ALTERNATE PROOF OF DFT CIRCULAR CONVOLUTION PROPERTY

ECE 5213

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10/24/2012

Lemma 1: Let $m, n, k, N \in \mathbb{N}$ such that $0 \le m \le N-1$ and $0 \le n \le N-1$.

Then $e^{-j2\pi(n-m)k/N} = e^{-j2\pi\langle n-m\rangle_N k/N}$.

Proof:

From the hypothesis, we have that either:

① $(n-m) \in [0, N-1]$ or

② $(n-m) \notin [0, N-1]$.

Case ①: if $(n-m) \in [0, N-1]$, then $(n-m) = \langle n-m\rangle_N$

and thus we have trivially that
$$e^{-j2\pi(n-m)k/N} = e^{-j2\pi\langle n-m\rangle_N k/N}.$$

Case ②: if $(n-m) \notin [0, N-1]$, then by the definition of "$(n-m) \bmod N$" $\equiv \langle n-m\rangle_N$;

we have that $\exists r \in \mathbb{Z}$ s.t. $n-m = \langle n-m\rangle_N + rN$.

$$\text{So } e^{-j2\pi(n-m)k/N} = e^{-j2\pi(\langle n-m\rangle_N + rN)k/N}$$
$$= e^{-j2\pi\langle n-m\rangle_N k/N} e^{-j2\pi rkN/N}$$
$$= e^{-j2\pi\langle n-m\rangle_N k/N} \underbrace{e^{-j2\pi rk}}_{\text{ONE}}$$
$$= e^{-j2\pi\langle n-m\rangle_N k/N}.$$

QED

Theorem:

if $N \in \mathbb{N}$ and $x[n]$ and $h[n]$ are two length-N discrete-time sequences, then

$$DFT\{x[n] \textcircled{N} h[n]\} = X[k]H[k]$$

where $X[k] = DFT\{x[n]\}$ and $H[k] = DFT\{h[n]\}$.

Proof: Let $y[n] = x[n] \textcircled{N} h[n] \equiv \sum_{m=0}^{N-1} x[m]\, h[\langle n-m \rangle_N]$

Then $Y[k] = \sum_{n=0}^{N-1} y[n]\, W_N^{nk}$

$$= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m]\, h[\langle n-m \rangle_N] \right) W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m]\, h[\langle n-m \rangle_N]\, e^{-j2\pi nk/N}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m]\, h[\langle n-m \rangle_N]\, e^{-j2\pi nk/N} \underbrace{\left(e^{j2\pi mk/N}\, e^{-j2\pi mk/N} \right)}_{\text{ONE}}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m]\, e^{-j2\pi mk/N} \quad h[\langle n-m \rangle_N]\, e^{-j2\pi (n-m)k/N}$$

Lemma 1

$$\sum_{m=0}^{N-1} x[m]\, e^{-j2\pi mk/N} \underbrace{\sum_{n=0}^{N-1} h[\langle n-m \rangle_N]\, e^{-j2\pi \langle n-m \rangle_N k/N}}_{} \qquad (*)$$

Now, the trick is to realize that for <u>any</u> $m \in [0, N-1]$, the **quantity** $\langle n-m \rangle_N$ in ☺ will generate some <u>permutation</u> of the integers $\{0, 1, \ldots, N-1\}$ in the sum from $n=0$ to $N-1$.

$\Rightarrow$ In other words, for any $m \in [0, N-1]$,

$$\sum_{n=0}^{N-1} h[\langle n-m \rangle_N] e^{-j2\pi \langle n-m \rangle_N k/N}$$

$$= h[0] e^{-j2\pi 0 k/N} + h[1] e^{-j2\pi 1 k/N} + h[2] e^{-j2\pi 2k/N} + \cdots + h[N-1] e^{-j2\pi (N-1)k/N}$$

(although the order of addition will be different except when $m=0$)

$$= \sum_{\ell=0}^{N-1} h[\ell] e^{-j2\pi \ell k/N}.$$

So we have

$$Y[k] = (*) = \sum_{m=0}^{N-1} x[m] e^{-j2\pi mk/N} \sum_{\ell=0}^{N-1} h[\ell] e^{-j2\pi \ell k/N}$$

$$= \sum_{m=0}^{N-1} x[m] W_N^{mk} \sum_{\ell=0}^{N-1} h[\ell] W_N^{\ell k} = X[k] H[k].$$

QED