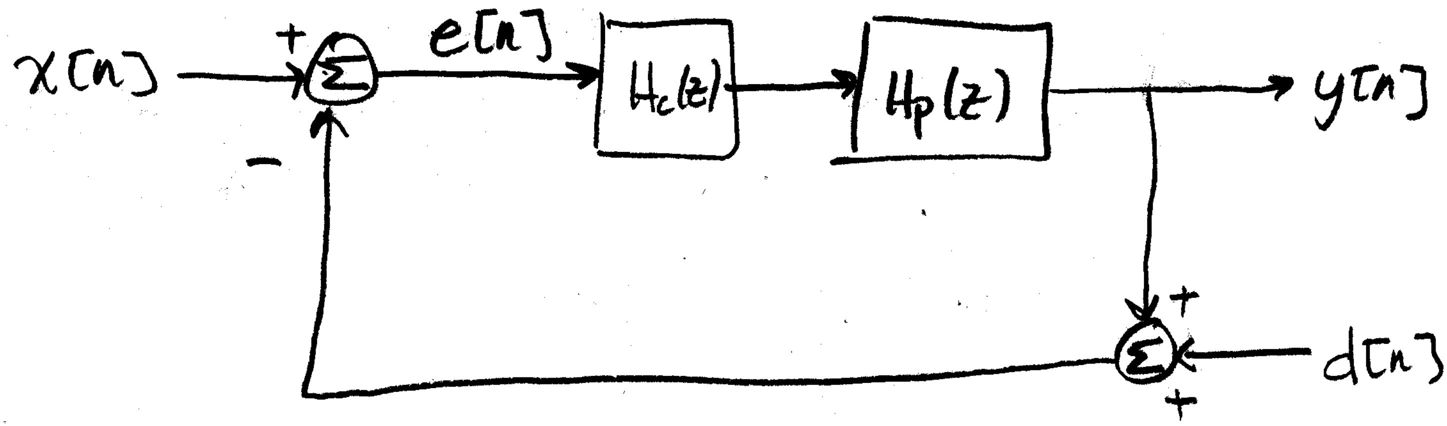
$$E(z) = \frac{1}{1+H(z)} X(z)$$

$$E(e^{j\omega}) = \frac{1}{1+H(e^{j\omega})} X(e^{j\omega})$$

$\Rightarrow$ To make $E(e^{j\omega})$ small, we want to design $H_c(e^{j\omega})$ to make $H(e^{j\omega})$ <u>large</u> wherever $X(e^{j\omega})$ is nonzero.

- But now consider a practical system where there is some measurement error in the sensors used for feedback.

- We model this as a "disturbance" $d[n]$:



- Now we have

$$Y(z) = H(z)E(z)$$
$$= H(z)\left[X(z) - D(z) - Y(z)\right]$$
$$Y(z)\left[1+H(z)\right] = H(z)X(z) - H(z)D(z)$$
$$Y(z) = \frac{H(z)}{1+H(z)}X(z) - \frac{H(z)}{1+H(z)}D(z)$$

→ Minimizing the error and rejecting the disturbance are conflicting goals.

→ Want $H_c(e^{j\omega})$ large to make $e[n]$ small

→ Want $H_c(e^{j\omega})$ small minimize disturbance.