Module 6 : The Z-Transform

- In modules 4 and 5, we learned how to write a signal $x[n]$ as a sum of the "spectral" basis functions

$$\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)} .$$

- We did this using the discrete-time Fourier transform (DTFT):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega .$$

- This made it much easier (than convolution) to find the output of an LTI system by using the DTFT convolution property:

$$\text{if} \quad y[n] = x[n] * h[n]$$

$$\text{then} \quad Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) .$$

- But we also saw on p. 5.100 that there are some "bad" signals like $x[n] = 2^n u[n]$ that <u>do not</u> have a DTFT...

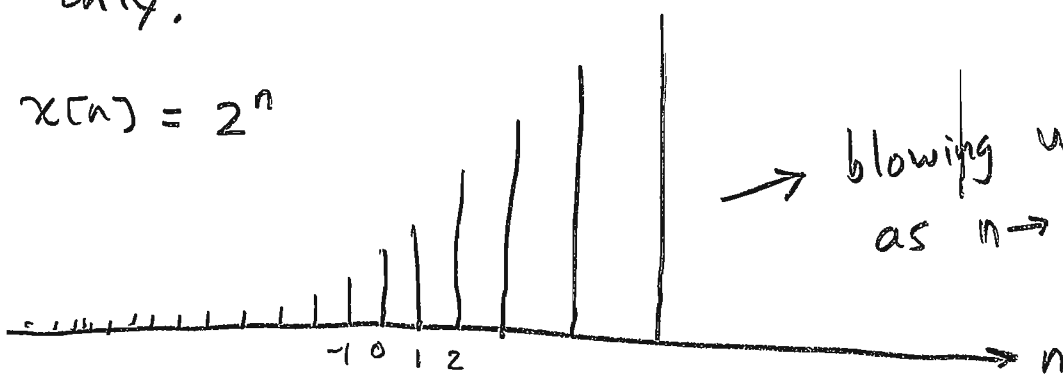
→ For such signals, the DTFT sum fails to converge... and so $X(e^{j\omega})$ <u>does not</u> exist.

- So what are we supposed to do when our system input is one of these "bad" signals?

<u>ANSWER</u>: in many cases, we can use a more powerful transform called the "z-transform."

- The z-transform will generally solve the problem if the "badness" is <u>at most</u> polynomial badness on one side only.

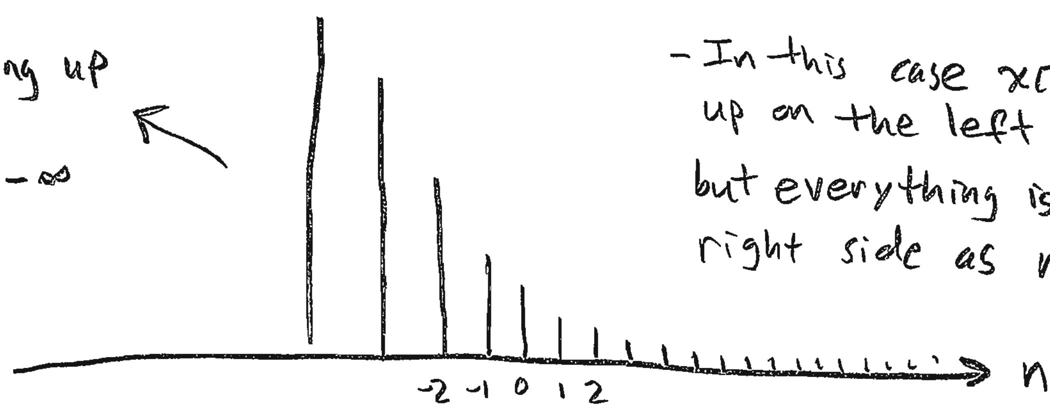
<u>EX</u>: $x[n] = 2^n$



→ In this case, $x[n]$ is blowing up on the right side as $n \to \infty$, but everything is okay on the left side as $n \to -\infty$.

<u>EX</u>: $x[n] = \left(\frac{1}{2}\right)^n$

blowing up as $n \to -\infty$



- In this case $x[n]$ is blowing up on the left side as $n \to -\infty$, but everything is okay on the right side as $n \to \infty$.

- The z-transform can help with both of these examples, because in both cases the badness is only on one side.

- For a super bad signal like $x[n] = 2^{|n|}$

that is blowing up on both sides (both as $n \to \infty$ and as $n \to -\infty$), the z-transform will not be able to help.

☆☆☆ Important Note: the fastest and easiest way to understand the z-transform is to not think about z-transforms at first!!

- Instead, let us focus on how to make a fixed up discrete Fourier transform first.

  - Then we will worry about the z-transform after that.

- So let's start by assuming that we have a <u>bad guy</u> $x[n]$ who is bad on one side only...

  - Like $x[n] = 2^n$ or $x[n] = 2^n u[n]$.

    → Bad on the <u>right side</u>

  - Or like $x[n] = (\frac{1}{2})^n$ or $x[n] = (\frac{1}{2})^n u[-n]$.

    → Bad on the <u>left</u> side.

- Our strategy will be to make a "fixed up" guy by multiplying $x[n]$ times a "fixer-upper" function.

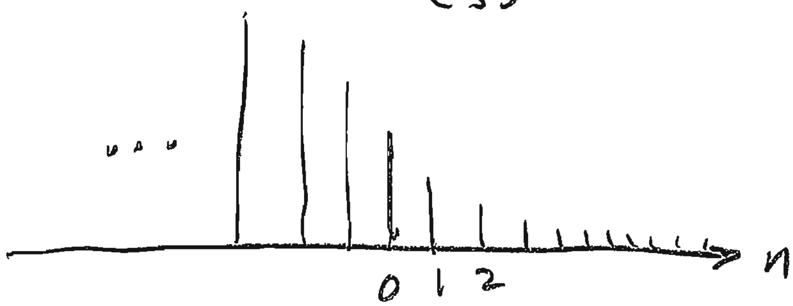
  → Then we will take the DTFT of the <u>fixed up guy</u>.

⟹ Our fixer-upper functions will be of the form r^{-n}, where the "fixer-upper parameter" r is a non-negative real number.

→ In other words, r will be a real number such that $r \geq 0$.

EX: if $r=3$, then the fixer-upper is

$$r^{-n} = 3^{-n} = \left(\tfrac{1}{3}\right)^n$$



- This can fix things up on the right side... because the fixer-upper function 3^{-n} is going to zero faster than any finite order polynomial in the limit as $n \to \infty$.

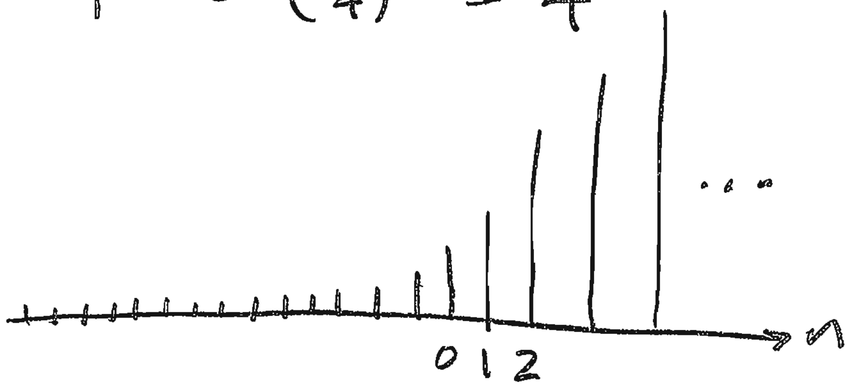
  $\rightarrow$ but this fixer-upper function is also blowing up on the left in the limit as $n \to -\infty$.

- So the fixer-upper 3^{-n} can fix up a bad guy $x[n]$ who is bad on the right.

  $\Rightarrow$ But for a bad guy who is bad on the left, this fixer upper (3^{-n}) will only make things worse.

EX: if $r = \frac{1}{4}$, then the fixer-upper is

$$r^{-n} = \left(\tfrac{1}{4}\right)^{-n} = 4^n$$



- This can fix things up on the left side... because the fixer-upper function $\left(\frac{1}{4}\right)^{-n} = 4^n$ is going to zero faster than any finite order polynomial in the limit as $n \to -\infty$.

→ But this fixer-upper function can not help with a bad guy who is bad on the right side!

→ Because this fixer-upper function $\left(\frac{1}{4}\right)^{-n} = 4^n$ is blowing up in the limit as $n \to \infty$.

→ For a bad guy $x[n]$ who is bad on the right, the fixer-upper $\left(\frac{1}{4}\right)^{-n} = 4^n$ will only make things worse!

- So we have the <u>important</u> fact that the fixer-upper function r^{-n} can only fix up on one side.

→ If the fixer-upper parameter $r > 1$, then the fixer-upper function r^{-n} can fix up bad guys $x[n]$ who are bad on the right... but only makes things worse for guys who are bad on the left.

→ if $0 < r < 1$, then the fixer-upper function r^{-n} can fix up bad guys $x[n]$ who are bad on the left... but only makes things worse for guys who are bad on the right.

⇒ The <u>fixed up guy</u> is obtained by multiplying the signal $x[n]$ times the fixer-upper r^{-n}:

The guy (signal): $x[n]$

The "fixed up" guy: $x[n]\, r^{-n}$

→ But we must choose the fixer-upper parameter r <u>carefully</u>... or else it will not fix things up at all... it will make them worse!

EX: $x[n] = 2^n u[n]$

- This guy is bad on the right.
- As we saw on p. 5.100, this guy does not have a DTFT $X(e^{j\omega})$ ... because he is so bad on the right that he makes the DTFT sum diverge.

- Now suppose we take the fixer-upper parameter to be $r = 3$.
- Then the fixer-upper function is $r^{-n} = 3^{-n} = \left(\frac{1}{3}\right)^n$
- The fixed up guy is
$$x[n]r^{-n} = 2^n\left(\tfrac{1}{3}\right)^n u[n] = \left(\tfrac{2}{3}\right)^n u[n].$$
- We will now take the DTFT of the fixed up guy.
- Because this fixed up DTFT is a transform that depends on both the frequency variable ω and the fixer-upper parameter r, we will write it as:

$$X(r,\omega) = \text{DTFT}\{x[n]r^{-n}\}$$

$\longrightarrow$

EX... we have:

$$X(r,\omega) = \text{DTFT}\{x[n]\,3^{-n}\}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\,3^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n u[n]\,3^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} 2^n 3^{-n} e^{-j\omega n} \quad \left(\begin{array}{c}\text{because } u[n]=0 \\ \forall\ n<0\end{array}\right)$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{2}{3}\right)^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\tfrac{2}{3}e^{-j\omega}\right)^n$$

$$= \lim_{A\to\infty} \sum_{n=0}^{A} \left(\tfrac{2}{3}e^{-j\omega}\right)^n$$

$$= \lim_{A\to\infty} \frac{\left(\tfrac{2}{3}e^{-j\omega}\right)^0 - \left(\tfrac{2}{3}e^{-j\omega}\right)^{A+1}}{1-\tfrac{2}{3}e^{-j\omega}}$$

$$= \frac{1-0}{1-\tfrac{2}{3}e^{-j\omega}} = \frac{1}{1-\tfrac{2}{3}e^{-j\omega}} \quad \checkmark$$

$\Rightarrow$ So the fixer-upper worked and the fixed up guy $x[n]\,r^{-n}$ does have a DTFT $X(r,\omega)$ when we choose $r=3$.

EX, ... But if we *instead* choose $r = \frac{1}{4}$, then it will *not* work ... the DTFT of the fixed up guy will still diverge:

$$X(r,\omega)\Big|_{r=\frac{1}{4}} = \text{DTFT}\left\{ x[n] \left(\tfrac{1}{4}\right)^{-n} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\tfrac{1}{4}\right)^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n u[n] \left(\tfrac{1}{4}\right)^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} 2^n \left(\tfrac{1}{4}\right)^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n 4^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} 8^n e^{-j\omega n} \longrightarrow \text{divergent!!!}$$

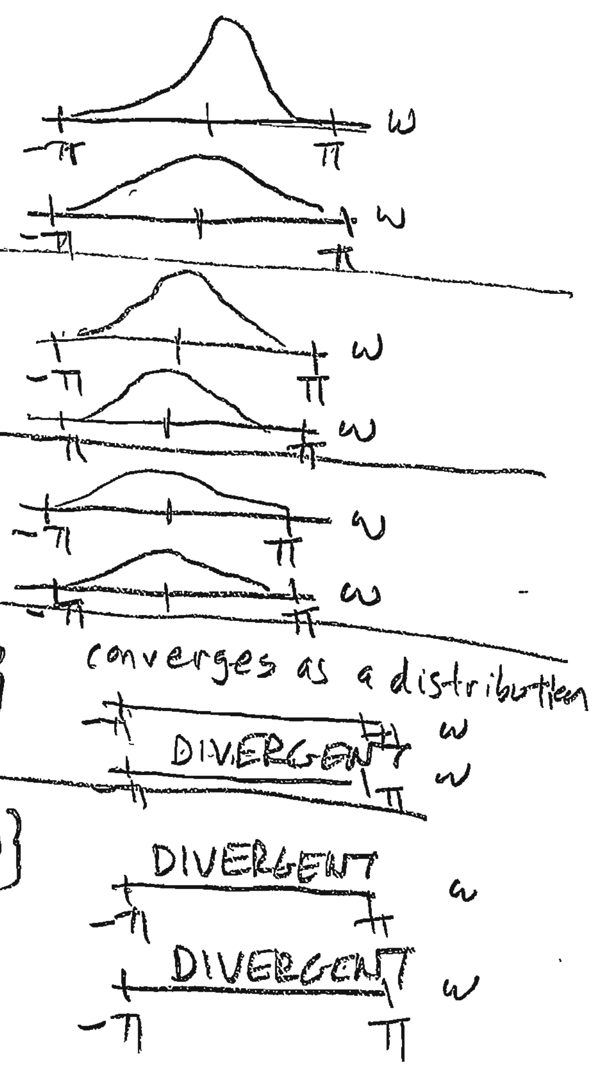
$\Rightarrow$ For this bad guy $x[n] = 2^n u[n]$,

- if we choose $r > 2$, then the fixer-upper will help, and $X(r,\omega) = \text{DTFT}\{x[n]\, r^{-n}\}$ will *converge*.

- But if we choose $0 < r < 2$, then the fixer-upper will *not* help, and $X(r,\omega)$ will *diverge*.

- So in general, we think of $X(r,\omega)$ as a <u>family</u> of "fixed up" transforms for all the different possible choices of r.

r	$X(r,\omega)$
$\vdots$	
5	$\text{DTFT}\{x[n]5^{-n}\}$
$4.5 = \frac{9}{2}$	$\text{DTFT}\{x[n](\frac{9}{2})^{-n}\}$
4	$\text{DTFT}\{x[n]4^{-n}\}$
$3.5 = \frac{7}{2}$	$\text{DTFT}\{x[n](\frac{7}{2})^{-n}\}$
3	$\text{DTFT}\{x[n]3^{-n}\}$
$2.5 = \frac{5}{2}$	$\text{DTFT}\{x[n](\frac{5}{2})^{-n}\}$
2	$\text{DTFT}\{x[n]2^{-n}\} = \text{DTFT}\{u[n]\}$
$1.5 = \frac{3}{2}$	$\text{DTFT}\{x[n](\frac{3}{2})^{-n}\}$
1	$\text{DTFT}\{x[n]1^{-n}\} = \text{DTFT}\{x[n]\}$
$0.5 = \frac{1}{2}$	$\text{DTFT}\{x[n](\frac{1}{2})^{-n}\}$
$\vdots$	



"A family of fixed up DTFTs."

- In reality, we do this for <u>all</u> real choices of $r > 0$, <u>not</u> just the ones shown here.

Some Important Observations about the fixer-upper:

1 - Increasing r makes the fixer-upper function r^{-n} fix up more on the right.

- As $n \to \infty$, $5^{-n} = \left(\frac{1}{5}\right)^n$ goes to zero faster than $2^{-n} = \left(\frac{1}{2}\right)^n$

2 - Decreasing r makes the fixer-upper function r^{-n} fix up more on the left.

- As $n \to -\infty$, $\left(\frac{1}{10}\right)^{-n} = 10^n$ goes to zero faster than $\left(\frac{1}{2}\right)^{-n} = 2^n$.

3 - When $r = 1$, the fixer-upper function is

$$r^{-n} = 1^{-n} = 1^n = 1 \quad \forall n \in \mathbb{Z}.$$

→ So the fixed up guy is $x[n] r^{-n} = x[n]$ and the fixer-upper does nothing.

⇒ In this case, we get

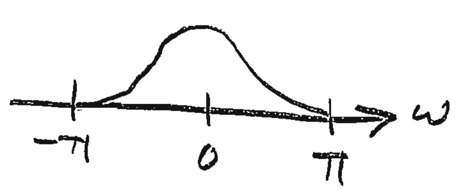
$$X(r,\omega) = \text{DTFT}\{x[n]\, 1^n\}$$
$$= \text{DTFT}\{x[n]\} = X(e^{j\omega})$$

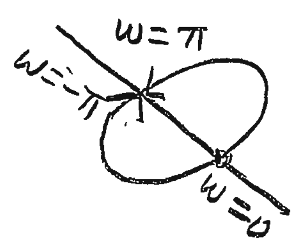
- Now, to look at the family of fixed up transforms $X(r,\omega)$, we do something that is going to seem pretty crazy at first.

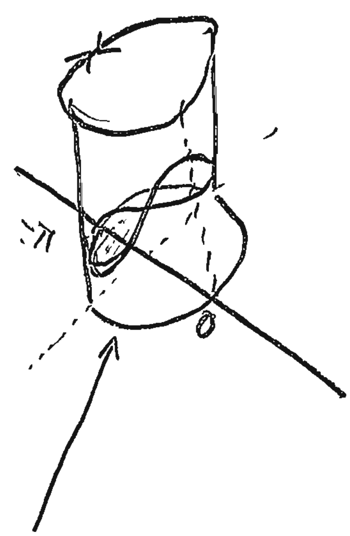
  - Recall from pp. 4.31 – 4.33 that any DTFT is 2π-periodic.
  - Even though the graphs of the functions $X(r,\omega)$ shown on p. 6.11 only go from $-\pi$ to π, in reality every one of those graphs is 2π-periodic in ω.
- To represent this, we take each graph on p. 6.11 and wrap the horizontal ω-axis around a circle... so that the point $\omega = -\pi$ touches the point $\omega = +\pi$...

EX: $r = 3$:

$X(3,\omega) = \text{DTFT}\{x(n)\,3^{-n}\}$

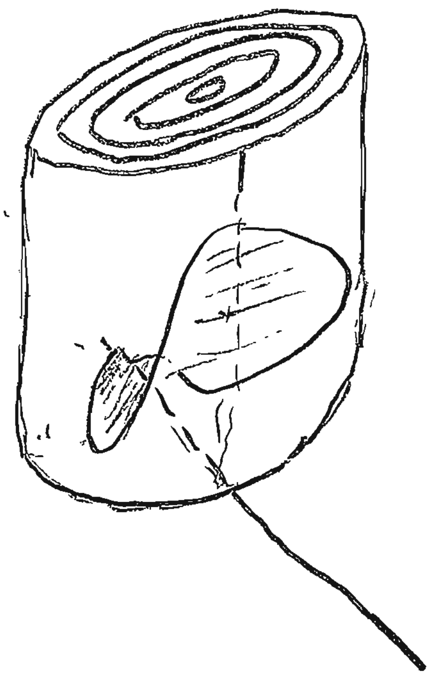






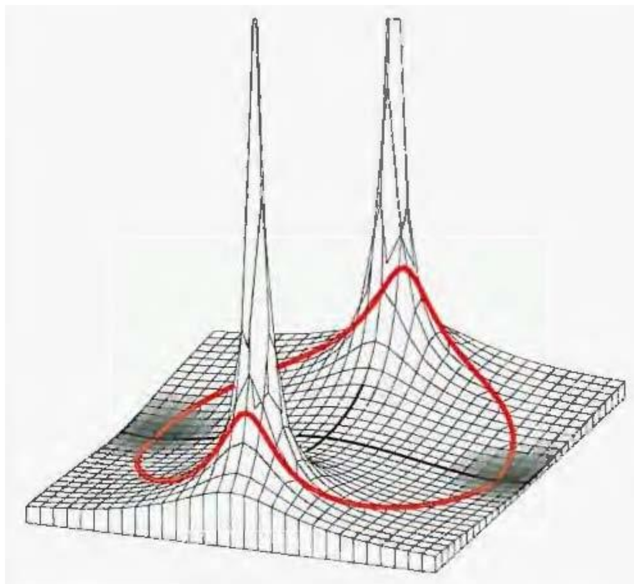
The graph is wrapped around a cylinder.

- We make the radius of each cylinder equal to r.
    - The graph of $X(5,\omega) = \text{DTFT}\{x[n]\,5^{-n}\}$ is wrapped around a cyclinder of radius 5.
    - The graph of $X(3,\omega)$ is wrapped around a cylinder of radius 3.
    - And we do this for all real choices of r in the range $r > 0$.
- Then we put the smaller radius cylinders inside the larger radius cylinders like Russian dolls:



- The graphs for <u>all</u> cylinders corresponding to $r > 0$ now form a <u>surface</u>... that is the 3D plot of the function $X(r,\omega)$.

- Here is an example of what the 3D plot of $X(r,\omega)$ might look like (without showing the cylinders):



- The orange line goes through the plot exactly above the circle of radius 1. It shows the graph of the DTFT of the "fixed up guy" when the fixer-upper parameter is r = 1.
- In other words, the orange line shows a graph of the DTFT of $x[n] \times 1^{-n}$, which is the graph of the DTFT of x[n]; i.e., it is $X(e^{j\omega})$.
- This graph of $X(e^{j\omega})$ is wrapped around the circle of radius 1 in the plane below the plot.
- Going around circles of other radii, like r=2, or r=3, or r=½ would give the graphs of the DTFT's of other fixed up guys $x[n] \times r^{-n}$… all wrapped around circles of radius r.

- Now let's extend these ideas and learn what the z-transform is all about.

- We have that $X(r,\omega)$ is actually a <u>collection</u> or <u>family</u> of DTFT's of fixed up guys $x[n]r^{-n}$ for <u>all</u> real choices of $r > 0$. Actually, $r \geq 0$.
  - It is a 3D surface made out of DTFT's of $x[n]r^{-n}$, all wrapped around circles with radius r.

- In other words,

$$X(r,\omega) = \text{DTFT}\{x[n]r^{-n}\}$$

$$= \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n}. \qquad (*)$$

- Now, $r \geq 0$ and both r and ω are <u>real</u>

- So $re^{j\omega}$ can be thought of as a complex number in <u>polar</u> form.

  - The magnitude is the fixer-upper parameter r.
  - The angle is the DTFT frequency variable ω.

- To get the z-transform, we simply name this complex number $re^{j\omega}$... we call it "z".

- Then, eq. (*) on p. 6.16 becomes:

$$X(r,\omega) = \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\equiv X(z)$$, the <u>z-transform</u> of $x[n]$

- The z-transform $X(z)$ is a complex-valued function that contains the DTFT's of the fixed up guys $x[n]r^{-n}$ for <u>all</u> choices of $r \geq 0$.
- These DTFT's are all graphed above a <u>plane</u> where the independent variable is $z = re^{j\omega}$
- This plane is called the "z-plane" or sometimes the "complex z-plane."
    - Above circles where the DTFT of the fixed up guy $x[n]r^{-n}$ does <u>not</u> blow up, the z-transform $X(z)$ <u>converges</u>.

- The set of circles in the z-plane where $X(z)$ converges (does <u>not</u> blow up) is called the "Region of Convergence" or "ROC".

- Above circles where the DTFT of $x[n]r^{-n}$ blows up, $X(z)$ also diverges obviously.
  $\Rightarrow$ These circles in the z-plane where $X(z)$ blows up are <u>not</u> part of the ROC.

- Remember: for a "bad" $x[n]$, there will generally be some set of choices for r that <u>help</u> and make the DTFT of $x[n]r^{-n}$ converge. The set of circles in the z-plane with these r for radii is the "region of convergence" or ROC of $X(z)$.

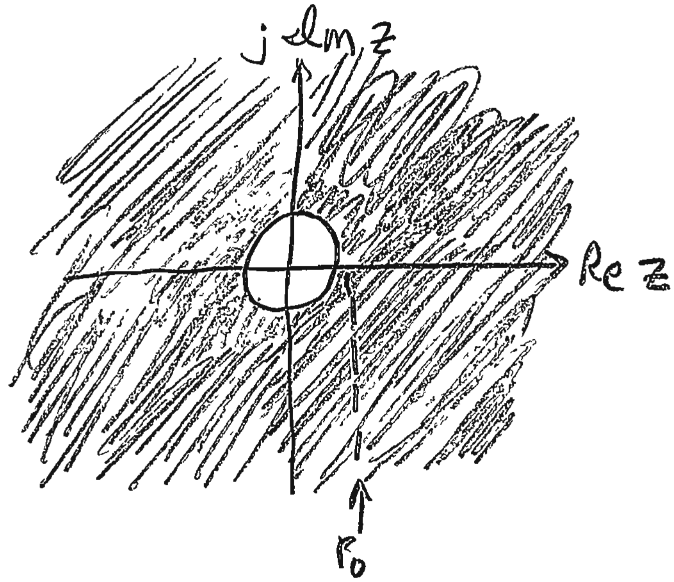
  - Likewise, there will generally be some set of choices for r that don't help or make things worse... for these r, the DTFT of $x[n]r^{-n}$ blows up... in other words, $X(z)$ blows up for these choices of r.
  - The ROC of $X(z)$ does <u>not</u> include the circles in the z-plane that have radii equal to these "bad" choices of r.

- We don't usually try to plot the z-transform $X(z)$ by hand.

- But we frequently <u>do</u> need to plot the ROC of $X(z)$ in the z-plane.

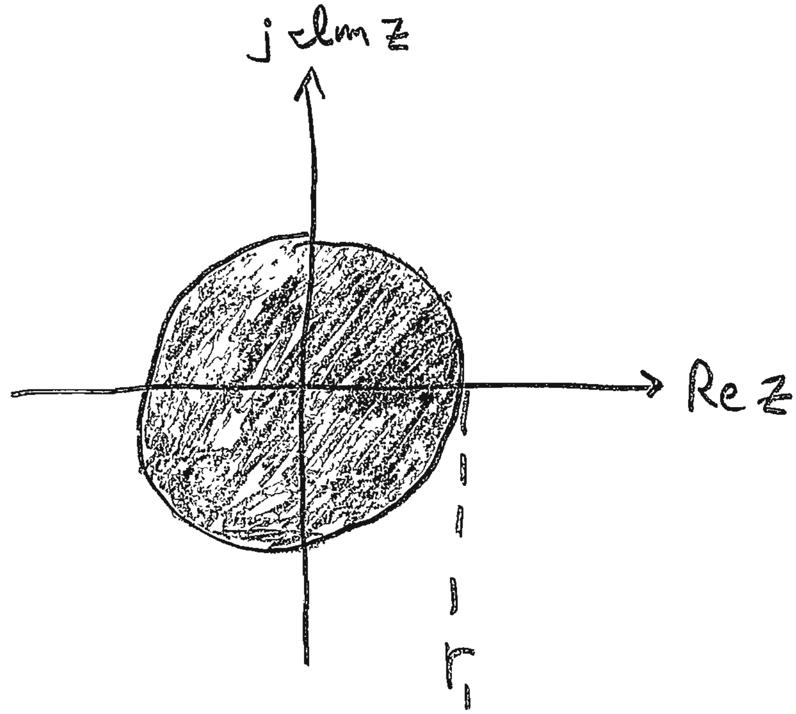
  - There are several possibilities:

    - The ROC can be the union of all circles with radii <u>greater</u> than some minimum value... call it r_0:



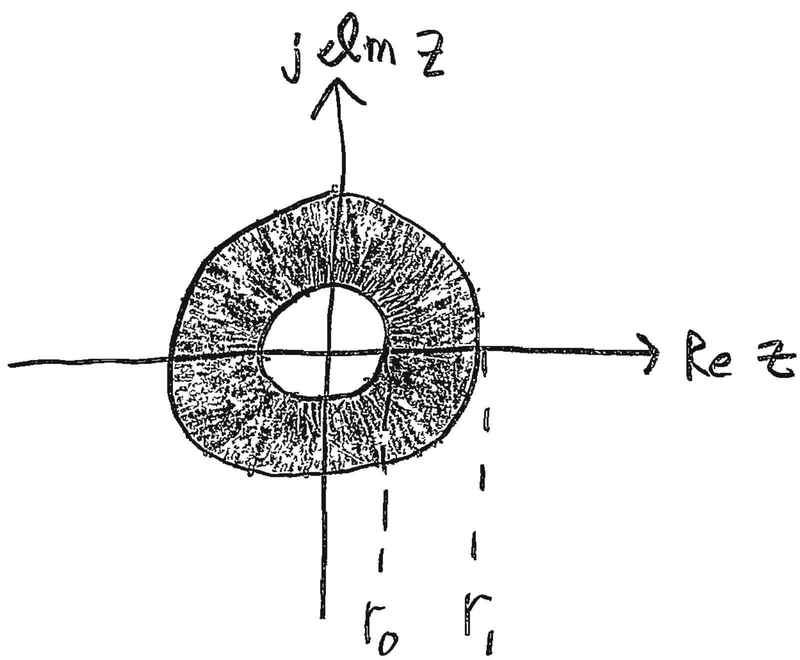
- This is called an <u>exterior</u> ROC.
- It includes all the circles with radius $> r_0$.

- Or the ROC can be the set of all circles with radius <u>less</u> than some maximum value r_1:



- This is called an <u>interior</u> ROC.
- It includes all the circles with radius $< r_1$.
- As we will see later, it <u>might</u> or <u>might not</u> include the circle of radius <u>zero</u>, which is really just the single point $z=0$.

- Or the ROC could be the union of all the circles with radii <u>between</u> some minimum value r_0 and some maximum value r_1:



- This is called an <u>annular</u> ROC.
- It contains all the circles in the z-plane such that $r_0 < r < r_1$.

- Finally, the ROC might be the <u>whole</u> z-plane, or the whole z-plane <u>except</u> the point $z = 0$.

ALWAYS REMEMBER: z is a complex variable. We think of it as $z = re^{j\omega}$, where r, the magnitude of z, is the fixer upper parameter and ω, the angle of z, is the DTFT frequency variable.

- To invert the z-transform, we can take the inverse DTFT around any circle in the z-plane where the DTFT of $x[n]r^{-n}$ does not blow up.
  - In other words, we can integrate $X(z)e^{j\omega n}$ around any circle that is in the ROC.
  - This will give us back $x[n]r^{-n}$. We have to multiply that by r^{n} to get back $x[n]$.
- In other words, we invert the z-transform by integrating $X(z)r^{n}e^{j\omega n}$ around any circle in the z-plane that is included in the ROC.

- This requires performing contour integration of $X(z)$.

  → You have probably not yet had a math class where contour integration was taught.

  → Therefore, in ECE 2713 we will <u>not</u> compute inverse z-transforms from the definition, which is a complex contour integral.

  → Instead, we will compute inverse z-transforms by table lookup.

- Now, finally, here are the formal definitions of the z-transform and the inverse z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- The formula sheets for Test 2 and the Final Exam that are available on the course web site have a table of z-transform pairs and a table of z-transform properties.

<u>Note</u>: the circle of radius $r = 1$ in the z-plane is <u>very</u> important. It is called the "unit circle."

<u>Recall</u>: the z-transform is the DTFT of $x[n]\, r^{-n}$, including all choices of $r > 0$.

- When $r = 1$, we get

$$X(z) = \text{DTFT}\{x[n]\, 1^{-n}\}$$
$$= \text{DTFT}\{x[n]\}$$
$$= X(e^{j\omega}).$$

$\Rightarrow$ So if $x[n]$ <u>does</u> have a DTFT $X(e^{j\omega})$, then the unit circle <u>is</u> included in the ROC of $X(z)$.

$\Rightarrow$ If $x[n]$ does <u>not</u> have a DTFT $X(e^{j\omega})$, then the unit circle is <u>not</u> included in the ROC of $X(z)$.

- Okay, let's work a z-transform.

<u>EX</u>: $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} \qquad \left(\text{because } u[n]=0 \ \forall n<0\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n \qquad (*)$$

- On our formula sheet, we have a sum formula that says

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha},$$

but <u>only</u> if $|\alpha| < 1$!!!

$\Rightarrow$ if $|\alpha| \geq 1$, the sum is <u>not</u> equal to $\frac{1}{1-\alpha}$!!!

$\Rightarrow$ if $|\alpha| > 1$, the sum blows up!!!

- In eq. $(*)$ above, we've got $\alpha = \frac{1}{2} z^{-1}$,

so $|\alpha| = \frac{1}{2} \cdot \frac{1}{|z|}$

- For this to be < 1, we've got to have

$\frac{1}{|z|} < 2 \Rightarrow |z| > \frac{1}{2}$!!!

- otherwise, if $|z| \leq \frac{1}{2}$, the sum in eq. (*) on page 6.25 blows up.

- Applying the sum formula, we get

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad \underbrace{|z| > \frac{1}{2}}_{\text{ROC}}$$

→ The ROC of $X_1(z)$ is $|z| > \frac{1}{2}$.

→ For all z outside the circle of radius $\frac{1}{2}$, $X(z)$ converges and is equal to $\frac{1}{1-\frac{1}{2}z^{-1}}$.

→ for all z on or inside the circle of radius $\frac{1}{2}$, $X(z)$ blows up.

– Now let's do another one:

<u>EX</u>: $x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} = -\sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[-n-1]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\tfrac{1}{2}\right)^n z^{-n} \qquad \left(\text{because } u[-n-1] = 0 \ \forall n > -1\right)$$

$$= -\sum_{n=-\infty}^{-1} \left(\tfrac{1}{2} z^{-1}\right)^n$$

Let $k = -n-1 \longrightarrow n = -k-1$

when $n = -\infty$, $k = \infty$

when $n = -1$, $k = 1-1 = 0$

$$X_2(z) = -\sum_{k=\infty}^{0} \left(\tfrac{1}{2} z^{-1}\right)^{-k-1}$$

$$= -\sum_{k=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^{-k-1} \qquad \left(\text{order of addition does not matter}\right)$$

$$= -\sum_{k=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^{-k} \left(\tfrac{1}{2} z^{-1}\right)^{-1}$$

$$= -\sum_{k=0}^{\infty} (2z)^k\, 2z = -2z \sum_{k=0}^{\infty} (2z)^k \qquad (*)$$

→ Sum formula: this sum converges if:

$$|2z| < 1$$

$$2|z| < 1$$

$$|z| < \tfrac{1}{2}$$

→ and blows up if $|z| \geq \frac{1}{2}$.

- Thus, provided that $|z| < \frac{1}{2}$, we get from eq. $(*)$ on p. 6.27 that:

$$X_2(z) = (*) = -2z \sum_{k=0}^{\infty} (2z)^k$$

$$= -2z \cdot \frac{1}{1-2z} = \frac{-2z}{1-2z}$$

$$= \frac{-2z}{1-2z} \cdot \frac{\frac{1}{2}z^{-1}}{\frac{1}{2}z^{-1}}$$

$$= \frac{-1}{\frac{1}{2}z^{-1} - 1} = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \underbrace{|z| < \frac{1}{2}}_{\text{ROC}}.$$

- Comparing this to p. 6.26, we have:

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$\Rightarrow$ This shows that the ROC is <u>very</u> important.

- Without the ROC, you might think that $X_1(z)$ and $X_2(z)$ are equal.

$\Rightarrow$ But they are <u>not</u>. $\longrightarrow$

$X_1(z)$ is equal to $\frac{1}{1-\frac{1}{2}z^{-1}}$, but only for z such that $|z| > \frac{1}{2}$.

→ on the other z's, $X_1(z)$ is not equal to $\frac{1}{1-\frac{1}{2}z^{-1}}$. On the z's such that $|z| \le \frac{1}{2}$, $X_1(z)$ blows up or diverges.

$X_2(z)$ is equal to $\frac{1}{1-\frac{1}{2}z^{-1}}$, but only for z such that $|z| < \frac{1}{2}$.

→ on the other z's, $X_2(z)$ is not equal to $\frac{1}{1-\frac{1}{2}z^{-1}}$. On the z's such that $|z| \geq \frac{1}{2}$, $X_2(z)$ blows up or diverges.

⟹ Thus, there are no values of z where $X_1(z) = X_2(z)$.

→ On the z's where $X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$, $X_2(z)$ blows up.

→ On the z's where $X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$, $X_1(z)$ blows up.

$\Rightarrow$ So even though $X_1(z)$ and $X_2(z)$ both have the same functional form (namely $\frac{1}{1-\frac{1}{2}z^{-1}}$), there is no z where the two transforms are equal... because the ROC's are different.

- In fact, they are <u>disjoint</u>. The set of z that are in <u>both</u> ROC's is <u>empty</u>.

- Now let's see how we would use the <u>table</u> to work these two transforms.
  - The table of z-transforms is found on p. 7 of the formula sheet for Test 2.

- It says:

Signal	Transform	ROC	
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$	(*)
$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$	(**)

- Given $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$, we use line (*) on p. 6.30 with $\alpha = \frac{1}{2}$ to write down the transform

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}.$$

- Given $x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$, we use line (**) on p. 6.30 with $\alpha = \frac{1}{2}$ to write down the transform

$$X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}.$$

- You can also use the table to write down inverse z-transforms... but notice that you <u>must</u> know the ROC to do this.

    - If you were just given that $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$, but you were not given the ROC, then you wouldn't know whether to use line (*) on p. 6.30 to find $x[n]$ or whether to use line (**).

→ But if you were given that $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$ and the ROC was $|z| > \frac{1}{2}$, then you would know that $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

→ Similarly, if you were given that $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$ and the ROC was $|z| < \frac{1}{2}$, then you would know that $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$.

☆☆ The Z-transforms that we will see in ECE 2713 will generally be a ratio of two polynomials in the "character" z^{-1}.

- This is very similar to what we saw for the DTFT... we saw that our discrete-time Fourier transforms were generally a ratio of two polynomials in the DTFT "character" $e^{-j\omega}$.

- As an example, suppose that we are given

$$X(z) = \frac{1-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}, \quad |z| > \frac{1}{2} \qquad (*)$$

- If we multiply out the denominator (using the "foil" rule), we get:

$$X(z) = \frac{1-z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2} \qquad (**)$$

→ a ratio of two polynomials in z^{-1}.

- As with the DTFT, you need to be able to go back and forth between the form (*) on p. 6.32, where the denominator (and numerator) are <u>factored</u>...
  - and the form (**) on p. 6.32, where the denominator (and numerator) are <u>multiplied out</u>.

$\Rightarrow$ You go back and forth between these two forms of the z-transform <u>exactly the same</u> as you did with the DTFT.

→ But with the z-transform, going from the "multiplied out form" (**) to the factored form (*) will often require you to do more arithmetic with fractions than was required with the DTFT.

- Nevertheless, you do it exactly the same way as you did for the DTFT.

- The nice thing about the factored form is that it explicitly shows the <u>roots</u> of the numerator and denominator polynomials.

<u>Note 1</u>: the <u>roots</u> of a polynomial are the values of the independent variable that make the polynomial equal to <u>zero</u>.

<u>Note 2</u>: An n'th order polynomial has n roots.

- If the order is 2 (quadratic), then there are 2 roots.

- In our example $X(z)$ from p. 6.32, the denominator is $(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1}) = 1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}$

$\Rightarrow$ If you plug in $z=\frac{1}{2}$, the term $(1-\frac{1}{2}z^{-1})$ becomes $(1-\frac{1}{2}\cdot 2) = (1-1) = 0$, so the denominator is <u>zero</u> when $z=\frac{1}{2}$.

$\Rightarrow$ If you plug in $z=-\frac{1}{3}$, the term $(1+\frac{1}{3}z^{-1})$ becomes $(1+\frac{1}{3}(-3)) = (1-1) = 0$, so the denominator is <u>zero</u> when $z=-\frac{1}{3}$.

- In general, when you write a z-transform like $X(z)$ in factored form, the numerator and denominator will each be a <u>product</u> of terms of the form:

$$(1 - \alpha z^{-1}).$$

→ This term is <u>zero</u> when $z = \alpha$...

→ Because if we plug in $z = \alpha$, we get

$$(1 - \alpha \cdot \tfrac{1}{\alpha}) = (1 - 1) = 0.$$

- So again looking back to the <u>factored</u> form of our example $X(z)$ in eq. (*) on p. 6.32, we have:

$$X(z) = \frac{\overbrace{1 - z^{-1}}^{\alpha = 1}}{\underbrace{(1 - \tfrac{1}{2} z^{-1})}_{\alpha = \frac{1}{2}}\underbrace{(1 + \tfrac{1}{3} z^{-1})}_{\alpha = -\frac{1}{3}}}$$

- The numerator has <u>one</u> root at $z = 1$

- The denominator has <u>two</u> roots at $z = \frac{1}{2}$ and $z = -\frac{1}{3}$.

DEF: a rational z-transform is a z-transform that is a ratio of two polynomials in z^{-1}.

→ Most if not all of the z-transforms you will see in ECE 2713 will be rational z-transforms.

⇒ The roots of the denominator are the values of z that make the denominator zero.

→ These values make $X(z)$ blow up, because they make the denominator of $X(z)$ equal to zero.

→ These values of z are called the poles of $X(z)$. ☆☆☆

⇒ The roots of the numerator are the values of z that make the numerator zero.

→ These values of z make $X(z)$ zero, because they make the numerator of $X(z)$ equal to zero.

→ These values of z are called the zeros of $X(z)$. ☆☆☆

Summary: The roots of the denominator of $X(z)$ are the poles of $X(z)$.

The roots of the numerator are the zeros of $X(z)$.

- The Poles of $X(z)$ are the values of z that make $X(z)$ blow up.
- The zeros of $X(z)$ are the values of z that make $X(z)$ equal to zero.
- When you write $X(z)$ in factored form, the poles and zeros are shown explicitly.

EX: from p. 6.32,

$$X(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

$$= \frac{1 - (1)z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - (-\frac{1}{3})z^{-1})}$$

zeros: $z = 1$

poles: $z = \frac{1}{2}, -\frac{1}{3}$

Note: For any rational z-transform, there may also be additional poles and or zeros at $z=0$ and $z=\infty$.

→ You always have to check for these.

→ The easiest way to do this is to convert the numerator and denominator from z^{-1} to z.

- Then it will be easy to see if there are additional poles and/or zeros at $z=0$ and/or $z=\infty$.

- To do this, you multiply $X(z)$ by $1=\frac{z^M}{z^M}$, where $M=$ highest power of z^{-1} that occurs in either the numerator or the denominator.

EX from p. 6.32:

$$X(z)=\frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

$$=\frac{1-z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}$$

- The highest power of z^{-1} that occurs is z^{-2} in the denominator... so $M=2$.
- To convert $X(z)$ from z^{-1} to z, multiply $X(z)$ by $1 = \frac{z^2}{z^2}$.

we get:

$$X(z) = \frac{1-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \cdot \frac{z^2}{z^2}$$

$$= \frac{z\left[z\left(1-z^{-1}\right)\right]}{\left[z\left(1-\frac{1}{2}z^{-1}\right)\right]\left[z\left(1+\frac{1}{3}z^{-1}\right)\right]}$$

$$= \frac{z(z-1)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

- From this, we can see:
  - The zero we already found at $z=1$
  - The poles we already found at $z=\frac{1}{2}, -\frac{1}{3}$
- In addition, we see that there is an <u>additional</u> zero at $z=0$... because this will make the numerator <u>zero</u>.

- There is <u>not</u> any extra pole or zero at $z=\infty$ because

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z(z-1)}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

$$= \lim_{z\to\infty} \frac{z^2}{z^2}$$

because;

$$\lim_{z\to\infty} (z-1) = \lim_{z\to\infty} z = \infty$$

$$\lim_{z\to\infty} (z-\tfrac{1}{2}) = \lim_{z\to\infty} = \infty$$

$$\lim_{z\to\infty} (z+\tfrac{1}{3}) = \lim_{z\to\infty} = \infty$$

$$= \lim_{z\to\infty} 1$$

$$= 1.$$

$\longrightarrow$ So $z=\infty$ is <u>not</u> a zero of $X(z)$ and it is <u>not</u> a pole.

$\Rightarrow$ So here is a <u>complete</u> list of <u>all</u> the poles and zeros of $X(z)$:

zeros: $z = 1, 0$

poles: $z = \frac{1}{2}, -\frac{1}{3}$

RECAP : to find the poles and zeros of a rational z-transform $X(z)$:

1. Write $X(z)$ in terms of z^{-1}.
2. read off the roots of the numerator. These are the values α appearing in the numerator terms $(1-\alpha z^{-1})$. They are zeros of $X(z)$.
3. read off the roots of the denominator. These are the values α appearing in the denominator terms $(1-\alpha z^{-1})$. They are poles of $X(z)$.
4. Convert $X(z)$ from z^{-1} to z as we demonstrated on p. 6.39.
5. Plug in the value $z=0$ to check if it gives you an additional pole or zero.
6. Plug in the value $z=\infty$ to check if it gives you an additional pole or zero.

- Often, we will need to make a pole-zero plot for $X(z)$. This is a graph that shows the poles and zeros in the z-plane
  - you plot the poles using the symbol "x"
  - you plot the zeros using the symbol "o"
  - you should <u>always</u> draw the unit circle on your plot.
- The pole-zero plot is also sometimes called a "p-z plot" or a "p/z plot".

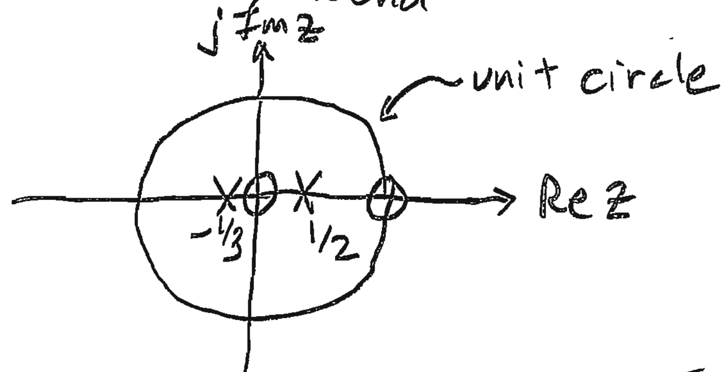
<u>NOTE</u>: the pole-zero plot does <u>not</u> depend on the ROC. You don't need to know the ROC to draw the pole-zero plot.

<u>EX</u>: (Same one from p. 6.32 again)

$$X(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

$$\left.\begin{array}{ll} \text{poles:} & z = \tfrac{1}{2}, -\tfrac{1}{3} \\ \text{zeros:} & z = 1, 0 \end{array}\right\}$$ As we already found

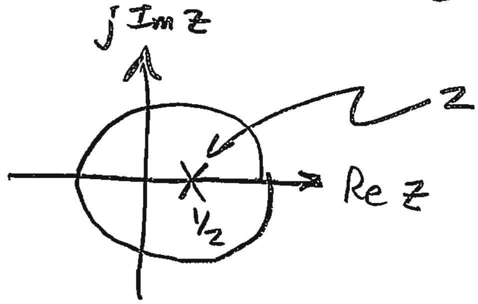
Give a pole-zero plot:



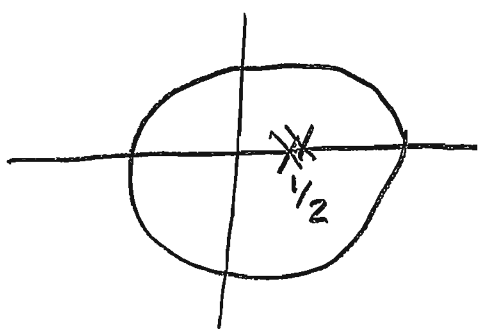
- There may be repeated roots in the numerator or denominator.

  - For example, if you had $(1-\frac{1}{2}z^{-1})^2$ in the denominator, this would be a repeated pole with multiplicity 2.

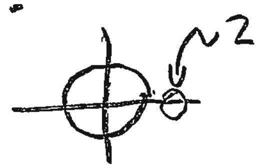
  - You draw it like this in the pole-zero plot:



  - Some people draw it with two X's instead like this:



    - But that gets messy, especially if the multiplicity is > 2.

- You draw repeated zeros the same way. 

More About the ROC of $X(z)$

- Here are some more facts about the ROC of $X(z)$. You will prove them in ECE 3793.

- If $x[n]$ is <u>right-sided</u>, like $x[n] = \left(\frac{1}{2}\right)^n u[n]$ (starts at $n=0$) or $x[n] = \left(\frac{1}{3}\right)^{n+2} u[n+2]$ (starts at $n=-2$) for example,
  - Then the ROC of $X(z)$ is <u>exterior</u>.
  - Generally, it's everything in the z-plane <u>outside</u> the largest magnitude pole.

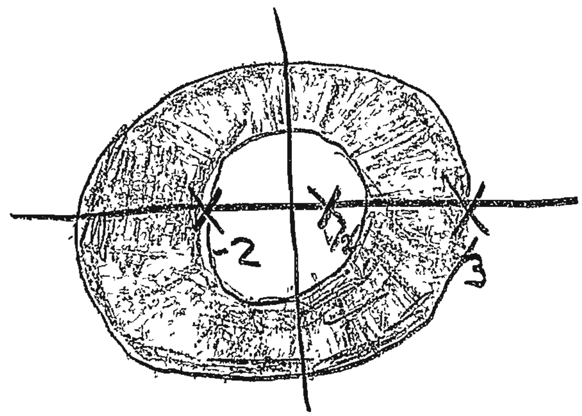
- If $x[n]$ is <u>left-sided</u>, like $x[n] = 2^n u[-n]$ or $x[n] = -3^n u[-n-1]$, then the ROC of $X(z)$ is <u>interior</u>.
  - Generally, it's everything in the z-plane <u>inside</u> the smallest magnitude pole.

More About The ROC...

- If $x[n]$ is <u>two-sided</u>, like $x[n] = \left(\frac{1}{2}\right)^{|n|}$ or $x[n] = \left(\frac{1}{2}\right)^n u[n] + 3^n u[-n]$, then the ROC of $X(z)$ is <u>annular</u>.
  - Generally, it's everything in the z-plane that lies in a donut <u>between</u> two of the poles.

<u>EX</u>:

Everything in a donut between the poles at $z=-2$ and $z=3$.



- Note that "between" here means that the donut is made of circles with radii r lying <u>between</u> the magnitude of the pole at $z=-2$ and the magnitude of the pole at $z=3$.

- For the z-transform, the ROC is <u>always</u> a union of <u>circles</u>.

More About ROC...

- If $x[n]$ is finite length, like

$$x[n] = \delta[n] + 2\delta[n-1] - 3\delta[n-2]$$ or

$$x[n] = \left(\tfrac{1}{2}\right)^n \{u[n] - u[n-5]\},$$ then

the ROC of $X(z)$ will be the whole z-plane except possibly the point $z=0$.

  - Remember: you always have to check the two points $z=0$ and $z=\infty$.
  - Any rational $X(z)$ could have a pole or a zero at $z=0$.
  - If there is a pole at $z=0$, then the point $z=0$ is not in the ROC.

$\Rightarrow$ The ROC of $X(z)$ is the set of z in the z-plane where $X(z)$ converges.

$\Rightarrow$ Poles are the values of z where $X(z)$ blows up.

$\Rightarrow$ There can't be any poles in the ROC. ★★★

- Here are some of the shorthand ways that we write "$x[n]$ and $X(z)$ are a z-transform pair":

$$X(z) = \mathcal{Z}\{x[n]\} \qquad x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

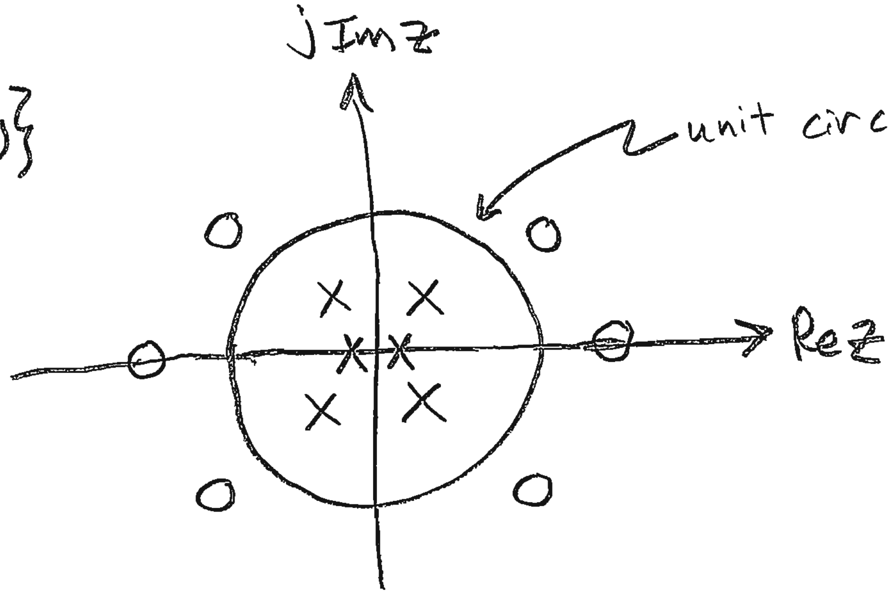
$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

<u>Note</u>: for all the z-transforms we have seen so far, the poles and zeros have been <u>real</u>.

- In ECE 2713 we will try to stick to cases where the poles and zeros <u>are</u> real.
- But you must be aware that this is not the case in general.
  - Especially for "real world" digital filters, the poles and zeros will generally be <u>complex</u>.

- Here is an example of what the pole-zero plot of $H(z)$ might look like for a real-world digital filter with impulse response $h[n]$:

$H(z) = \mathcal{Z}\{h[n]\}$



- Here, $H(z)$ has:
  - Two real poles
  - Four complex poles
  - Two real zeros
  - Four complex zeros

FACT: if $X(z)$ is a rational z-transform,
- and if the numerator and denominator polynomials have real coefficients,
- Then the poles are real or occur in complex conjugate pairs.
- The zeros are also real or occur in complex conjugate pairs.

- Now here is an example of finding a z-transform when there are two terms:

Given: $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

Table w/ $\alpha = \frac{1}{2}$: $\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \ |z| > \frac{1}{2}$

Table w/ $\alpha = -\frac{1}{3}$: $\left(-\frac{1}{3}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \ |z| > \frac{1}{3}$

So $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \underbrace{\frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}}_{\text{one}} + \frac{1}{1+\frac{1}{3}z^{-1}} \cdot \underbrace{\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}}_{\text{one}}$$

one ← to get a common denominater → one

$$= \frac{1+\frac{1}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} + \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

$$= \frac{1+1+\frac{1}{3}z^{-1}-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{2+\frac{2}{6}z^{-1}-\frac{3}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

$$= \frac{2-\frac{1}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{2(1-\frac{1}{12}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} \longrightarrow$$

- So the "factored" form of $X(z)$ is:

$$X(z) = \frac{z(1 - \frac{1}{12}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

- What about the ROC?

  - $X(z)$ was a sum of two terms:

$$\frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \qquad (\alpha = \tfrac{1}{2} \text{ for this term})$$

$$\frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3} \qquad (\alpha = -\tfrac{1}{3} \text{ for this term})$$

  - The overall ROC of $X(z)$ is the set of all z that make <u>both</u> terms converge.
  - In other words, it is the <u>intersection</u> of the two ROC's of the individual terms.
  - In other words,

$$\begin{aligned} ROC\{X(z)\} &= \{|z| > \tfrac{1}{2}\} \cap \{|z| > \tfrac{1}{3}\} \\ &= \{|z| > \tfrac{1}{2}\}. \end{aligned}$$

$\longrightarrow$

- So the answer is:

$$X(z) = \frac{2(1-\frac{1}{12}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{2}$$

- Now let's do a pole-zero plot for this $X(z)$.
  - From the factored form shown above, we can just read off:
    zeros: $z = \frac{1}{12}$
    poles: $z = \frac{1}{2}, -\frac{1}{3}$
  - But don't forget:
    - we must always <u>check</u> the two points $z=0$ and $z=\infty$.
    - To do this, it's easiest to convert $X(z)$ from "z^{-1}" to "z".
    - We see that the highest power of z^{-1} upstairs or downstairs is z^{-2} in the denominator,
    - So multiply $X(z)$ by $\frac{z^2}{z^2}$

$\longrightarrow$

$$X(z) = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \cdot \frac{z^2}{z^2}$$

$$= \frac{2z\left[z\left(1-\frac{1}{12}z^{-1}\right)\right]}{\left[z\left(1-\frac{1}{2}z^{-1}\right)\right]\left[z\left(1+\frac{1}{3}z^{-1}\right)\right]}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}, \quad |z| > \frac{1}{2}$$

- plugging in $z=0$, we see that there is an <u>additional</u> zero at $z=0$.

- plugging in $z=\infty$, we get:

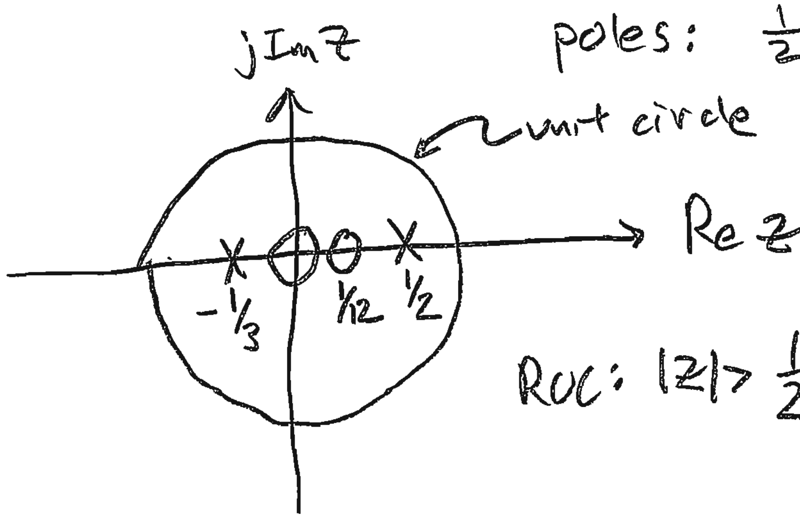
$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{2z^2}{z^2} = \lim_{z\to\infty} 2 = 2$$

→ So $z=\infty$ is <u>not</u> a pole or a zero.

- All together then, we have: zeros: $\frac{1}{12}$, 0

poles: $\frac{1}{2}$, $-\frac{1}{3}$

p/z-plot:



ROC: $|z| > \frac{1}{2}$

- Now let's work this the other way... in other words, given $X(z)$ let's find $x[n]$.

- The "multiplied out" form of $X(z)$ is:

$$X(z) = \frac{2(1-\frac{1}{12}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{2-\frac{1}{6}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-1}-\frac{1}{6}z^{-2}}$$

$$= \frac{2-\frac{1}{6}z^{-1}}{1-\frac{3}{6}z^{-1}+\frac{2}{6}z^{-1}-\frac{1}{6}z^{-2}}$$

$$= \frac{2-\frac{1}{6}z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2}.$$

- So the problem would be:

Given: $X(z) = \dfrac{2-\frac{1}{6}z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2}$

Find: $x[n]$.

- To solve this, we must:

1. Factor the denominator
2. Do a PFE
3. Deduce the ROC's for the individual terms
4. Find $x[n]$ by table lookup.

Solution

$$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{2 - \frac{1}{6}z^{-1}}{(1 - az^{-1})(1 - bz^{-1})}$$

- we need a and b such that:

(*) $-(a+b) = -\frac{1}{6}$

(**) $ab = -\frac{1}{6}$ (this means a and b must have different signs)

- From (**), our choices are

$a = 1 \quad b = -\frac{1}{6} \qquad a = \frac{1}{2} \quad b = -\frac{1}{3}$

$a = -1 \quad b = \frac{1}{6} \qquad a = -\frac{1}{2} \quad b = \frac{1}{3}$

- Quickly checking all the sums, we see that <u>only</u> $a = \frac{1}{2}$, $b = -\frac{1}{3}$ can give us $-(a+b) = -(\frac{1}{2} - \frac{1}{3}) = -\frac{1}{6}$ ✓

- So $$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{2}$$

→

- Next, we do the PFE using the Heaviside cover up method:

$$X(z) = \frac{2 - \frac{1}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{3}z^{-1}}$$

- Writing θ instead of z^{-1} to simplify things (just like when we wrote θ for $e^{-j\omega}$ with the DTFT), we get:

$$\frac{2-\frac{1}{6}\theta}{(1-\frac{1}{2}\theta)(1+\frac{1}{3}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1+\frac{1}{3}\theta}$$

$$A = \left.\frac{2-\frac{1}{6}\theta}{1+\frac{1}{3}\theta}\right|_{\underbrace{\theta=2}_{\text{this is the value of } \theta \text{ that makes } 1-\frac{1}{2}\theta = 0.}} = \frac{2-\frac{2}{6}}{1+\frac{2}{3}} = \frac{\frac{12}{6}-\frac{2}{6}}{\frac{3}{3}+\frac{2}{3}} = \frac{\frac{10}{6}}{5/3}$$

$$= \frac{3}{5}\cdot\frac{10}{6}$$

$$= \frac{3}{5}\cdot\frac{5}{3} = \underline{\underline{1}}$$

$$B = \left.\frac{2-\frac{1}{6}\theta}{1-\frac{1}{2}\theta}\right|_{\underbrace{\theta=-3}_{\text{makes } (1+\frac{1}{3}\theta)=0}} = \frac{2+\frac{3}{6}}{1+\frac{3}{2}} = \frac{2+\frac{1}{2}}{1+\frac{3}{2}} = \frac{5/2}{5/2} = \underline{\underline{1}}$$

– So $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} + \dfrac{1}{1+\frac{1}{3}z^{-1}}$, $|z| > \frac{1}{2}$

→ Now we must figure out the two ROCs, for the individual terms so that they <u>intersect</u> to the overall ROC $|z| > \frac{1}{2}$.

→ Looking at p.7 of the formula sheet for Test 2, we see that

$$\frac{1}{1-\alpha z^{-1}}, \ |z| > |\alpha| \quad \overset{Z}{\longleftrightarrow} \quad \alpha^n u[n]$$

$$\frac{1}{1-\alpha z^{-1}}, \ |z| < |\alpha| \quad \overset{Z}{\longleftrightarrow} \quad -\alpha^n u[-n-1]$$

- For the first term $\frac{1}{1-\frac{1}{2}z^{-1}}$, we've got $\alpha = \frac{1}{2}$. So the choices for the ROC of this term are $|z| > \frac{1}{2}$ or $|z| < \frac{1}{2}$.
- For the second term $\frac{1}{1+\frac{1}{3}z^{-1}}$, we've got $\alpha = -\frac{1}{3}$. So the choices for the ROC of this term are $|z| > \frac{1}{3}$ and $|z| < \frac{1}{3}$.

$\Rightarrow$ The only way we can get the intersection to be $|z| > \frac{1}{2}$ (The given overall ROC for $X(z)$) is if:

- The ROC for the first term is $|z| > \frac{1}{2}$
- The ROC for the second term is $|z| > \frac{1}{3}$

- So, our PFE with ROCs is:

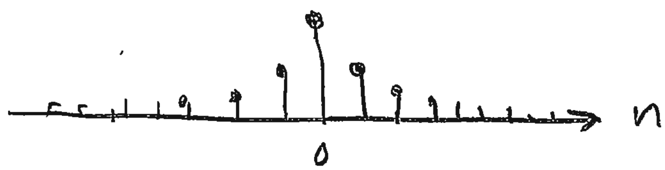
$$X(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{1}{1+\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}}$$

- Table:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \quad \checkmark$$

- Let's do another example where $x[n]$ is <u>two sided</u>.

Given: $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$



- Note that $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$. So $x[n] = \left(\frac{1}{2}\right)^n$ for $n \geq 0$.

- Also, $u[-n-1] = \begin{cases} 1, & n \leq -1 \\ 0, & n > -1 \end{cases}$. So $x[n] = 2^n$ for $n \leq -1$.

→ Looking in the table, we see that we can handle the first term with the transform pair $\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \; |z| > |\alpha|$

→ For the second term, we will have to use the transform pair

$$-\alpha^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \; |z| < |\alpha|.$$

NOTE: This means:

$$-[\alpha^n] u[-n-1], \quad \checkmark$$

<u>NOT</u> $(-\alpha)^n u[-n-1]$ XXX

- So we need to re-write $x[n]$ as

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] - \quad -2^n u[-n-1]$$

Table: $\left(\tfrac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$

Table: $-2^n u[-n-1] \overset{z}{\longleftrightarrow} \dfrac{1}{1-2z^{-1}}, \quad |z| < 2$

So $$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}$$

$$= \frac{1-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} - \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$$= \frac{1-2z^{-1}-1+\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{-2z^{-1}+\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

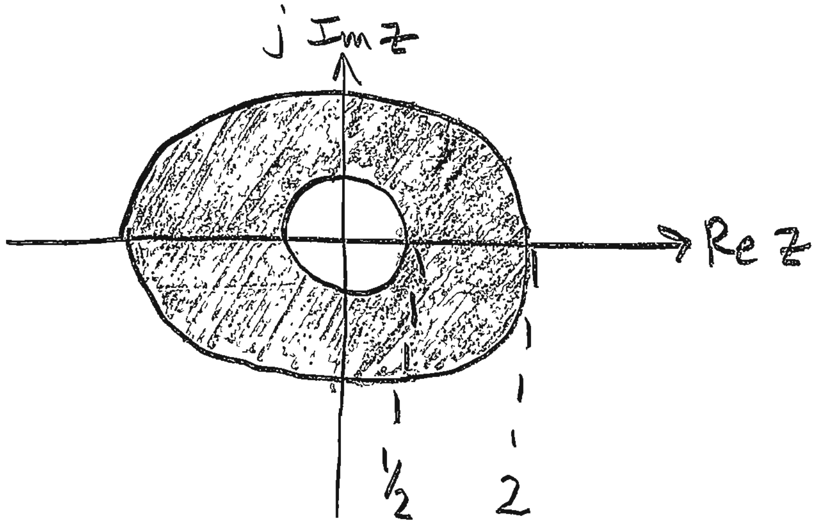
$$= \frac{-\frac{4}{2}z^{-1}+\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

- The ROC of $X(z)$ is the intersection of the ROCs, for the individual terms... i.e., it's all the z's that make both terms converge.
- This is: $\{|z| > \frac{1}{2}\} \cap \{|z| < 2\} = \frac{1}{2} < |z| < 2.$

- So the answer is:

$$X(z) = \frac{-\frac{3}{2} z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

- Here is what the ROC looks like:



two-sided $x[n] \longleftrightarrow$ annular ROC ✓

- Now let's do a pole-zero plot.

- Reading off the α's from the numerator and denominator of $X(z)$, we get:

→ There are no $(1-\alpha z^{-1})$ terms upstairs!!

→ Downstairs, we get: poles: $z = \frac{1}{2}, 2$

⟶

- Convert from "z^{-1}" to "z" and check the two special points $z=0$ and $z=\infty$:

$$X(z) = \frac{-\frac{3}{2} z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \cdot \frac{z^2}{z^2}$$

$$= \frac{-\frac{3}{2} z}{(z-\frac{1}{2})(z-2)}$$

- plugging in $z=0$, we get

$$X(0) = \frac{0}{1} = 0$$

- So there is a zero at $z=0$.

- plugging in $z=\infty$, we get

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{-z}{z^2} = \lim_{z\to\infty} -\frac{1}{z} = 0$$

- So there is also a zero at $z=\infty$.

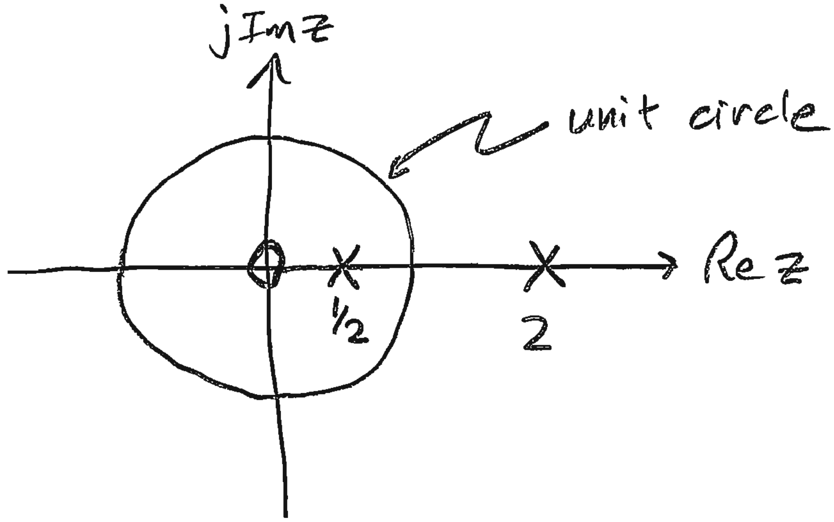
☆☆
☆☆☆ <u>NOTE</u>: For the pole-zero plot, we do <u>not</u> show poles and zeros that may occur at $z=\infty$. →

- For the pole-zero plot, we only show the finite poles and zeros.

- So we've got: zeros: $z = 0, \infty$

  poles: $z = \frac{1}{2}, 2$

p/z plot:



- Now let's work this one the other way... as an inverse transform example.

Given: $$X(z) = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

Find: $x[n]$

$\longrightarrow$

- Since $X(z)$ is already in factored form, we can jump straight to the PFE:

$$X(z) = \frac{-\frac{3}{2} z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-2z^{-1}}$$

$$A = \left.\frac{-\frac{3}{2}\theta}{1-2\theta}\right|_{\theta=2} = \frac{-3}{1-4} = \frac{-3}{-3} = 1$$

$$B = \left.\frac{-\frac{3}{2}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=\frac{1}{2}} = \frac{-3/4}{1-\frac{1}{4}} = \frac{-3/4}{3/4} = -1$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}, \quad \frac{1}{2} < |z| < 2$$

→ From the Table, our choices for the ROC of the first term are $|z| > \frac{1}{2}$ or $|z| < \frac{1}{2}$

→ Our choices for the ROC of the second term are $|z| > 2$ or $|z| < 2$.

⟹ For the intersection to be $\frac{1}{2} < |z| < 2$, they <u>must</u> be $|z| > \frac{1}{2}$ and $|z| < 2$. →

- With the ROC's, our PFE is now:

$$X(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{1}{1-2z^{-1}}}_{|z|<2}, \quad \frac{1}{2}<|z|<2$$

Table: $x[n] = \left(\frac{1}{2}\right)^n u[n] - -2^n u[-n-1]$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1] \checkmark$$

Important Note: Remember, the unit circle of the z-plane is where $r = 1$. This is where the fixer-upper is $r^{-n} = 1^{-n} = 1^n = 1$.

- It is also where $z = re^{jw} = e^{jw}$

- So, above the unit circle in the z-plane, $X(z) = X(e^{jw})$, the DTFT of $x[n]$.

$\Rightarrow$ If the <u>unit circle</u> is included in the ROC of $X(z)$, then $X(z)$ converges for $r = 1 \longrightarrow z = e^{j\omega}$.

$\Rightarrow$ This means that $X(e^{j\omega})$ **exists**.

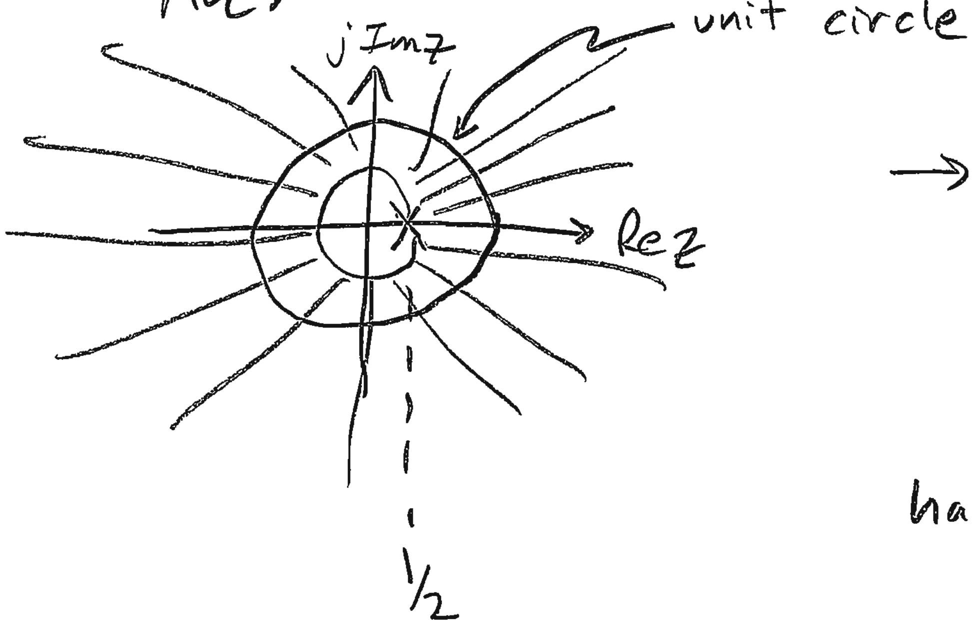
★★ $\Rightarrow$ In other words, if the ROC of $X(z)$ includes the unit circle, then $x[n]$ has a DTFT $X(e^{j\omega})$. ★★★

- In the example starting on page 6.49, we had

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]$$

$$X(z) = \frac{2\left(1 - \frac{1}{12}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}, \quad |z| > \frac{1}{2}$$

ROC:



$\rightarrow$ Since the ROC <u>does</u> include the unit circle, this $x[n]$ <u>does</u> have a DTFT $X(e^{j\omega})$.

- It is given by (just set $z = e^{j\omega}$ in $X(z)$):

$$X(e^{j\omega}) = \frac{2(1 - \frac{1}{12}e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}.$$

- In the example starting on p. 6.58, we had

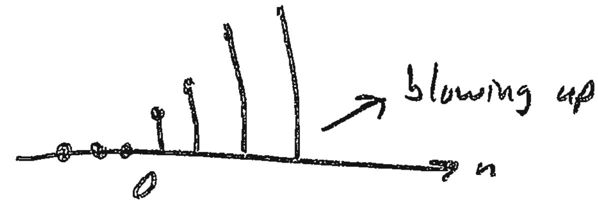
$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

$$X(z) = \frac{-3/2 \; z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \tfrac{1}{2} < |z| < 2$$

- See the plot of the ROC on p. 6.60.
  - The ROC <u>does</u> include the unit circle.
  - So this $x[n]$ also <u>does</u> have a DTFT.
  - It is given by:

$$X(e^{j\omega}) = \frac{2(1 - \frac{1}{12}e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})(1-2e^{-j\omega})}$$

EX: $x[n] = 2^n u[n]$

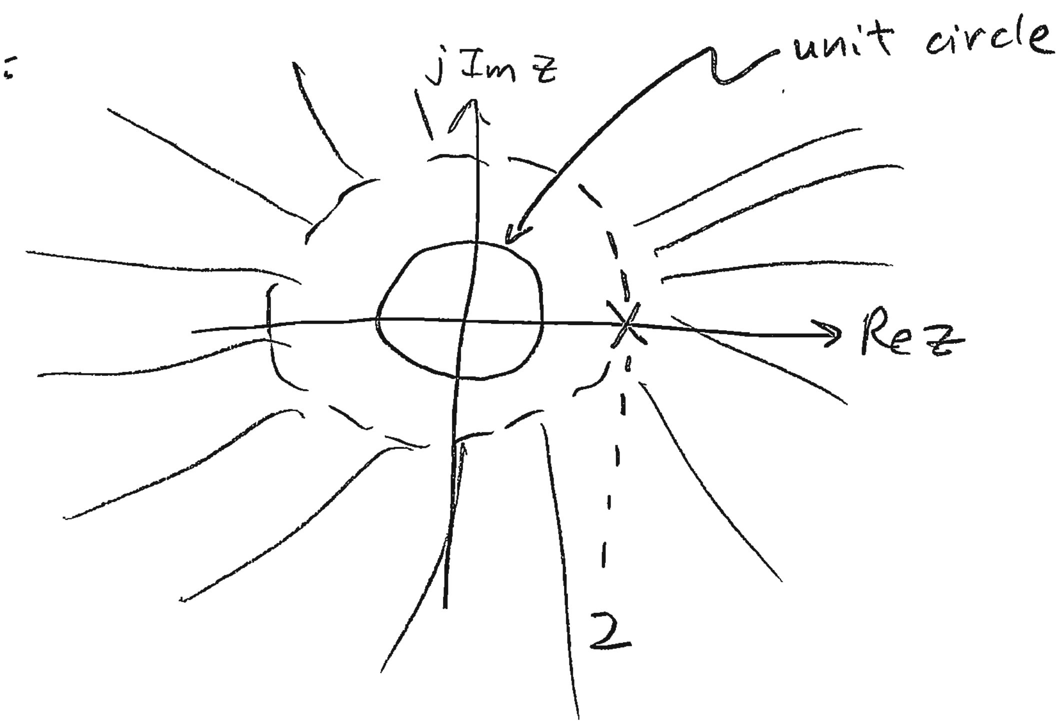


- This $x[n]$ does __not__ have a DTFT $X(e^{j\omega})$ because he is blowing up on the right side as $n \to \infty$.
- But he __does__ have a z-transform.

Table: $X(z) = \dfrac{1}{1-2z^{-1}}, \quad |z| > 2$

- In this case, the ROC of $X(z)$ is everything __outside__ the circle of radius 2.
- This ROC does __not__ contain the unit circle, which again shows that $X(e^{j\omega})$ does __not__ exist.

__p-z plot:__



<u>EX</u>: $x[n] = -\left(\frac{1}{3}\right)^n u[-n-1] - 2^n u[-n-1]$

Table: $-\left(\frac{1}{3}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| < \frac{1}{3}$

Table: $-2^n u[-n-1] \overset{Z}{\longleftrightarrow} \dfrac{1}{1-2z^{-1}}, \quad |z| < 2$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-2z^{-1}}$$

$$= \frac{1-2z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} + \frac{1-\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

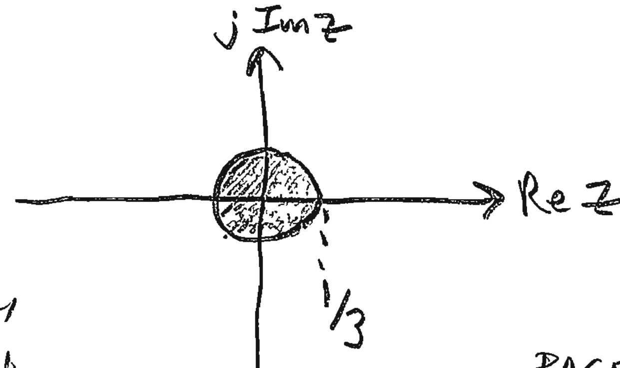
$$= \frac{1+1-2z^{-1}-\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

$$= \frac{2-\frac{6}{3}z^{-1}-\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

$$= \frac{2-\frac{7}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

ROC: $\{|z| < \frac{1}{3}\} \cap \{|z| < 2\} = \{|z| < \frac{1}{3}\}$

ROC:



- Since the ROC does <u>not</u> include the unit circle, $X(e^{j\omega})$ does <u>not</u> exist.

- So $X(z) = \dfrac{2\left(1-\frac{7}{6}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-2z^{-1}\right)}$, $|z| < \frac{1}{3}$

zeros : $z = \frac{7}{6}$

poles : $z = \frac{1}{3}, 2$

- Multiply $X(z)$ by $\frac{z^2}{z^2}$ to convert from "z^{-1}" to "z":

$$X(z) = \frac{2\left(1-\frac{7}{6}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-2z^{-1}\right)} \cdot \frac{z^2}{z^2}$$

$$= \frac{2z\left(z-\frac{7}{6}\right)}{\left(z-\frac{1}{3}\right)\left(z-2\right)}$$

- Check the two special points $z=0$ and $z=\infty$:

$z=0$: there is an additional zero at $z=0$.

$z=\infty$: $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{2z^2}{z^2} = \lim_{z\to\infty} 2 = 2$

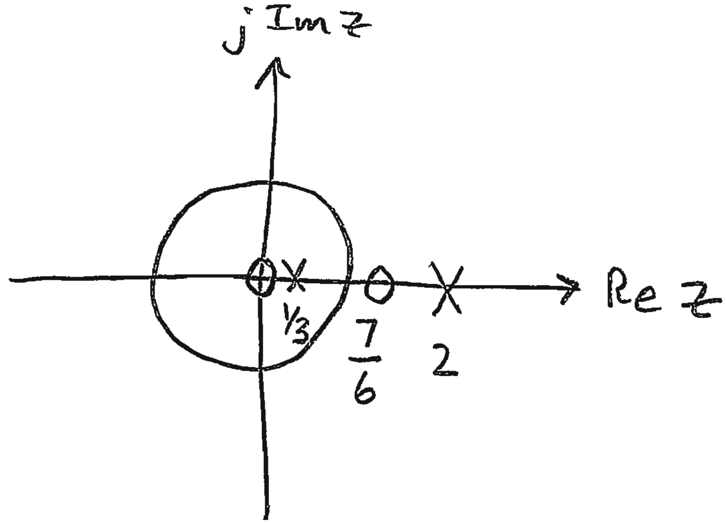
$\rightarrow$ $z=\infty$ is <u>not</u> a zero and <u>not</u> a pole.

$\longrightarrow$

– So all together: zeros: $z = \frac{7}{6}, 0$

poles: $z = \frac{1}{3}, 2$

p-z plot:



– Now let's do this same one again as an inverse z-transform problem:

Given: $X(z) = \dfrac{2(1-\frac{7}{6}z^{-1})}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}, \quad |z| < \frac{1}{3}$

Find $x[n]$.

PFE: $\dfrac{2(1-\frac{7}{6}\theta)}{(1-\frac{1}{3}\theta)(1-2\theta)} = \dfrac{A}{1-\frac{1}{3}\theta} + \dfrac{B}{1-2\theta}$

$$A = \left.\frac{2(1-\frac{7}{6}\theta)}{1-2\theta}\right|_{\theta=3} = \frac{2(1-\frac{7}{2})}{1-6} = \frac{2(-\frac{5}{2})}{-5} = \frac{-5}{-5} = 1$$

$$B = \left.\frac{2(1-\frac{7}{6}\theta)}{1-\frac{1}{3}\theta}\right|_{\theta=\frac{1}{2}} = \frac{2(1-\frac{7}{12})}{1-\frac{1}{6}} = \frac{2(\frac{5}{12})}{5/6} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| < \frac{1}{3}$$

- The possible ROC's for $\frac{1}{1-\frac{1}{3}z^{-1}}$ are $|z| > \frac{1}{3}$ and $|z| < \frac{1}{3}$

- The possible ROC's for $\frac{1}{1-2z^{-1}}$ are $|z| > 2$ and $|z| < 2$

- Since the overall ROC is $|z| < \frac{1}{3}$, the individual ROC's for the two terms must intersect to $|z| < \frac{1}{3}$.

  $\rightarrow$ They must be $|z| < \frac{1}{3}$ and $|z| < 2$.

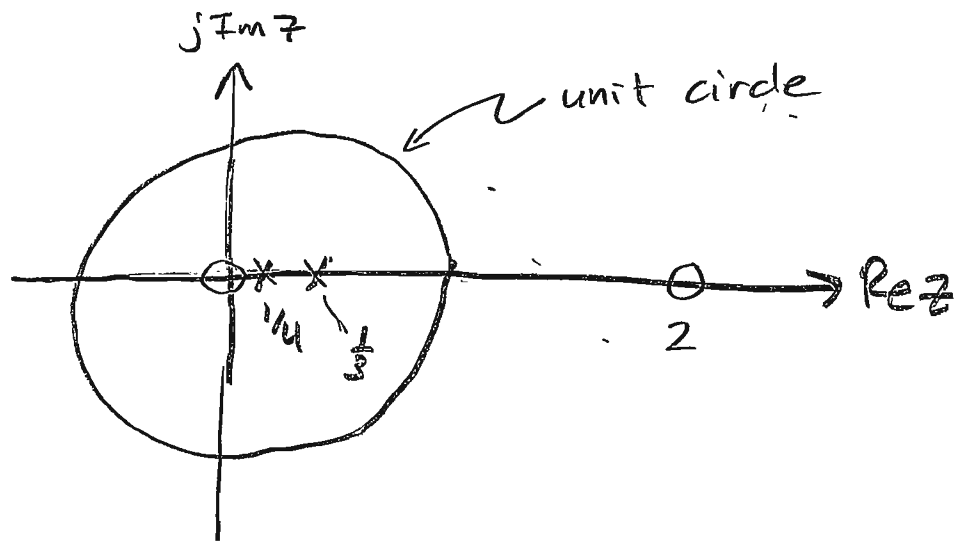
So
$$X(z) = \underbrace{\frac{1}{1-\frac{1}{3}z^{-1}}}_{|z| < \frac{1}{3}} + \underbrace{\frac{1}{1-2z^{-1}}}_{|z| < 2}, \quad |z| < \frac{1}{3}$$

Table: $x[n] = -\left(\frac{1}{3}\right)^n u[-n-1] - 2^n u[-n-1]$ ✓

FACT: the pole-zero plot determines $X(z)$ up to a multiplicative constant.

- Suppose $X(z)$ has zeros at $z=2$ and $z=0$ and poles at $z=\frac{1}{3}$ and $z=\frac{1}{4}$.

- Then the p-z plot is:



- Because of the zero at $z=0$, the numerator of $X(z)$ must have a term $(1-0z^{-1}) = 1$.
- Because of the zero at $z=2$, the numerator must also contain a term $(1-2z^{-1})$.
- Because of the pole at $z=\frac{1}{4}$, the denominator must contain a term $(1-\frac{1}{4}z^{-1})$.
- Because of the pole at $z=\frac{1}{3}$, the denominator must also contain a term $(1-\frac{1}{3}z^{-1})$.

- So it must be the case that

$$X(z) = \frac{K(1-2z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})},$$

where "K" is an unknown constant.

- But what is the ROC ?

  - From the information given so far, it is impossible to tell.

- However, remember :

  - poles are where X(z) blows up.
  - The ROC is where X(z) converges... i.e., the ROC is where X(z) does not blow up.

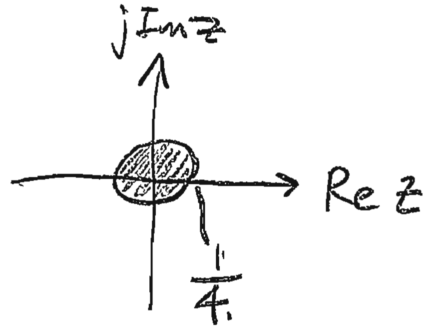
  $\Rightarrow$ So there can not be any poles in the ROC.

- For this example, this means that there are three possible ROC's, that are bounded by the poles.

$\longrightarrow$

- They are:

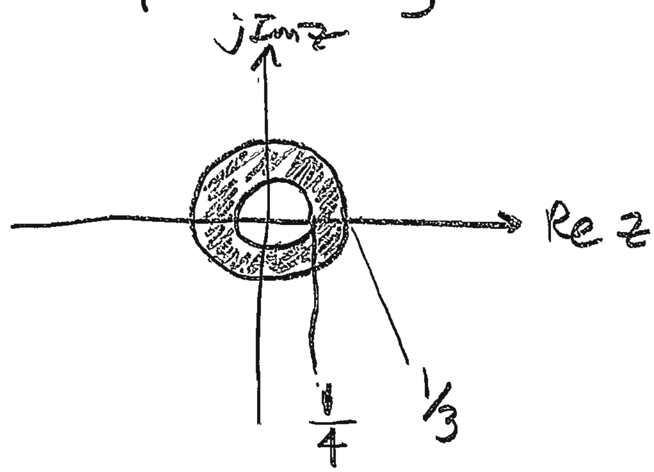
① everything <u>inside</u> a circle through the <u>smallest</u> pole ($z=\frac{1}{4}$) : $|z| < \frac{1}{4}$



→ Since this ROC is <u>interior</u>, it corresponds to a <u>left-sided</u> $x[n]$.

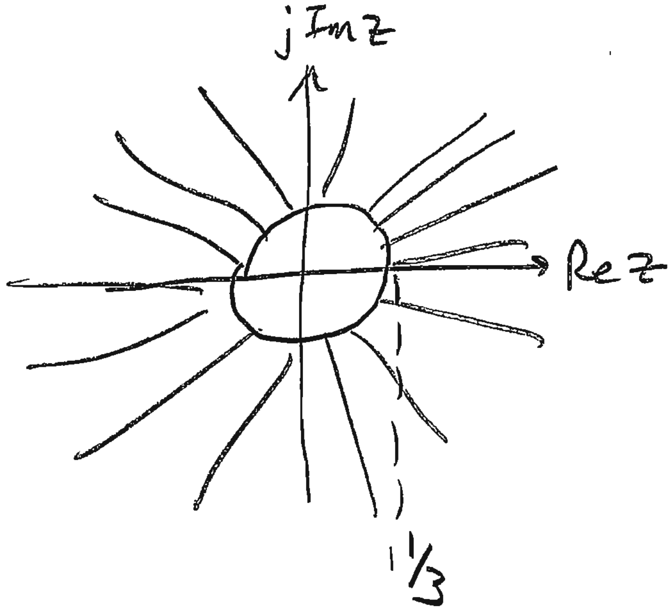
② everything <u>between</u> a circle through the pole at $z=\frac{1}{3}$ and a circle through the pole at $z=\frac{1}{4}$:

$$\frac{1}{4} < |z| < \frac{1}{3}$$



→ Since this ROC is <u>annular</u>, it corresponds to a <u>two-sided</u> $x[n]$.

(3) everything outside a circle through the largest pole $(z=\frac{1}{3})$: $|z|>\frac{1}{3}$



→ Since this ROC is exterior, it corresponds to a right-sided $x[n]$.

- Since there are three possible ROCs, there are three $x[n]$ that have this pole-zero plot and $X(z)=\dfrac{K(1-2z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$.

⇒ Each one has a different ROC.

→

- Since ROC 3, $|z| > \frac{1}{3}$, is the <u>only</u> one that contains the unit circle, the right-sided $x[n]$ corresponding to ROC 3 is the only one that has a DTFT $X(e^{j\omega})$.

- How can we figure out what these three $x[n]$ are?

  - To answer that, we would have to do a PFE on $X(z)$.

  - Since we don't know what "K" is, we we can't actually do the PFE.

  - However, we know that if we <u>could</u> do the PFE, we would get something like:

$$X(z) = \frac{K(1-2z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

- The possible ROC's for $\frac{A}{1-\frac{1}{4}z^{-1}}$ are $|z| > \frac{1}{4}$ and $|z| < \frac{1}{4}$

- The possible ROC's for $\frac{B}{1-\frac{1}{3}z^{-1}}$ are $|z| > \frac{1}{3}$ and $|z| < \frac{1}{3}$ $\longrightarrow$

So $X(z) = \underbrace{\frac{A}{1-\frac{1}{4}z^{-1}}}_{\substack{|z|>\frac{1}{4} \\ \text{or} \\ |z|<\frac{1}{4}}} + \underbrace{\frac{B}{1-\frac{1}{3}z^{-1}}}_{\substack{|z|>\frac{1}{3} \\ \text{or} \\ |z|<\frac{1}{3}}}$

- we cannot take $|z|<\frac{1}{4}$ and $|z|>\frac{1}{3}$ ...
  - because this would make the overall ROC (the intersection) equal to the empty set.
- If we take $|z|<\frac{1}{4}$ and $|z|<\frac{1}{3}$, then we get ROC ① on p. 6.74. Both terms invert left-sided and we get $x[n] = -A\left(\frac{1}{4}\right)^n u[-n-1] - B\left(\frac{1}{3}\right)^n u[-n-1]$.
- If we take $|z|>\frac{1}{4}$ and $|z|<\frac{1}{3}$, then we get ROC ② on p. 6.74. The first term inverts right-sided and the second term inverts left-sided. We get: $x[n] = A\left(\frac{1}{4}\right)^n u[n] - B\left(\frac{1}{3}\right)^n u[-n-1]$.
- Finally, if we take $|z|>\frac{1}{4}$ and $|z|>\frac{1}{3}$, then we get ROC ③ on p. 6.75. Both terms invert right-sided and we get: $x[n] = A\left(\frac{1}{4}\right)^n u[n] + B\left(\frac{1}{3}\right)^n u[n]$.

EX:

$$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})}$$

$x[n]$ does have a DTFT $X(e^{j\omega})$.

} GIVEN

→ Find $x[n]$.

Solution: Since $X(e^{j\omega})$ exists, the unit circle must be included in the ROC of $X(z)$.

- Do a PFE:

$$\frac{1}{(1-\frac{1}{2}\theta)(1+2\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1+2\theta}$$

$$A = \left.\frac{1}{1+2\theta}\right|_{\theta=2} = \frac{1}{1+2\cdot 2} = \frac{1}{1+4} = \frac{1}{5}$$

$$B = \left.\frac{1}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{2}} = \frac{1}{1-(\frac{1}{2})(-\frac{1}{2})} = \frac{1}{1+\frac{1}{4}} = \frac{1}{5/4} = \frac{4}{5}$$

$$X(z) = \underbrace{\frac{1/5}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2} \text{ or } |z|<\frac{1}{2}} + \underbrace{\frac{4/5}{1+2z^{-1}}}_{|z|>2 \text{ or } |z|<2}$$

→

- Since $X(e^{j\omega})$ exists, the ROC must contain the unit circle.

  - The individual ROC's must be $|z| > \frac{1}{2}$ and $|z| < 2$.

  - This makes the overall ROC of $X(z)$ equal to $\frac{1}{2} < |z| < 2$, which does contain the unit circle ✓

- So we've got

$$X(z) = \underbrace{\frac{1/5}{1 - \frac{1}{2}z^{-1}}}_{|z| > \frac{1}{2}} + \underbrace{\frac{4/5}{1 + 2z^{-1}}}_{|z| < 2}$$

- For the first term, $\alpha = \frac{1}{2}$
- For the second term, $\alpha = -2$

Table: $x[n] = \frac{1}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{5}(-2)^n u[-n-1]$

EX: $$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})}$$

$x[n]$ is right-sided.

Find $x[n]$.

- The PFE is the same as in the last example:

$$X(z) = \frac{1/5}{1-\frac{1}{2}z^{-1}} + \frac{4/5}{1+2z^{-1}}$$

- since we are given that $x[n]$ is right-sided, both terms must invert right-sided.

$\Rightarrow$ The ROC's must be:

$|z| > \frac{1}{2}$ and $|z| > 2$

Table: $$\underline{\underline{x[n] = \frac{1}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{5}(-2)^n u[n]}}$$

EX:
$$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})}$$

$x[n]$ is left-sided.

Find $x[n]$.

- The PFE is the same as in the last two examples:

$$X(z) = \frac{1/5}{1-\frac{1}{2}z^{-1}} + \frac{4/5}{1+2z^{-1}}$$

- Since we are given that $x[n]$ is left-sided, both terms must invert left-sided.

$\Rightarrow$ The ROC's <u>must</u> be:

$$|z| < \frac{1}{2} \quad \text{and} \quad |z| < 2$$

Table: $x[n] = -\frac{1}{5}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{4}{5}(-2)^n u[-n-1]$.

The z-transform Convolution Property

- If $x[n] \overset{z}{\longleftrightarrow} X(z)$ with ROC R_1

  and $h[n] \overset{z}{\longleftrightarrow} H(z)$ with ROC R_2

  and $y[n] = x[n] * h[n]$,

- then $\underline{\underline{Y(z) = X(z)H(z)}}$ with ROC <u>at least</u>

  $R_1 \cap R_2$.

- As with the DTFT, this property is <u>widely</u> used for analyzing and designing discrete-time LTI systems ... and digital filters in particular.

- What does it mean "with ROC at least $R_1 \cap R_2$"?

  - Generally, the ROC of $Y(z)$ will be the intersection of the ROC's of $X(z)$ and $H(z)$.
  - However, in some cases one of them ($X(z)$ or $H(z)$) may have a <u>zero</u> where the other one has a <u>pole</u>. →

- When this happens, multiplying them together to get $Y(z) = X(z)H(z)$ results in a <u>pole-zero cancellation</u>.

- This causes the cancelled pole to "disappear."

- Since the ROC is bounded by the poles, this could cause the ROC of $Y(z)$ to be increased to something larger than $R_1 \cap R_2$.

- Here is an example of that:

  - Let H be a discrete-time LTI system with impulse response

$$h[n] = \frac{5}{3}\left(\frac{1}{4}\right)^n u[n] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n].$$

  - Let the system input be

$$x[n] = \left(\frac{2}{3}\right)^n u[n].$$

  - Find the system output $y[n]$.

$\longrightarrow$

Solution:

Table: $$H(z) = \frac{5/3}{1-\frac{1}{4}z^{-1}} - \frac{2/3}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{5/3\left(1-\frac{1}{2}z^{-1}\right) - 2/3\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{5/3 - 5/6\,z^{-1} - 2/3 + \frac{1}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{1 - 2/3\,z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad |z| > \frac{1}{2}$$

Table: $$X(z) = \frac{1}{1-\frac{2}{3}z^{-1}}, \quad |z| > 2/3$$

Note: the intersection of the ROCs is

$$\left\{|z| > \frac{1}{2}\right\} \cap \left\{|z| > \frac{2}{3}\right\} = \left\{|z| > \frac{2}{3}\right\}$$

– Now, by the convolution property, we have

$$Y(z) = X(z)H(z)$$

$$= \frac{1}{1-\frac{2}{3}z^{-1}} \cdot \frac{1-\frac{2}{3}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$

pole-zero cancellation

$$= \frac{(1-\frac{2}{3}z^{-1})}{(1-\frac{2}{3}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$

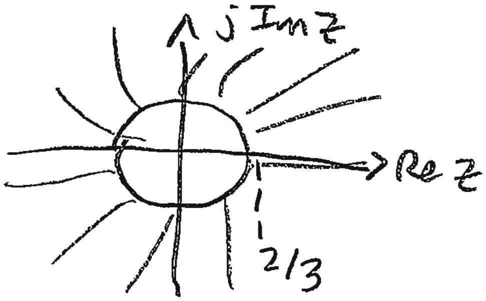
$$= \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{2}$$

→ The intersection of the ROC's was $|z| > \frac{2}{3}$

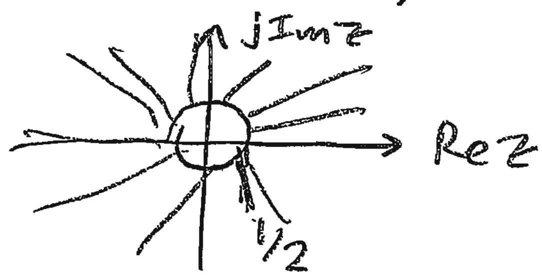
→ But due to the pole-zero cancellation, the ROC of $Y(z)$... which is everything outside of a circle through the largest pole of $Y(z)$... is $|z| > \frac{1}{2}$

⇒ which is larger than the intersection.

The intersection:



ROC of $Y(z)$:



- As with the DTFT,

the z-transform convolution property

$$Y(z) = X(z) H(z)$$

leads to three main types of problems:

① Analysis or Convolution:

- given $h[n]$ and $x[n]$, find $y[n]$.

→ Use $Y(z) = X(z) H(z)$

② Deconvolution:

- given $h[n]$ and $y[n]$, find $x[n]$.

→ Use $X(z) = \dfrac{Y(z)}{H(z)}$

③ System Identification:

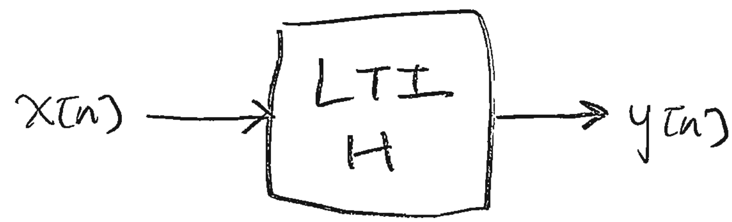
- given $x[n]$ and $y[n]$, find $h[n]$.

→ Use $H(z) = \dfrac{Y(z)}{X(z)}$

- All three are very similar to what we have already seen for the DTFT, except that with the z-transform you must also keep track of the ROC.

Transfer Function

- Let H be an LTI system with impulse response $h[n]$;



- The z-transform $H(z) = \mathcal{Z}\{h[n]\}$ is called the <u>transfer function</u> of the system.
- The transfer function tells you a lot about the system.
- The transfer function <u>contains</u> the frequency response.
  - Above the unit circle of the z-plane, $H(z)$ is equal to $H(e^{j\omega})$
    - Recall that $z = re^{j\omega}$
    - Above the unit circle, we have $r=1$ and $z = e^{j\omega}$
    - So $H(z)\Big|_{|z|=1} = H(e^{j\omega})$

- Recall from p. 5.91: A discrete-time LTI system H is <u>causal</u> iff $h[n] = 0 \ \forall n < 0$.

→ This means that $h[n]$ can <u>not</u> be left-sided.

⇒ For a causal system, the ROC of $H(z)$ can <u>not</u> be interior.

→ This also means that $h[n]$ can <u>not</u> be two-sided.

⇒ For a causal system, the ROC of $H(z)$ can <u>not</u> be annular.

☆☆☆ For a <u>causal</u> discrete-time LTI system H, $h[n]$ must be either:

- right-sided [the ROC of $H(z)$ is <u>exterior</u> in this case], or
- finite-length [in this case the ROC of $H(z)$ is the entire z-plane... except possibly the points $z=0$ and $z=\infty$ which must always be checked].

- Important Note:

  - For H to be causal, it is necessary for the ROC of $H(z)$ to be either exterior or the whole z-plane.

→ But this alone is not sufficient for H to be causal.

→ For H to be causal, you need more.

⇒ You also need that $h[n] = 0 \ \forall n < 0$.

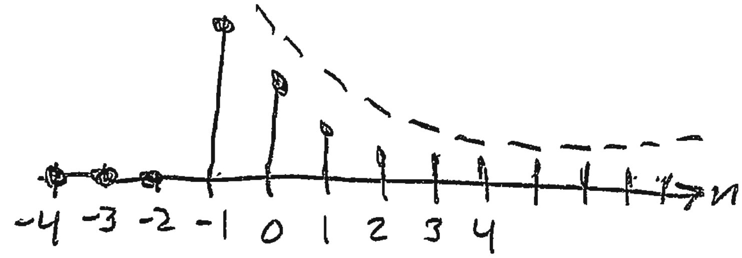
⇒ In other words, you need $h[n]$ to be:

① right-sided or finite-length [ROC of $H(z)$ is exterior or the entire z-plane], AND

② $h[n]$ "starts" at or after $n=0$.

→

- For example, suppose $h[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1]$



→ This $h[n]$ is <u>right-sided</u>.

→ So the ROC of $H(z)$ is <u>exterior</u>

[it's $|z| > \frac{1}{2}$]

→ But the system is <u>not</u> causal, because $h[-1] = 1 \neq 0$, so it's <u>not</u> true that $h[n] = 0 \ \forall \ n < 0$

$\Longrightarrow$ <u>NOT CAUSAL</u>.

- Similarly, if $h[n] = \frac{1}{3}\delta[n+1] + \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1]$,

then the ROC of $H(z)$ turns out to be $\{$all z except $z=0$ and $z=\infty\}$,

→ But the system is <u>NOT CAUSAL</u> because $h[-1] = \frac{1}{3} \neq 0$.

→ i.e., $h[n]$ "starts" <u>before</u> $n=0$.

RECAP:

- If H is a causal LTI system, then the ROC of $H(z)$ must be exterior or the whole z-plane (except possibly the two points $z=0$ and $z=\infty$).

→ However, if you have an $H(z)$ and you know that the ROC is exterior... or you know that the ROC is the whole z-plane,

⇒ This is not enough to say that the system is causal...

⇒ You also need to check that $h[n] = 0 \;\; \forall\, n<0$.

EX: You are given that:

- H is a discrete-time LTI system
- The transfer function is

$$H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$$

- But you are not given the ROC
- H is causal.

Find: the impulse response $h[n]$.

Solution:

- From the given $H(z)$, we can see immediately that there are poles at $z=\frac{1}{2}$ and $z=2$.
- We must also check the two points $z=0$ and $z=\infty$.
- Converting $H(z)$ from "z^{-1}" to "z", we get

$$H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} \cdot \frac{z^2}{z^2}$$

$$= \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-2\right)}$$

$\longrightarrow$

- So there is also a 2^{nd}-order zero at $z=0$.
- plugging in $z=\infty$, we get

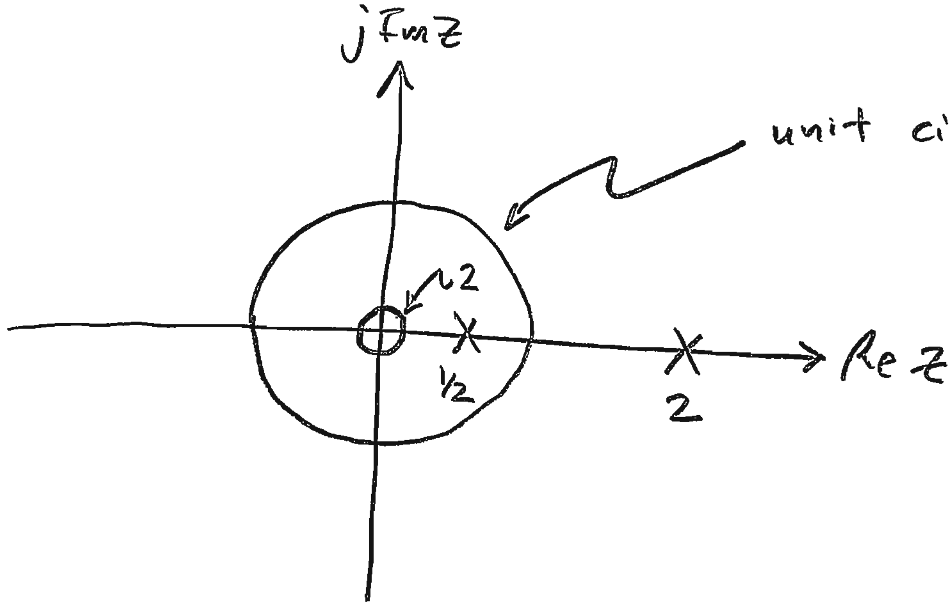
$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{z^2}{z^2} = \lim_{z\to\infty} 1 = 1$$

$\Rightarrow$ The point $z=\infty$ is <u>not</u> a zero and it is <u>not</u> a pole.

- So the complete list is: zeros: $z=0$ (2^{nd} order)

poles: $z=\frac{1}{2}, 2$

p/z - plot:



- For this pole-zero plot, the possible ROC's are:
  a) $|z| < \frac{1}{2}$
  b) $\frac{1}{2} < |z| < 2$ $\longrightarrow$
  c) $|z| > 2$

- Because we are given that H is causal, the ROC must be either exterior or the whole z-plane.

$\Longrightarrow$ The ROC of $H(z)$ must be $\underline{\underline{|z| > 2}}$.

- Next we do a PFE on $H(z)$:

$$\frac{1}{(1-\frac{1}{2}\theta)(1-2\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-2\theta}$$

$$A = \left.\frac{1}{1-2\theta}\right|_{\theta=2} = \frac{1}{1-4} = \frac{1}{-3} = -\frac{1}{3}$$

$$B = \left.\frac{1}{1-\frac{1}{2}\theta}\right|_{\theta=\frac{1}{2}} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3}$$

$$H(z) = -\underbrace{\frac{1/3}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{4/3}{1-2z^{-1}}}_{|z|>2}, \quad |z| > 2$$

$$\boxed{\text{Table: } h[n] = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{4}{3}2^n u[n]}$$

$h[n]$ is right-sided ✓

$h[n] = 0 \quad \forall\, n < 0$ ✓

$\Longrightarrow$ CAUSAL ✓

STABILITY

- Recall from p. 5.96:

  - A discrete-time LTI system H is stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

  - In other words, if and only if the impulse response $h[n]$ is absolutely summable.

- Recall from p. 5.98:

  - If $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$,

    Then $H(e^{j\omega})$ exists... i.e., the DTFT sum converges.

$\Longrightarrow$ Thus, we have the following very important property:

- If H is a stable discrete-time LTI system,
  - then $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
    - then $H(e^{j\omega})$ converges
    - then the ROC of $H(z)$ must include the unit circle of the z-plane.

<u>Note</u>: technically, this property is <u>not</u> "if and <u>only</u> if." It's just "if".

⇒ In other words,

- If H is <u>stable</u>, then the ROC of $H(z)$ <u>must</u> include the unit circle.
- But it doesn't technically go the other way...
  - It is theoretically possible to construct a system H such that $H(e^{j\omega})$ exists as a distribution, but the system is <u>not</u> strictly stable.

⇒ However, we will <u>not</u> see such systems in ECE 2713.

⇒ For ECE 2713, you can assume that:

★★ H is a stable discrete-time LTI system $\overset{\text{if and only if}}{\Longleftrightarrow}$ The ROC of $H(z)$ includes the unit circle ★★

EX: You are given that:

- H is a discrete-time LTI system
- The transfer function is

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

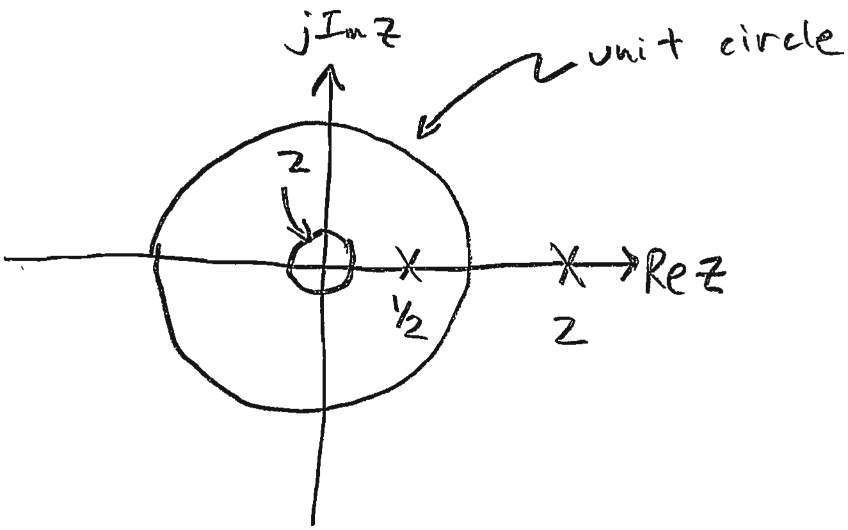
(same as the example on p. 6.92)

- But you are not given the ROC.
- H is stable (different from the example on p. 6.92)

Find: the impulse response $h[n]$.

Solution:

- The pole-zero plot is the same as what we got back on p. 6.92:



The possible ROCs are also the same as we got on p. 6.92:

a) $|z| < \frac{1}{2}$

b) $\frac{1}{2} < |z| < 2$

c) $|z| > 2$

- But in this case, we are given that the system is <u>stable</u>.

$\Rightarrow$ So the ROC of $H(z)$ <u>must</u> include the unit circle.

$\Longrightarrow$ The ROC of $H(z)$ <u>must</u> be:

$$\underline{\underline{\tfrac{1}{2} < |z| < 2}}$$

- The PFE is the <u>same</u> as what we got back on p. 6.93, but the ROC is <u>different</u>:

$$H(z) = -\underbrace{\frac{1/3}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{4/3}{1-2z^{-1}}}_{|z|<2}, \quad \tfrac{1}{2}<|z|<2$$

$$\boxed{\text{Table: } h[n] = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n u[n] - \tfrac{4}{3}2^n u[-n-1]}$$

- In this case, $h[n]$ is <u>two-sided</u>, so the system is <u>not</u> causal.
- But it is <u>stable</u>

<u>Note</u>: the <u>different</u> solution we got on p. 6.94 <u>was</u> causal... but it was <u>not</u> stable because $2^n u[n]$ is <u>not</u> absolutely summable.

CAUSAL PLUS STABLE

- In many practical filtering situations, we require a digital filter H that is both causal and stable.
- These requirements place important limitations on the locations of the poles.

⇒ For the system to be causal, the ROC of $H(z)$ must be exterior or the whole z-plane (except possibly $z=0$ and/or $z=\infty$).

→ But as we will see later, any practical digital filter (except the identity filter with $h[n]=\delta[n]$) will have at least one pole in the finite z-plane.

→ So even if the ROC of $H(z)$ is the whole z-plane, there will still be at least one pole at $z=0$.

→ Thus, we can still think of such a ROC as being "exterior" in the sense that it is exterior to the point $z=0$.

- In other words, for any practical digital filter that is causal, the ROC of $H(z)$ will be exterior to the largest magnitude pole.... (which might be at $z=0$).

$\Rightarrow$ For the system to be stable, the ROC of $H(z)$ must include the unit circle.

$\Rightarrow$ There can not be any poles in the ROC!

- poles are where $H(z)$ blows up... does not converge.
- The ROC is where $H(z)$ converges... does not blow up.
- So there can't be any poles in the ROC.

- Therefore, for the system to be causal and stable, the ROC must be exterior to the largest pole...

and

- the ROC must start inside the unit circle... so that the ROC can be exterior and include the unit circle.

- This means that, for a discrete-time LTI system H that is <u>both</u> causal and stable,

⇒ All of the poles of $H(z)$ must be strictly <u>inside</u> the unit circle of the z-plane.

⇒ If $H(z)$ has any poles <u>outside</u> the unit circle, then the system <u>cannot</u> be both causal and stable.

- In the examples on p. 6.92 and p. 6.97, we had

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

- The pole at $z=2$ is <u>outside</u> the unit circle.

⇒ A system with this $H(z)$ can <u>not</u> be both causal and stable.

⇒ For the example on p. 6.92, the ROC was $|z|>2$.

→ The resulting system was <u>causal</u>, but <u>not</u> <u>stable</u> ... because the ROC did <u>not</u> include the unit circle.

⇒ For the example on p. 6.97, the ROC was $\frac{1}{2} < |z| < 2$.

→ In this case, the resulting system was stable... because the ROC did include the unit circle...

→ But it was <u>not</u> causal... look back at p. 6.98. We got a two-sided impulse response $h[n]$... <u>not</u> <u>causal</u>.

<u>NOTE</u>:

- For a stable system, the ROC of $H(z)$ must include the unit circle. This means that the frequency response $H(e^{j\omega})$ is <u>convergent</u>.

  ⇒ A discrete-time LTI system H that is <u>stable</u> has a convergent frequency response... in other words, $H(e^{j\omega})$ <u>exists</u>.

- For an unstable system, the ROC of $H(z)$ does <u>not</u> contain the unit circle. This means that $H(e^{j\omega})$ is <u>divergent</u>.

  ⇒ A discrete-time LTI system H that is <u>unstable</u> has a divergent frequency response... in other words, $H(e^{j\omega})$ does <u>not</u> exist.

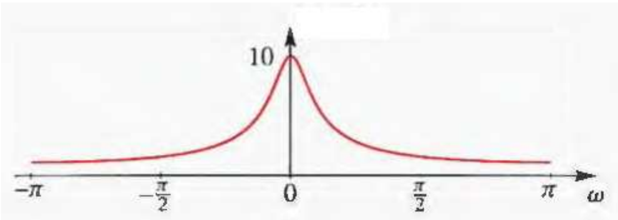
- <u>EX:</u> suppose H is a causal, stable discrete-time LTI system (filter) with impulse response

$$h[n] = \left(\frac{4}{5}\right)^n u[n].$$

- Using the DTFT table from the formula sheet on the course web site, we can write down the frequency response:

$$H(e^{j\omega}) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}$$

- Here is a plot of the filter magnitude response $|H(e^{j\omega})|$:

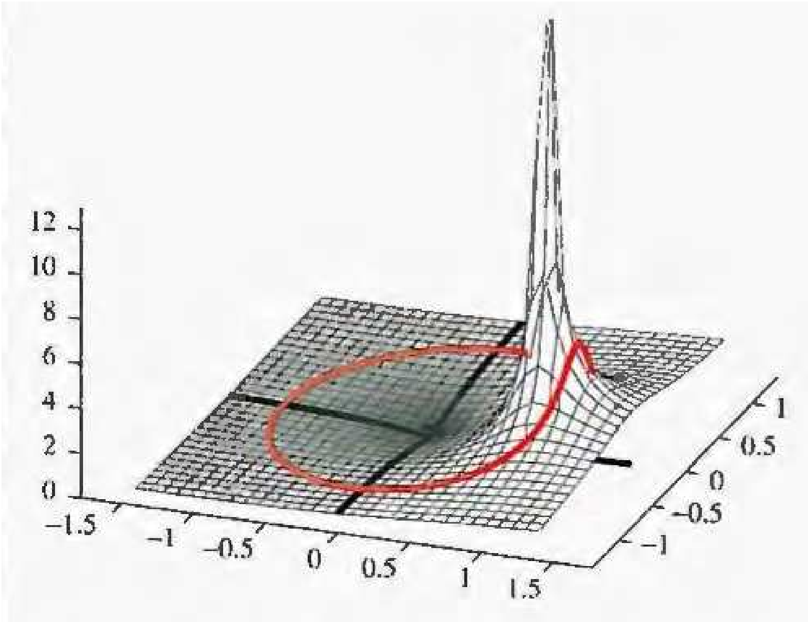


- From this plot, we see that H is a low-pass filter.
- Using the z-transform table from the formula sheet on the course web site, we can write down the transfer function:

$$H(z) = \frac{1}{1 - \frac{4}{5}z^{-1}} \;, \qquad \text{ROC: } |z| > \frac{4}{5}$$

- From the denominator, we see immediately that there is one pole at $z = \frac{4}{5}$.
- Notice that this pole is *inside* the unit circle, the ROC is *exterior*, and the ROC *includes* the unit circle.

- Here is a 3D plot of $|H(z)|$:



- Above the unit circle of the z-plane, $H(z) = H\left(e^{j\omega}\right)$.
- This is shown by the orange curve in the figure. This orange curve is the same graph of $\left|H\left(e^{j\omega}\right)\right|$ that we just saw on page 6.103, but now it is wrapped around the unit circle of the z-plane as the part of the graph of $H(z)$ where $|z|$ = r = 1.
- Notice how the pole at $z = \frac{4}{5}$, just inside the unit circle, pulls up the whole surface $H(z)$ and thus *shapes* $H\left(e^{j\omega}\right)$.
- You may have wondered why we need to have both a discrete-time Fourier transform (DTFT) and a z-transform.
- Here is one reason: a filter designer *designs* the poles and zeros of $H(z)$ to shape the frequency response $H\left(e^{j\omega}\right)$.

- The poles pull the surface $H(z)$ *up* towards ∞; the zeros pull the surface $H(z)$ *down* to zero.
- But if we only had a DTFT $H(e^{j\omega})$, and *not* a z-transform $H(z)$, the designer could not use any poles to shape $H(e^{j\omega})$…
  - because designing a pole directly into $H(e^{j\omega})$ would make the frequency response fail to converge.
  - The filter would then be *unstable*.
  - So instead, we design the poles into $H(z)$, in the z-plane but off the unit circle.
  - ***Recall:*** for the filter to be both causal and stable, the poles must be placed strictly *inside* the unit circle of the z-plane.
  - There is no such restriction on the zeros – the designer is free to place zeros anywhere in the z-plane, including inside the unit circle, outside the unit circle, and even on the unit circle.

The z-Transform Time Shift Property

- If $x[n] \overset{Z}{\longleftrightarrow} X(z)$
  - Then $x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$
- In other words, shifting a signal in time brings out powers of z^{-1} in the z-transform.
- This does not generally change the ROC.
  $\Rightarrow$ However, the points $z=0$ and $z=\infty$ must <u>always</u> be checked.

<u>EX</u>: $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$

- According to our table,

$$\left(\tfrac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$$

- Applying the time shift property with $n_0 = 1$, we get

$$X(z) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Converting from "z^{-1}" to z, we have

$$X(z) = \frac{z}{z-\frac{1}{2}}$$

- So there is still a pole at $z=\frac{1}{2}$, but no new poles because of the time shift $\longrightarrow$

- So the ROC of $X(z)$ is still the same as the ROC of $(\frac{1}{2})^n u[n]$.

<u>Answer</u>: $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$

<u>EX</u>: $x[n] = (\frac{1}{2})^n u[n-1]$

- We have to be careful here. The term $u[n-1]$ tells us that we are going to have to apply a time shift.

- But we've got $(\frac{1}{2})^n$, not $(\frac{1}{2})^{n-1}$!

$\Rightarrow$ However, $(\frac{1}{2})^n = \frac{1}{2}(\frac{1}{2})^{n-1}$

$\Rightarrow$ So $x[n] = \frac{1}{2}(\frac{1}{2})^{n-1} u[n-1]$

- Applying the time shift property to the z-transform pair $(\frac{1}{2})^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}$, $|z|>\frac{1}{2}$ with $n_0 = 1$, we get

$$X(z) = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

EX: $x[n] = \left(\frac{1}{2}\right)^{n+2} u[n+2]$

- Applying the time shift property to the z-transform pair $\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$ with $n_0 = -2$, we get

$$X(z) = \frac{z^2}{1-\frac{1}{2}z^{-1}}$$

- converting from "z^{-1}" to z, we see that

$$X(z) = \frac{z^3}{z - \frac{1}{2}}$$

- So the time shift introduced a 2nd order pole at $z=\infty$, but this doesn't change the ROC.

<u>Answer</u>: $$X(z) = \frac{z^2}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

EX: $X(z) = \dfrac{z^{-3}}{(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}$, $|z| > \frac{2}{3}$

Find $x[n]$.

Solution: since there are two finite poles (at $z=\frac{1}{2}$ and $z=\frac{2}{3}$), we know we need to do a PFE.

- But $X(z)$ is an improper fraction in z^{-1}, so we can't do the PFE directly on $X(z)$.
- However, we can apply the time shift property:

$$X(z) = \underbrace{z^{-3}}_{\text{a time shift}} \Bigg[\underbrace{\frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}}_{\text{a proper fraction in } z^{-1}} \Bigg]$$

- So let $X_2(z) = \dfrac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}$, $|z| > \frac{2}{3}$.

→ We will do a PFE to find $x_2[n]$.

→ Then $x[n] = x_2[n-3]$ by the time shift property.

- Doing the PFE on $X_2(z)$, we get

$$\frac{1}{(1-\frac{1}{2}\theta)(1-\frac{2}{3}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-\frac{2}{3}\theta}$$

$$A = \left.\frac{1}{1-\frac{2}{3}\theta}\right|_{\theta=2} = \frac{1}{1-4/3} = \frac{1}{-1/3} = -3$$

$$B = \left.\frac{1}{1-\frac{1}{2}\theta}\right|_{\theta=\frac{3}{2}} = \frac{1}{1-\frac{3}{4}} = \frac{1}{1/4} = 4$$

$$X_2(z) = \underbrace{\frac{-3}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{4}{1-\frac{2}{3}z^{-1}}}_{|z|>\frac{2}{3}}, \quad |z|>\frac{2}{3}$$

Table: $x_2[n] = -3\left(\frac{1}{2}\right)^n u[n] + 4\left(\frac{2}{3}\right)^n u[n]$

Time Shift property: $x[n] = -3\left(\frac{1}{2}\right)^{n-3} u[n-3] + 4\left(\frac{2}{3}\right)^{n-3} u[n-3]$

- Recall from Module 2:

$$x[n] * \delta[n] = x[n]$$

$$x[n] * \delta[n-1] = x[n-1]$$

- A discrete-time LTI system H with impulse response $h[n] = \delta[n-1]$ is called the "unit delay" system:

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \longrightarrow y[n] = x[n] * h[n]$$

$$= x[n] * \delta[n-1]$$

$$= x[n-1]$$

- According to our z-transform table,

$$\delta[n] \overset{Z}{\longleftrightarrow} 1, \quad \text{all } z.$$

- To find the transfer function $H(z)$ of the unit delay system, apply the time shift property:

$$H(z) = \mathcal{Z}\{\delta[n-1]\} = z^{-1} = \frac{1}{z}$$

→ We see that the time shift has introduced a pole at $z=0$, so the ROC is $\underline{\underline{|z| > 0}}$.

- In the time domain, the I/O equation for this system is:

$$y[n] = x[n-1]$$

- In the z-transform domain, this becomes:

$$Y(z) = X(z)\,H(z) = z^{-1} X(z)$$

- For this reason, the unit delay system is usually drawn like this:

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n] = x[n-1].$$

- It is an FIR filter, because $h[n]$ has finite length.
- It is causal because $h[n] = 0 \;\; \forall \; n < 0$.
- It is stable because $\sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty$.

- The unit delay system comes up frequently in the implementation of digital filters.

Relationship Between Trasfer Function and Input-Output Equation

- As we saw on p. 6.82, for a discrete-time LTI system H with impulse response $h[n]$ and transfer function $H(z)$

$$x[n] \longrightarrow \boxed{\begin{matrix} H \\ \text{LTI} \end{matrix}} \longrightarrow y[n]$$

  - the z-transform convolution property tells us that
$$Y(z) = X(z)\, H(z).$$

  - Rearranging, we get
$$H(z) = \frac{Y(z)}{X(z)}$$

  - Similar to the DTFT, this can be used to find the transfer function $H(z)$ from the I/O equation (recall: for a discrete-time LTI system, the I/O equation is a linear constant coefficients difference equation).

- However, more information is generally needed in order to find the ROC of $H(z)$.

- To see how this works, suppose H is a discrete-time LTI system with I/O equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1]$$

- Use the time shift property to take the z-transform on both sides:

$$Y(z) - \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{5}{2}z^{-1} + z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}}$$

- To find the poles, we need to factor the denominator of $H(z)$ to get it in the form $(1-az^{-1})(1-bz^{-1})$.

- We need $a+b = \frac{5}{2}$ and $ab = 1$.

→ $ab=1$ implies a and b have the same sign.

→ $a+b=\frac{5}{2}$ then implies they are both positive.

→ To get the sum to be $\frac{5}{2}$, one of them must have a 2 downstairs. To get the product to be one, the other one must have a 2 upstairs.

→ This leads us immediately to $a=\frac{1}{2}$, $b=2$.

$$H(z) = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

Zeros: $z=1$

poles: $z=\frac{1}{2}, 2$

- To check the two points $z=0$ and $z=\infty$, convert from z^{-1} to z:

$$H(z) = \frac{z(z-1)}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

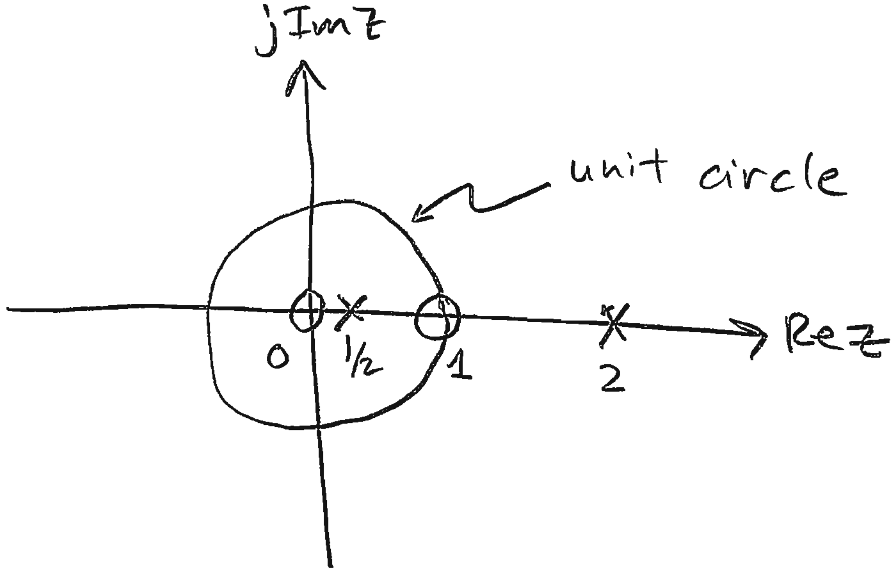
→ There is an additional zero at $z=0$.

⟶

- So all together, we have: zeros: $z = 1, 0$

poles: $z = \frac{1}{2}, 2$

P/Z-plot:



- There are <u>three</u> possible ROC's:

① $|z| > 2$

② $\frac{1}{2} < |z| < 2$

③ $|z| < 2$

⇒ This means that there are actually <u>three</u> discrete LTI systems that have this difference equation and this $H(z)$.

⇒ But they all have different ROC's, different impulse responses $h[n]$, and different effects on the input signal $x[n]$.

- The system with ROC 1, $|z| > 2$, has a right-sided $h[n]$. It is the only one of the three systems that can be causal. Let's call this system H_1.

- The system with ROC 2, $\frac{1}{2} < |z| < 2$, has a two-sided $h[n]$. Because ROC 2 contains the unit circle, this system is the only one of the three that is stable. Let's call this system H_2. It is the only one that has a frequency response $H(e^{j\omega})$.

- The system with ROC 3, $|z| < \frac{1}{2}$, has a left-sided $h[n]$. Because of this, it can <u>not</u> be causal. Because ROC 3 also does <u>not</u> contain the unit circle, this system can <u>not</u> be stable. Let's call this system H_3.

- Given just the difference equation, we need more information before we can determine which one of these three systems we've actually got.

- Irrespective of which one of these three systems we've actually got, we still need to do a PFE on $H(z)$:

$$\frac{1-\theta}{(1-\frac{1}{2}\theta)(1-2\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-2\theta}$$

$$A = \left.\frac{1-\theta}{1-2\theta}\right|_{\theta=2} = \frac{1-2}{1-4} = \frac{-1}{-3} = \frac{1}{3}$$

$$B = \left.\frac{1-\theta}{1-\frac{1}{2}\theta}\right|_{\theta=\frac{1}{2}} = \frac{1-\frac{1}{2}}{1-\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{4}{3}\cdot\frac{1}{2} = \frac{2}{3}$$

$$H(z) = \frac{1/3}{1-\frac{1}{2}z^{-1}} + \frac{2/3}{1-2z^{-1}} \qquad (*)$$

→ ROC can not be determined without more information

→ $h[n]$ can not be determined without more information.

- Now suppose we are given:

    - that H is causal
    - and/or that $h[n]$ is right-sided.

$\Rightarrow$ Then it <u>must</u> be ROC ①, the <u>exterior</u> ROC.

$\Rightarrow$ This means that <u>both</u> terms in the PFE (*) on p. 6.118 must have exterior ROC's and they must both invert right-sided.

- we get:

$$H(z) = \underbrace{\frac{1/3}{1 - \frac{1}{2}z^{-1}}}_{|z| > \frac{1}{2}} + \underbrace{\frac{2/3}{1 - 2z^{-1}}}_{|z| > 2}, \quad |z| > 2$$

Table: $\boxed{h[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3} 2^n u[n]}$

$\longrightarrow$

- We can see that this system <u>is</u> causal, because $h[n]=0 \ \forall \ n<0$.

- But it is <u>not</u> stable:

  - The ROC of $H(z)$ does <u>not</u> contain the unit circle $\Longrightarrow$ <u>not</u> <u>stable</u>

  - The term $2^n u[n]$ in $h[n]$ is blowing up as $n \to \infty$. Because of this, $h[n]$ is <u>not</u> absolutely summable... <u>not</u> <u>stable</u>.

- The frequency response $H(e^{j\omega})$ does <u>not</u> exist. The DTFT sum $\sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$ is <u>divergent</u>, again because of the term $2^n u[n]$ in the impulse response $h[n]$.

  - This is equivalent to the fact that the unit circle is <u>not</u> included in the ROC of $H(z)$.

- Next, suppose that <u>instead</u>, we are given:

  - that H is <u>stable</u>
  - and/or that $H(e^{j\omega})$ exists
  - and/or that $h[n]$ is <u>two-sided</u>.

$\Rightarrow$ Then it <u>must</u> be ROC ②, the <u>annular</u> ROC (making $h[n]$ two-sided)...

- and the <u>only</u> ROC that includes the unit circle (making H stable and making $H(e^{j\omega})$ converge [exist]).

$\Rightarrow$ This means that, in the PFE (*) on p. 6.118,

- the term for the pole @ $z=\frac{1}{2}$ must have an <u>exterior</u> ROC and must invert right-sided.
- the term for the pole @ $z=2$ must have an <u>interior</u> ROC and must invert left sided.

$\longrightarrow$

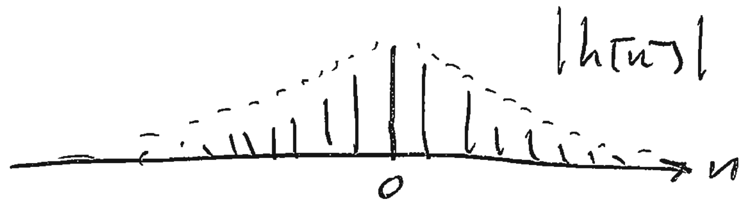
- we get:

$$H(z) = \underbrace{\frac{1/3}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{2/3}{1-2z^{-1}}}_{|z|<2}, \quad \frac{1}{2}<|z|<2$$

Table: $\boxed{h[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{3}2^n u[-n-1]}$

- we see immediately that this system is <u>not</u> causal: $h[n]$ is two-sided, so there are lots of places $n<0$ where $h[n] \neq 0$.

- But this time $h[n]$ <u>is</u> absolutely summable



- So out of the <u>three</u> LTI systems that <u>all</u> have the difference equation back on p. 6.114,

- This is the only one that is <u>stable</u>

→

- this is the only one that has a convergent frequency response $H(e^{j\omega})$.
  - to find the frequency response, remember that $z = re^{j\omega}$.
  - the frequency response is given by $H(z)$ <u>above</u> the unit circle... where $r=1$ (the fixer-upper parameter that does <u>nothing</u>... see ③ on p. 6.12)
- Looking back to $H(z)$ on p. 6.115, we get

$$H(e^{j\omega}) = H(z)\Big|_{|z|=r=1} = \frac{1-e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-2e^{-j\omega})}$$

- Finally, suppose <u>instead</u> that we are given:

  - that $h[n]$ is <u>left-sided</u>.

$\Rightarrow$ Then it <u>must</u> be ROC 3, the <u>interior</u> ROC.

$\Rightarrow$ This means that <u>both</u> terms in the PFE (*) on p. 6.118 must have interior ROCs, and they must both invert left-sided.

- we get:

$$H(z) = \underbrace{\frac{1/3}{1-\frac{1}{2}z^{-1}}}_{|z|<\frac{1}{2}} + \underbrace{\frac{2/3}{1-2z^{-1}}}_{|z|<2}, \quad |z| < \frac{1}{2}$$

Table:

$$\boxed{h[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{2}{3}2^n u[-n-1]}$$

- we see that this system is <u>not</u> causal... because $h[n]$ is left sided... $h[n]$ is nonzero for <u>all</u> $n<0$.

$\longrightarrow$

- This system is also <u>not</u> stable
  - The ROC of $H(z)$ does <u>not</u> include the unit circle of the z-plane
  - The term $\left(\frac{1}{2}\right)^n u[-n-1]$ is blowing up like $2^{|n|}$ as $n \to -\infty$. Because of this, $h[n]$ is <u>not</u> absolutely summable.

- The frequency response $H(e^{j\omega})$ does not exist.
  - again because of the term $\left(\frac{1}{2}\right)^n u[-n-1]$ in $h[n]$, the DTFT sum $\sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$ is <u>divergent</u>.

- So looking back to the difference equation

$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n] - x[n-1]$$

from p. 6.114,

- we see that there are <u>two</u> poles... because the greatest shift of $y[n]$ is by <u>2</u>... $y[n-2]$.
- This means there are <u>three</u> possible ROC's for $H(z)$.
- This means there are <u>three</u> different LTI systems H that all share this difference equation.

- At most <u>one</u> of them is causal. The one with the <u>exterior</u> ROC is the only one that has a possibility of being causal.
  - It will be causal if $h[n]$ starts at $n \geq 0$ so that $h[n] = 0 \;\; \forall n < 0$.
  - But if $h[n]$ had started <u>before</u> $n=0$, then <u>none</u> of the three systems would have been causal.

- <u>Exactly</u> <u>one</u> of the three systems is stable.
  - It is the one that has the ROC that includes the unit circle (it was ROC 2 in this example).
  - It is also the only one of the three systems that has a convergent frequency response $H(e^{j\omega})$.
    - For the other two systems, the frequency response $H(e^{j\omega})$ is <u>divergent</u> and fails to exist.

  $\Rightarrow$ However, <u>all</u> <u>three</u> systems <u>do</u> have a transfer function $H(z)$.

- More generally, the I/O-equation for an LTI system H is a linear constant coefficients difference equation.

  - This means that a linear combination of the <u>shifts</u> of the output signal $y[n]$

    - is equal to a linear combination of the <u>shifts</u> of the input signal $x[n]$.

  - For some numbers $a_0, a_1, a_2, \dots a_{N-1}$ and some numbers $b_0, b_1, \dots, b_{M-1}$

    - The I/O-equation is given by:

$$a_0 y[n] + a_1 y[n-1] + \dots + a_{N-1} y[n-(N-1)] \qquad (*)$$

$$= b_0 x[n] + b_1 x[n-1] + \dots + b_{M-1} x[n-(M-1)].$$

  - We usually assume that $a_0 = 1$.

    - In case you ever get one where $a_0 \neq 1$, just divide both sides by a_0 ... then you will have a new equivalent equation where a_0 <u>is</u> equal to 1.

- For the z-transform, this all works very similar to what we already saw on pages 5.46 - 5.73 for the DTFT.

- Taking the z-transform on both sides of the general difference equation (*) on p. 6.127, we get:

$$\mathcal{Z}\{a_0 y[n] + a_1 y[n-1] + \ldots + a_{N-1}\, y[n-(N-1)]\}$$
$$= \mathcal{Z}\{b_0 x[n] + b_1 x[n-1] + \ldots + b_{M-1} x[n-(M-1)]\}$$

$$a_0 Y(z) + a_1 z^{-1} Y(z) + \ldots + a_{N-1} z^{N-1} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + \ldots + b_{M-1} z^{M-1} X(z)$$

$$Y(z)\left[a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{N-1}\right] = X(z)\left[b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{M-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{M-1}}{a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{N-1}} \qquad (**)$$

<u>Note</u>: we could have used "capital Σ" do loops to save a lot of writing on p. 6.128... like this:

- Difference Equation:

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{m=0}^{M-1} b_m x[n-m]$$

- Take z-transform and use the time shift property:

$$Z\left\{\sum_{k=0}^{N-1} a_k y[n-k]\right\} = Z\left\{\sum_{m=0}^{M-1} b_m x[n-m]\right\}$$

$$\sum_{k=0}^{N-1} a_k Z\{y[n-k]\} = \sum_{m=0}^{M-1} b_m Z\{x[n-m]\}$$

$$\sum_{k=0}^{N-1} a_k z^{-k} Y(z) = \sum_{m=0}^{M-1} b_m z^{-m} X(z)$$

$$Y(z)\sum_{k=0}^{N-1} a_k z^{-k} = X(z)\sum_{m=0}^{M-1} b_m z^{-m}$$

$$H(z) = \frac{\sum_{m=0}^{M-1} b_m z^{-m}}{\sum_{k=0}^{N-1} a_k z^{-k}} \qquad (***)$$

⟹ which is <u>exactly</u> the same as eq. (**) on p. 6.128.

- From eq. (**) on p. 6.128 (equivalently, from eq. (***) on p. 6.129),

⇒ We see that for a discrete time LTI system H with difference equation (*) on p. 6.127,

⇒ The transfer function $H(z)$ is <u>always</u> a ratio of two polynomials in the "character" z^{-1}.

→ we say that $H(z)$ is a <u>rational</u> function of z^{-1}.

⇒ The denominator coefficients of $H(z)$ come <u>directly</u> from the "y-side" of the difference equation.

☆☆ ⇒ The "y-side" of the difference equation determines the <u>poles</u> of $H(z)$.

- This is important!!!

⇒ The numerator coefficients of $H(z)$ come directly from the "x-side" of the difference equation.

☆☆ ⇒ The "x-side" of the difference equation determines the zeros of $H(z)$.

→ This is also important !!!

EX:
$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{4}x[n-1]$$

$a_0 = 1$ $\qquad b_0 = 1$

$a_1 = -\frac{5}{6}$ $\qquad b_1 = -\frac{1}{4}$

$a_2 = \frac{1}{6}$

$$H(z) = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

[compare to the DTFT example on pages 5.47-5.48]

- Factoring as on pages 5.50 - 5.51, we get

$$H(z) = \frac{1 - \frac{1}{4}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

EX:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + \frac{1}{6}x[n-1]$$

$$H(z) = \frac{1 + \frac{1}{6}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

⇒ Compare to the 2nd DTFT example on p. 5.57.

⇒ This will become very easy to do after you practice a little bit.

⇒ Remember: the difference equation does not specify the system uniquely.

- For any given difference equation, there will generally be more than one LTI system that shares the difference equation.
  - → The number of systems is equal to the number of possible ROCs for $H(z)$.
- To determine the system uniquely, you need more information... →

- For example, you might also be told one or more of the following:

    - H is causal / not causal
    - H is stable / unstable
    - $H(e^{j\omega})$ exists / does <u>not</u> exist
    - $h[n]$ is right-sided / two-sided / left-sided

- But, for any given difference equation,

    - At <u>most</u> one of the systems will be causal (the one with the overall exterior ROC is the only one that might be causal ... if and only if $h[n] = 0 \ \forall\, n < 0$).

    - <u>Exactly</u> <u>one</u> of the systems will be stable... it is the one with the ROC that includes the unit circle.

    - <u>Exactly</u> <u>one</u> of the systems will have a frequency response $H(e^{j\omega})$... it is also the one with the ROC that includes the unit circle.

- It might or might not be true that one of the systems can be both causal and stable.

  → Recall from p. 6.101: for an LTI system to be both causal and stable, all of the poles must lie strictly inside the unit circle.

  → So, for any given difference equation there will generally be several possible ROCs... depending on the locations of the poles.

⇒ If the ROC that is overall exterior also includes the unit circle of the z-plane,

⇒ Then it is possible for the system corresponding to this ROC to be both causal and stable... iff $h[n] = 0 \ \forall n < 0$.

EX: H is a causal, stable discrete-time LTI system with input-output equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 2x[n] - \frac{5}{6}x[n-1]$$

- Find the impulse response $h[n]$.

Solution: take z-transform on both sides to find the transfer function $H(z)$:

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z)$$
$$= 2X(z) - \frac{5}{6}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right] = X(z)\left[2 - \frac{5}{6}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - \frac{5}{6}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

$$= \frac{2\left(1 - \frac{5}{12}z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \qquad (*)$$

$\longrightarrow$

- From (*) on p. 6.135 we can see that there are poles @ $z = \frac{1}{3}$ and $z = \frac{1}{2}$, as well as a zero @ $z = \frac{5}{12}$.

- Converting from z^{-1} to z (multiply $H(z)$ by $1 = \frac{z^2}{z^2}$), we get

$$H(z) = \frac{2z\left(z - \frac{5}{12}\right)}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

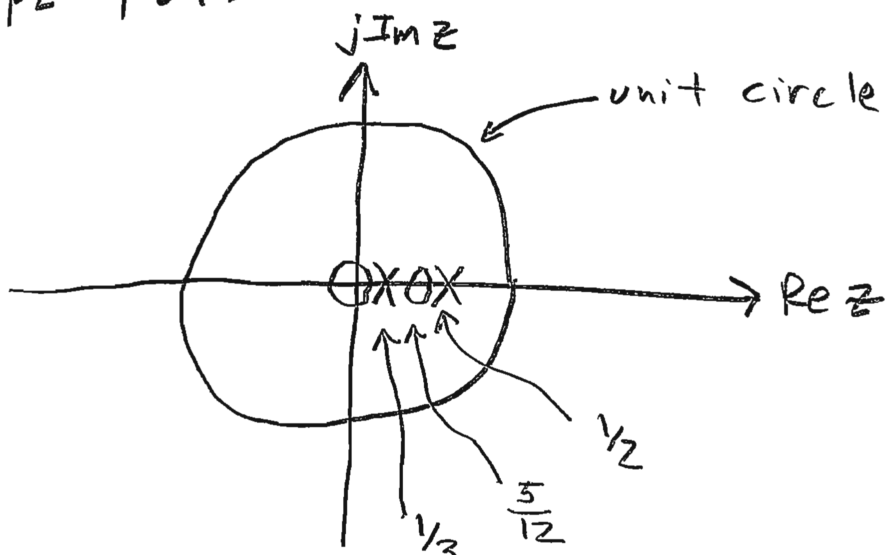
→ This shows us an additional zero @ $z=0$.

→ At $z=\infty$ we get

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{2z^2}{z^2} = 2$$

→ So $z=\infty$ is not a pole and is not a zero.

- Here is the pz-plot:



- The possible ROC's are:

$$|z| < \frac{1}{3}$$

$$\frac{1}{3} < |z| < \frac{1}{2}$$

$$|z| > \frac{1}{2}$$

→ Since the system is given to be stable, the ROC must include the unit circle

→ Since the system is given to be causal, the ROC must be exterior

$\Longrightarrow$ The ROC <u>must</u> be $\underline{\underline{|z| > \frac{1}{2}}}$

- To invert and find $h[n]$, we need a PFE:
- From (*) on p. 6.135, we have

$$\frac{2 - \frac{5}{6}\theta}{(1-\frac{1}{3}\theta)(1-\frac{1}{2}\theta)} = \frac{A}{1-\frac{1}{3}\theta} + \frac{B}{1-\frac{1}{2}\theta}$$

$$A = \left.\frac{2 - \frac{5}{6}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=3} = \frac{2-\frac{5}{2}}{1-\frac{3}{2}} = \frac{\frac{4}{2}-\frac{5}{2}}{-\frac{1}{2}} = \frac{-\frac{1}{2}}{-\frac{1}{2}} = \underline{1}$$

→

$$B = \left.\frac{2-\frac{5}{6}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{2-\frac{5}{3}}{1-\frac{2}{3}} = \frac{\frac{6}{3}-\frac{5}{3}}{1/3} = \frac{1/3}{1/3} = 1$$

– So,

$$H(z) = \underbrace{\frac{1}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

Table: $\boxed{h[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n]}$

- $h[n] = 0 \quad \forall\, n < 0$: causal ✓

- $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left| \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n \right|$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{1-\frac{1}{3}} + \frac{1}{1-1/2} = \frac{1}{2/3} + \frac{1}{1/2}$$

$$= \frac{3}{2} + 2 = \frac{7}{2} < \infty$$ ✓ <u>STABLE</u>

FIR Filters in the z-domain

- As we saw on pages 5.67 - 5.68, for an FIR filter the difference equation has no shifts of $y[n]$.

    $\Rightarrow$ This is equivalent to saying that $h[n]$ has finite length, although that fact is not obvious and takes some work to prove rigorously.

<u>EX</u>: $h[n] = \delta[n] - \frac{5}{6}\delta[n-1] + \frac{1}{6}\delta[n-2]$ finite length

- Using "convolution with deltas" from our course notes on convolution, we get

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= x[n] * \left\{ \delta[n] - \tfrac{5}{6}\delta[n-1] + \tfrac{1}{6}\delta[n-2] \right\} \\
&= x[n] - \tfrac{5}{6}x[n-1] + \tfrac{1}{6}x[n-2]
\end{aligned}
$$

$\longrightarrow$

- So the I/O equation is

$$y[n] = x[n] - \frac{5}{6}x[n-1] + \frac{1}{6}x[n-2]$$

No shifts of $y[n]$

⇒ This means that the denominator of $H(z)$ is equal to $\underline{\underline{1}}$.

⇒ This means there are no "nontrivial" poles... because the denominator of $H(z)$ is a <u>zeroth</u> order polynomial and has <u>no</u> <u>roots</u>.

⇒ For an FIR filter, the poles are all located at $\underline{\underline{z=0}}$.

- From the difference equation above, or by transforming $h[n]$ on p. 6.139 (Either way is fine), we get

$$H(z) = 1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2} \qquad (*)$$

⟶

- From (*) on p. 6.140, it's clear that $z=0$ is a pole, because

$$H(z) = 1 - \frac{5}{6} \cdot \frac{1}{z} + \frac{1}{6} \cdot \frac{1}{z^2} .$$

- So the ROC is $|z| > 0$.

- But to understand this more clearly, let's write $H(z)$ as a rational function in factored form:

$$H(z) = \frac{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}{1} = \frac{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})}{1}$$

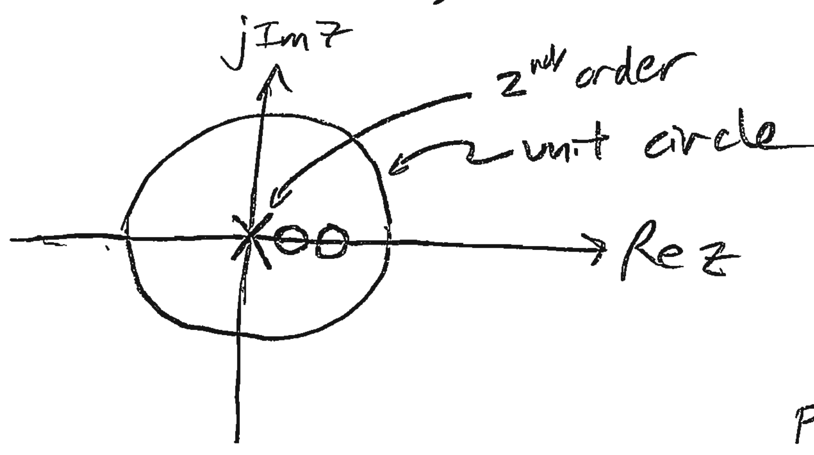
- Multiply by $\frac{z^2}{z^2}$ to convert from z^{-1} to z:

$$H(z) = \frac{(z - \frac{1}{2})(z - \frac{1}{3})}{z^2}$$

zeros: $z = \frac{1}{2}, \frac{1}{3}$

poles: $z = 0$ (2nd order)

p-z plot:



- For an FIR filter, this will always be the case.
  - the poles will all be located @ $z=0$.
  - the ROC will always be $|z|>0$
  - $\Rightarrow$ The ROC will always include the unit circle
    $\Rightarrow$ The filter will always be stable
    $\Rightarrow$ $H(e^{j\omega})$ will always exist (converge).
  - The filter might or might not be causal, depending on whether or not $h[n]=0 \ \forall n<0$.

EX: $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$

$\rightarrow h[n] = 0 \quad \forall \quad n<0$

$\Rightarrow$ causal

EX: $h[n] = \frac{1}{4}\delta[n+1] + \frac{1}{2}\delta[n] + \frac{1}{4}\delta[n-1]$

$h[-1] = \frac{1}{4} \neq 0$

$\Rightarrow$ NOT causal

IIR Filters in the z-domain

- As we saw for the DTFT on pages 5.70 – 5.73, if there are any shifts of $y[n]$ in the difference equation,
  - then the length of the impulse response $h[n]$ is <u>infinite</u>.
  - the system is an IIR filter.
- Thus, the transfer function $H(z)$ of an <u>IIR</u> filter will have a "nontrivial" denominator...
  - This means that the denominator will be a nontrivial polynomial in z^{-1}... it will <u>not</u> be equal to 1.

<u>EX</u>:

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{1}{4} x[n-1] \qquad (*)$$

→ The difference equation has shifts of $y[n]$

→ This is an IIR filter

→ The denominator of $H(z)$ will not be 1

→ There will be nontrivial poles at locations <u>other</u> than $z=0$.

→

- Applying the z-transform to both sides of the difference equation on p. 6.143 (and using the time shift property), we get:

$$\mathcal{Z}\left\{y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2]\right\} = \mathcal{Z}\left\{x[n] - \frac{1}{4}x[n-1]\right\}$$

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = X(z) - \frac{1}{4}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right] = X(z)\left[1 - \frac{1}{4}z^{-1}\right]$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}}$$

→ compare to the DTFT example on p. 5.71.

- Without more information, we cannot determine the ROC.

- Factoring the denominator, we get

$$H(z) = \frac{1 - \frac{1}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

→ There is a zero @ $z = \frac{1}{4}$

→ There are poles @ $z = \frac{1}{2}, \frac{1}{3}$

→

- Multiplying $H(z)$ by $\frac{z^2}{z^2}$ to convert from z^{-1} to z, we get

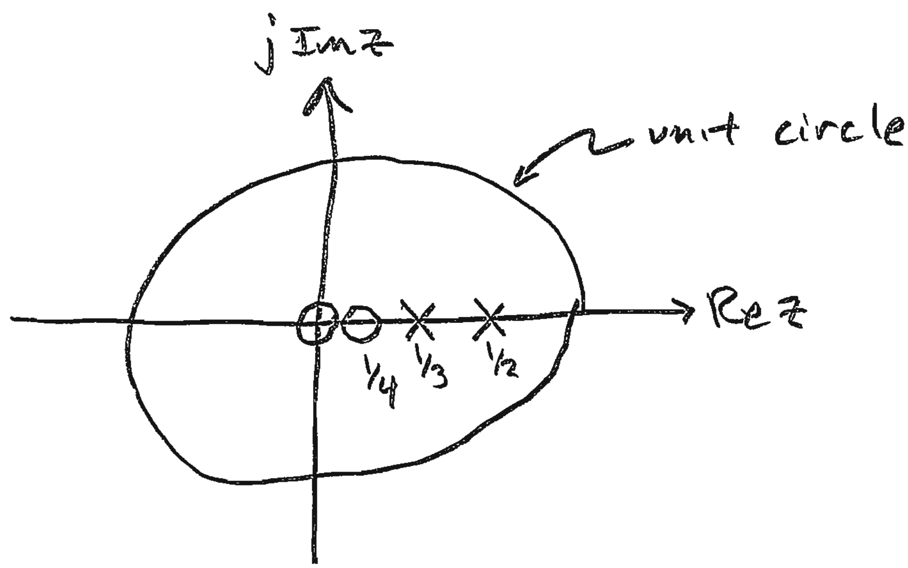
$$H(z) = \frac{z\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

→ There is an additional zero @ $z=0$

→ $\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{z^2}{z^2} = \lim_{z\to\infty} 1 = 1$

- so $z=\infty$ is not a pole and is not a zero.

PZ-plot:



- The possible ROC's for $H(z)$ are:

$$|z| < \frac{1}{3}$$

$$\frac{1}{3} < |z| < \frac{1}{2}$$

$$|z| > \frac{1}{2}$$

- This tells us that there are <u>three</u> IIR filters that have this difference equation and this $H(z)$.

  → But they all have <u>different</u> ROCs!

  ⇒ <u>Only</u> the one with ROC $|z| > \frac{1}{2}$ can be causal ($h[n]$ right-sided).

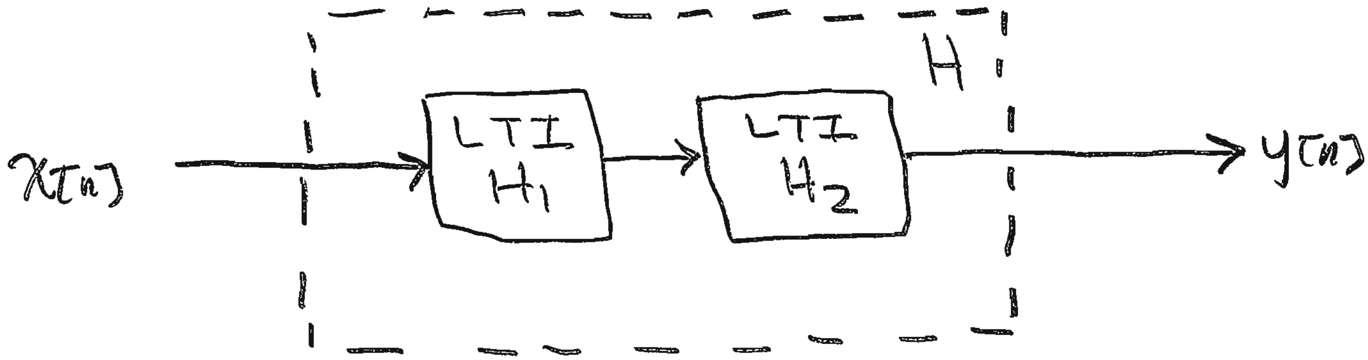
  ⇒ <u>Only</u> the one with ROC $|z| > \frac{1}{2}$ is stable... because this is the ROC that contains the unit circle.

  ⇒ Only the one with ROC $|z| > \frac{1}{2}$ has a frequency response $H(e^{j\omega})$... again because this is the only ROC that contains the unit circle... and the unit circle is where $H(z)\Big|_{|z|=r=1} = H(e^{j\omega})$.

z-domain System Interconnections

- This is almost identical to what we saw for the DTFT on pages 5.33 – 5.38.

- "Series" or "cascade" connection:



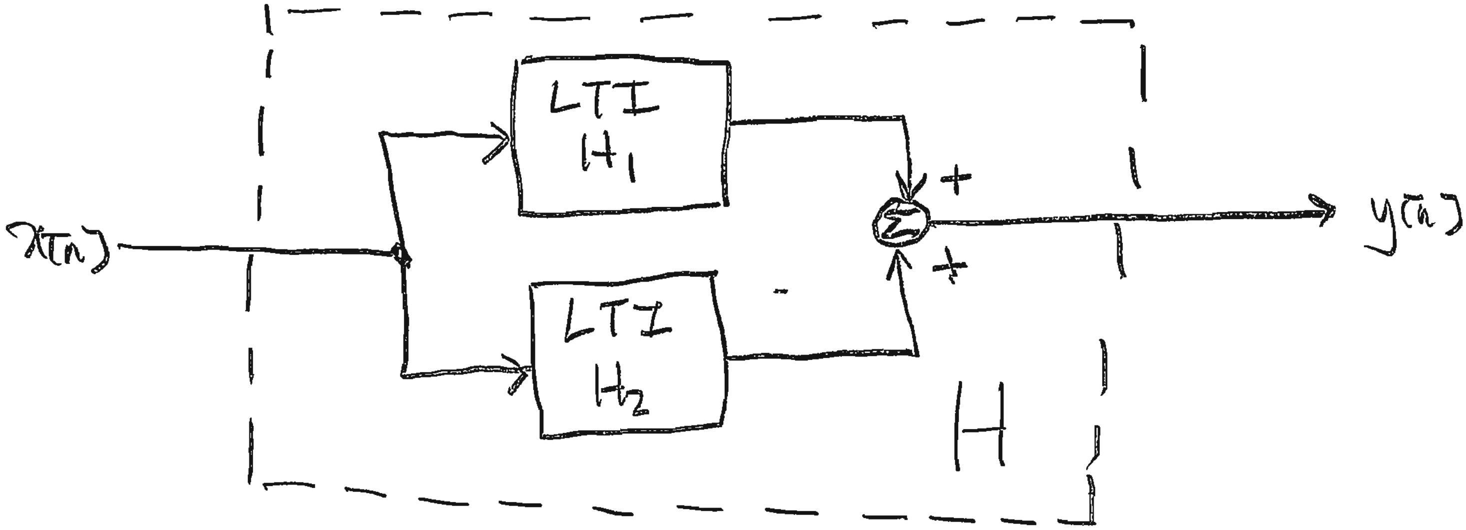
$$h[n] = h_1[n] * h_2[n]$$

$$H(z) = H_1(z)\, H_2(z)$$

- The ROC is <u>at</u> <u>least</u> $R_1 \cap R_2$

- Generally, the ROC will be the intersection of the ROCs' of $H_1(z)$ and $H_2(z)$.
- However, if mutiplying $H_1(z)$ times $H_2(z)$ causes a pole-zero cancellation, then the ROC might be expanded.

- Parallel Connection:

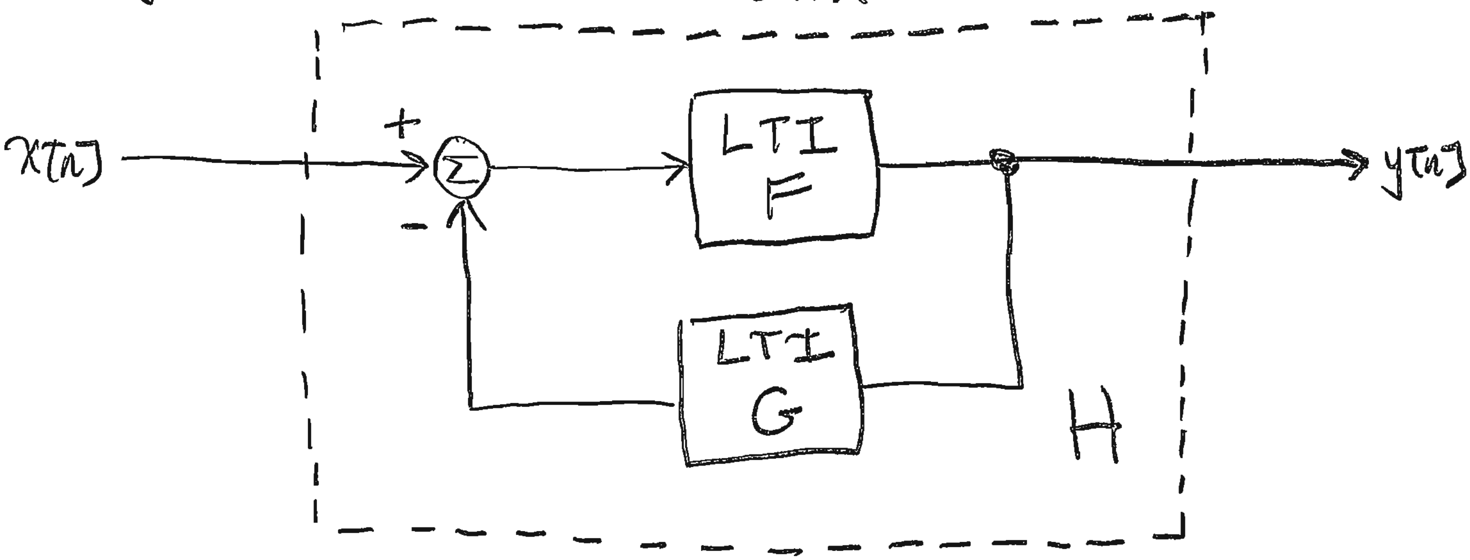


$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

- The ROC is <u>at</u> <u>least</u> $R_1 \cap R_2$
- Generally, the ROC will be the intersection of the ROCs' of $H_1(z)$ and $H_2(z)$.
- However, if adding $H_1(z)$ and $H_2(z)$ causes a pole-zero cancellation, then the ROC of $H(z)$ might be expanded.

- Negative Feedback Connection:



h[n]: no general closed form in the time domain.

$$H(z) = \frac{F(z)}{1 + F(z)G(z)}$$

- This connection will generally cause the poles of F(z) and G(z) to change location in H(z).
- So there is no simple rule to determine the ROC of H(z) from the ROC's of F(z) and G(z).
- To determine the ROC of H(z), you must find the poles and then use causality and/or stability information to deduce the correct ROC.

- If F and G are <u>both</u> causal, then H will also be causal.
  - In this case, the ROC of $H(z)$ will be exterior to the largest magnitude pole.

- In most practical cases, you will only be interested in the case where the system H is both causal and stable... so the correct ROC will be the one that contains the unit circle and it will be exterior to the largest magnitude pole.