ECE 4213/5213

MODULE 1

Index

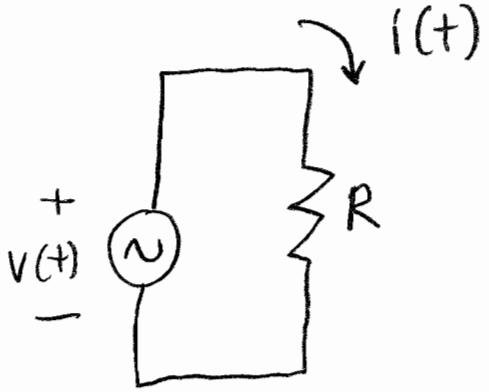
<u>NOTE</u>: unlike ECE 3793 (signals & systems), you will need to read the book for ECE 4213/5213.

→ I will try to make the notes as complete and clear as possible, but some things you will have to get yourself out of the book.

<u>DEF</u>: A <u>signal</u> is a manifestation (or "tabulation" or "listing") of one quantity with respect to another.

→ You are already well familiar with this idea intuitively.

<u>EX</u>:



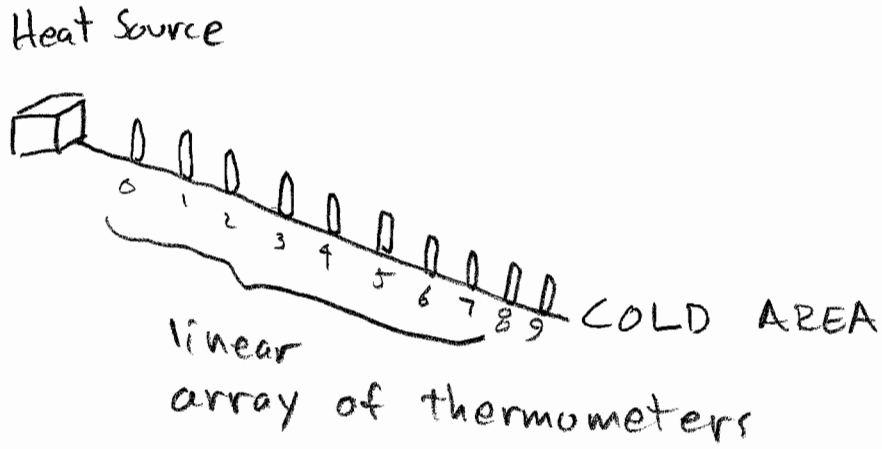
$v(t)$: voltage with respect to time.

$i(t)$: current with respect to time.

- In this class, we will usually talk about the variation of some quantity with respect to <u>time</u>.

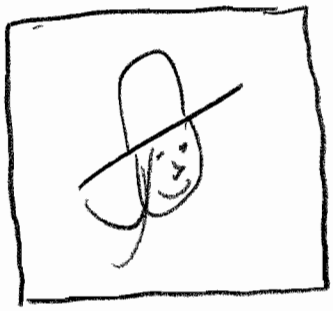
  → This is for convenience, and many signal processing applications are concerned with other types of signals.

<u>EX</u>:



$t[x]$ could be temperature with respect to displacement for $x = 0, 1, 2, \ldots, 9$.

<u>EX</u>: A digital image (ECE 5273):



$I(m,n)$ = optical brightness with respect to spatial position.

- We model signals with <u>functions</u>.

<u>DEF</u>: A function is a <u>rule</u> that matches each member of one set (the domain) with one and <u>only one</u> member of a second set (the range).

<u>EX</u>: $x(t) = 120\cos(60t + \pi/8)$

Domain $= \mathbb{R}$; $t \in \mathbb{R}$.

Range $= \mathbb{R}$; $x(t) \in \mathbb{R}$.

<u>EX</u>: $x(t) = e^{-2t} e^{j60t} u(t)$

Domain $= \mathbb{R}$; $t \in \mathbb{R}$.

Range $= \mathbb{C}$; $x(t) \in \mathbb{C}$.

<u>EX</u>: $x[n] = \left(\frac{1}{4}\right)^n u[n]$

Domain $= \mathbb{Z}$; $n \in \mathbb{Z}$.

Range $= \mathbb{R}$; $x[n] \in \mathbb{R}$.

<u>EX</u>: Temperature experiment from p. 1-2:

$$\text{Domain} = \{0, 1, \ldots, 9\},$$

a subset of $\mathbb{Z}$.

$$\text{Range} = \mathbb{R}.$$

<u>EX</u>: Lena Image from p. 1-2:

$$\text{Domain} = \{(m,n) : 0 \le m, n \le 255\},$$

a subset of $\mathbb{Z}^2$.

$$\text{Range} = \{0, 1, 2, \ldots, 255\},$$

a subset of $\mathbb{Z}$.

→ If the domain of the signal (more precisely, the domain of the function used to model the signal) is "continuous" or "uncountable", then it is a <u>continuous-time signal</u>.

(aka, continuous-domain signal).

- Most of the time in this class, continuous-time signals will have domain $\mathbb{R}$, as in a voltage $v(t)$ considered as a function $v: \mathbb{R} \to \mathbb{R}$ or $v: \mathbb{R} \to \mathbb{C}$.

- If the domain is countable or finite, the signal is a "discrete-time signal" or "discrete-domain signal".

  - Most discrete-time signals we will see in this class will have domain $\mathbb{Z}$ or a finite subset of $\mathbb{Z}$, as in

$$x[n] = \left(\tfrac{1}{4}\right)^n u[n], \qquad n \in \mathbb{Z},$$

or

$$x[n] = \left(\tfrac{1}{4}\right)^n, \qquad 0 \le n \le 15.$$

<u>DEF</u>: A discrete-time signal that has a countable or finite <u>range</u> is a <u>digital signal</u>.

<u>EX</u>: The Lena image from p. 1-2.

<u>EX</u>: An array of 4-byte integers in a computer program.

→ For most of this class, we'll be concerned with discrete-time signals.

→ Theoretical treatment of true digital signals is difficult.

→ The usual way to model a <u>true</u> digital signal $x_{dig}[n]$ is

$$x_{dig}[n] = x_{dt}[n] + q[n],$$

where $x_{dt}[n]$ is a discrete-time signal
$q[n]$ is a random signal that models the "quantization error."

Often, we refer to a signal as being 1D, 2D, or nD based on the <u>dimension</u> of an element of the domain.

<u>EX</u>: $x[n] = \left(\frac{1}{4}\right)^{-|n|}$, $n \in \mathbb{Z}$.

- here "n" is a scalar... 1D.
  So we call $x[n]$ a 1D signal.
  i.e. digital audio, digital speech.

<u>EX</u>: $I(m,n)$, a digital image.

- here, a domain element is an ordered pair of integers $(m,n) \in \mathbb{Z}^2$,
  So a domain element is 2D and we call $I(m,n)$ a 2D signal.

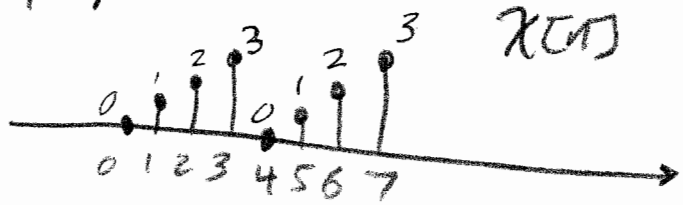
<u>EX</u>: Digital Video $x(m,n,t)$

$\longrightarrow$ Domain is 3D, x is a 3D signal

☆ <u>NOTE</u>: As we shall soon see, this is <u>not</u> the <u>only</u> way that the idea of dimension is used to describe a signal.

→ So you have to realize there is more than one meaning of "the dimension" of a signal and always keep it in mind which meaning is being used in any given discussion.

→ consider this signal $x[n]$ defined for $0 \le n \le 7$:



- because n is a scalar, we can describe this as a 1-D signal.

- but you can also think of this signal as a vector $x[n] = [0\ 1\ 2\ 3\ 0\ 1\ 2\ 3]^T$, which is a point in $\mathbb{R}^8$; e.g., you can think of $x[n]$ as 8-D.

Symbols

$\forall$	"for all"
$\in$	"in" or "is an element of"
$\exists$	"there exists"
s.t.	"such that"

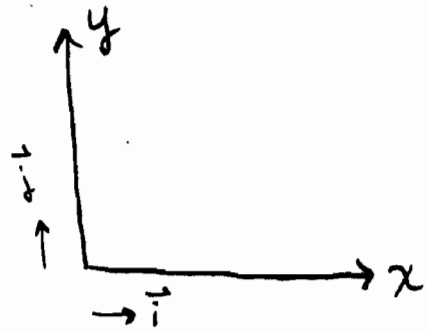
EX:

$\forall x \in \mathbb{R}$ "for all x in the reals"

$\exists x \in \mathbb{R}$ s.t. $x > 5$ "there exists an x in the reals such that x is greater than 5" (6, for example)

Signal Representations

- First, we review some linear algebra on $\mathbb{R}^2$.



$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $\mathbb{R}^2$ is the set of ordered pairs (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$.
- Each such ordered pair describes a point in the plane.

- The set of points, or vectors, $\{\vec{i}, \vec{j}\}$ is an orthonormal basis for $\mathbb{R}^2$.

- This means:

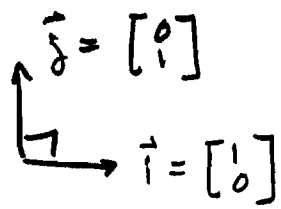
1. Any vector $\vec{v} \in \mathbb{R}^2$ can be written as a linear combination of the basis vectors:

$$\vec{v} = c_1 \vec{i} + c_2 \vec{j}$$

where c_1 and c_2 are constants.

2. Each basis vector has unit length.

3. All of the vectors in the basis are mutually orthogonal:



<u>DEF</u>:

- Given a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ and a vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, the dot product is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 = \sum_{n=1}^{2} x_n y_n$$

<u>EX</u>: $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\vec{x} \cdot \vec{y} = 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = \underline{\underline{11}}.$$

a <u>number</u>, not a vector.

<u>NOTES</u>

- In general, the dot product is written using angle brackets:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$$

- If one or both of the vectors have complex-valued entries, then you must <u>conjugate</u> the entries of the second vector when computing the dot product:

$\vec{x}, \vec{y} \in \mathbb{C}^n$,

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i^*$$

<u>Question</u>: given a vector $\vec{x} \in \mathbb{R}^2$, how do we write $\vec{x}$ as a linear combination of the basis $\{\vec{i}, \vec{j}\}$?

<u>Answer:</u> Use the dot product.

$$\vec{x} = c_1 \vec{i} + c_2 \vec{j}$$

where $c_1 = \langle \vec{x}, \vec{i} \rangle$

$c_2 = \langle \vec{x}, \vec{j} \rangle$

In other words,

$$\vec{x} = \langle \vec{x}, \vec{i} \rangle \vec{i} + \langle \vec{x}, \vec{j} \rangle \vec{j}.$$

<u>EX</u>: $\vec{x} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$$\begin{aligned}
\vec{x} &= \langle \vec{x}, \vec{i} \rangle \vec{i} + \langle \vec{x}, \vec{j} \rangle \vec{j} \\
&= \left\langle \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= (5 \cdot 1 + 8 \cdot 0) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (5 \cdot 0 + 8 \cdot 1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark
\end{aligned}$$

- This may seem overly simple, but the approach works in general.

<u>EX</u>: $\vec{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

is also an orthonormal basis for $\mathbb{R}^2$.

- To write a vector $\vec{x} \in \mathbb{R}^2$ as a linear combination of this basis, use the same strategy:

$$\vec{x} = \langle \vec{x}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{x}, \vec{e}_2 \rangle \vec{e}_2 .$$

- The space $\mathbb{R}^3$ is the set of ordered triples $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1, x_2, x_3 \in \mathbb{R}$.

- One orthonormal basis for $\mathbb{R}^3$ is the set $\{\vec{i}, \vec{j}, \vec{k}\}$ where

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The dot product in $\mathbb{R}^3$ is defined just like you expert. For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, the dot product is

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$
$$= \sum_{i=1}^{3} x_i y_i$$

- To write a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as a linear combination of the basis $\{\vec{i}, \vec{j}, \vec{k}\}$, proceed as before:

$$\vec{x} = \underbrace{\langle \vec{x}, \vec{i} \rangle}_{c_1} \vec{i} + \underbrace{\langle \vec{x}, \vec{j} \rangle}_{c_2} \vec{j} + \underbrace{\langle \vec{x}, \vec{k} \rangle}_{c_3} \vec{k}$$

(do an example)

$\Rightarrow$ Don't forget to conjugate the entries of the second vector if they are complex-valued.

- The procedure works in higher dimensional spaces as well.
  - Consider $\mathbb{R}^{3793}$.
  - A vector $\vec{x} \in \mathbb{R}^{3793}$ looks like

$$\vec{x} = [x_1 \ x_2 \ x_3 \ \dots \ x_{3793}]^T$$

  - If $\vec{x}, \vec{y} \in \mathbb{R}^{3793}$, then the dot product is

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{3793} x_i y_i^*$$

$\Rightarrow$ Don't forget to conjugate if the vectors have complex entries.

  - if the set of vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \dots, \vec{e}_{3793}\}$ is an orthonormal basis for $\mathbb{R}^{3793}$, then any vector $\vec{x} \in \mathbb{R}^{3793}$ can be written as

$$\vec{x} = \langle \vec{x}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{x}, \vec{e}_2 \rangle \vec{e}_2 + \dots + \langle \vec{x}, \vec{e}_{3793} \rangle \vec{e}_{3793}$$

$$= \sum_{n=1}^{3793} \langle \vec{x}, \vec{e}_n \rangle \vec{e}_n .$$

- This idea extends easily to infinite dimensional vector spaces too.

- Consider vectors of the form

$$\vec{x} = [\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots]^T$$

where the index runs over all the integers from $-\infty$ to ∞.

- This kind of vector is identical to a discrete-time signal $x[n]$ of the type you studied in signals & systems

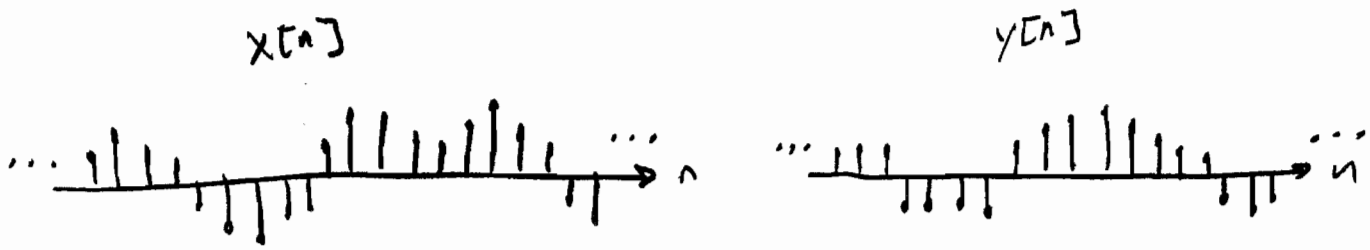
- For two such signals $\vec{x}, \vec{y}$ (or $x[n], y[n]$), the dot product is just

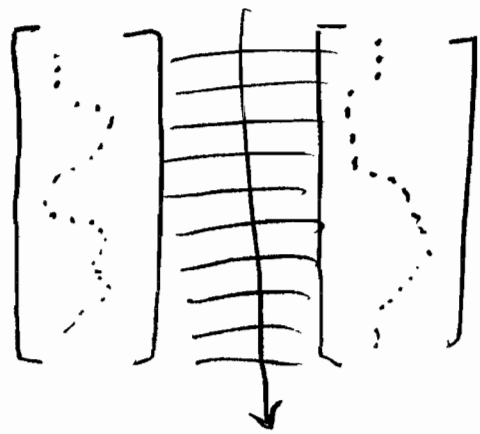
$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=-\infty}^{\infty} x_i y_i^* = \sum_{i=-\infty}^{\infty} x[i] y^*[i]$$

- The idea of "dot product" for two signals $x[n]$ and $y[n]$ is:
  1. Load the signal values into vectors
  2. Stand the vectors up beside each other
  3. If the entries are complex, conjugate the second vector
  4. Multiply corresponding entries
  5. Add up the products.

$\longrightarrow$

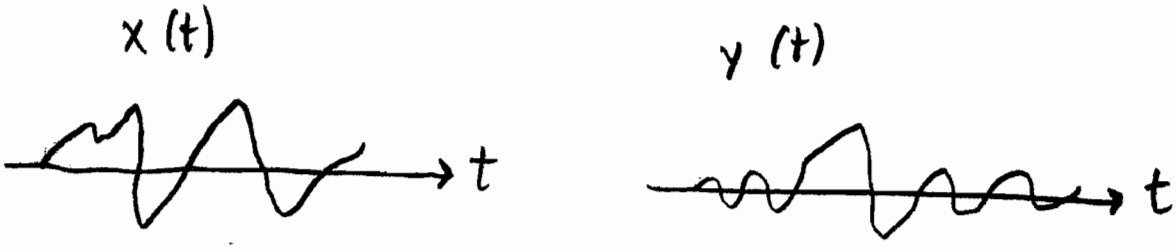
- In pictures:

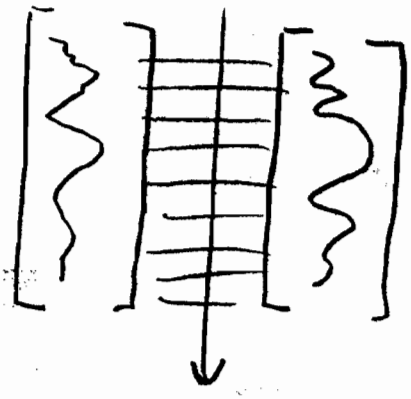




a number, $\langle x[n], y[n] \rangle = \sum_{i=-\infty}^{\infty} x[i] y^*[i]$

- The concept extends to continuous-time signals as well:



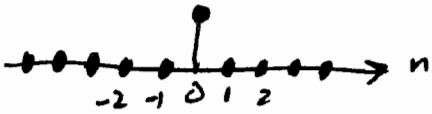


$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt$$

So What?

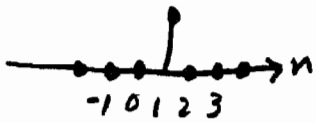
- Recall the discrete-time signal (Kronecker delta)

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{other} \end{cases}$$



- The signal is "turned on" at $n=0$.
- write it as a vector: $[\ldots\ 0\ 0\ \underset{\substack{\uparrow \\ n=0}}{1}\ 0\ 0\ \ldots]^T$

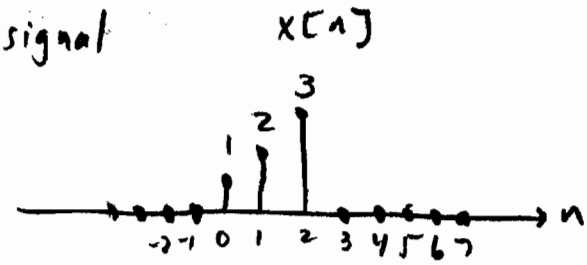
- The signal $\delta[n-1] = \begin{cases} 1 & n=1 \\ 0 & \text{other} \end{cases}$



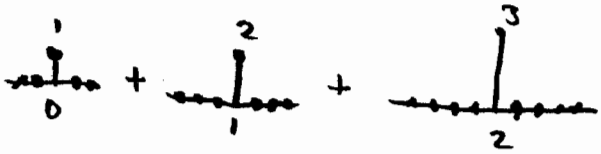
- The signal $\delta[n-k] = \begin{cases} 1, & n=k \\ 0 & \text{other} \end{cases}$ is turned on at $n=k$.

$\Rightarrow$ So, for discrete-time signals $x[n]$, the set $\{\delta[n-k]\}_{k\in\mathbb{Z}}$ plays the same role as the set $\{\vec{i}, \vec{j}, \vec{k}\}$ plays in $\mathbb{R}^3$.

- Consider the signal



- Obviously, $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

$=$ 

- But let's use linear algebra to write $x[n]$ as a linear combination of the basis $\{\delta[n-k]\}_{k\in\mathbb{Z}}$:

$$x[n] = \ldots + c_{-1}\underbrace{\delta[n-(-1)]}_{\delta[n+1]} + c_0\underbrace{\delta[n-0]}_{\delta[n]} + c_1\delta[n-1] + \ldots$$

- The coefficients c_k are given by

$$\begin{aligned} c_k &= \langle x[n], \delta[n-k] \rangle \\ &= \sum_{n=-\infty}^{\infty} x[n]\,\delta[n-k] \\ &= x[k] \quad \text{(why?)} \end{aligned}$$

- So $c_0 = 1$, $c_1 = 2$, $c_2 = 3$

  and the rest are zero.

- So $x[n] = 1\delta[n] + 2\delta[n-1] + 3\delta[n-2]$ ✓

$\Rightarrow$ This works in general.

we will often need to write a signal in terms of more than one basis.

- why?
  - just like changing coordinates (say, rectangular to spherical) sometimes makes a calculus problem easier,
  - changing basis will sometimes make a problem easier in this class.

- Recall: The set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ is, as we have seen, an orthonormal basis for the set of discrete-time signals $x[n]$.

- FACT: The set $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$ is also a basis.

  Note: $e^{j\omega n} = \cos \omega n + j \sin \omega n$

  → There is one basis "vector" for each $\omega \in [-\pi, \pi]$

  → The basis is not orthonormal. Each basis vector has length $\sqrt{2\pi}$ (not one!).

  → But this basis is orthogonal.

- Given a signal $x[n]$, let's write $x[n]$ as a linear combination of the basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$

<u>Step 1</u> : Use the dot product to find the required coefficients:

For any particular value $\omega \in [-\pi, \pi]$, we have the basis "signal" $e^{j\omega n}$ with required coefficient c_ω given by

$$c_\omega = \langle x[n], e^{j\omega n} \rangle$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(e^{j\omega n}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$\Rightarrow$ This is called the <u>discrete-time Fourier transform</u>.

$\rightarrow$ Usually, we write all of the coefficients for all of the basis signals together as a function of "ω" (indexing set) using the notation $X(e^{j\omega})$:

$$X(e^{j\omega}) = \langle x[n], e^{j\omega n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$\longrightarrow$ a number for each ω ...

e.g., a <u>function</u>.

Step 2: Add up all the coefficients times the corresponding basis signals to get our signal $x[n]$.

Note: Since the basis signals have length $\sqrt{2\pi}$ instead of 1, we have to scale the coefficients by $\frac{1}{2\pi}$.

$$x[n] = \frac{1}{2\pi} \underset{\omega \in [-\pi,\pi]}{\text{Add}} \left\{ (\text{coef})(\text{basis signal}) \right\}$$

$$= \frac{1}{2\pi} \underset{\omega \in [-\pi,\pi]}{\text{Add}} \left\{ C_\omega e^{j\omega n} \right\}$$

$$= \frac{1}{2\pi} \underset{\omega \in [-\pi,\pi]}{\text{Add}} \left\{ X(e^{j\omega}) e^{j\omega n} \right\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

- This is called the discrete-time inverse Fourier transform.

- Go back to $\mathbb{R}^2$ for a moment:
  - consider a basis $\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}$.
  - this basis is orthogonal, but not orthonormal, since each basis vector has length <u>three</u>, not <u>one</u>.
  - For the vector $\vec{x} = \begin{bmatrix}5\\8\end{bmatrix} \in \mathbb{R}^2$, we have

$$\left\langle \vec{x}, \begin{bmatrix}3\\0\end{bmatrix} \right\rangle = \begin{bmatrix}5\\8\end{bmatrix} \cdot \begin{bmatrix}3\\0\end{bmatrix} = 5\cdot 3 + 8\cdot 0 = 15$$

$$\left\langle \vec{x}, \begin{bmatrix}0\\3\end{bmatrix} \right\rangle = \begin{bmatrix}5\\8\end{bmatrix} \cdot \begin{bmatrix}0\\3\end{bmatrix} = 5\cdot 0 + 8\cdot 3 = 24$$

Too big by a factor of 3.

- Adding up dot products times basis vectors, we get

$$\left\langle \vec{x}, \begin{bmatrix}3\\0\end{bmatrix} \right\rangle \begin{bmatrix}3\\0\end{bmatrix} + \left\langle \vec{x}, \begin{bmatrix}0\\3\end{bmatrix} \right\rangle \begin{bmatrix}0\\3\end{bmatrix}$$

$$= 15 \begin{bmatrix}3\\0\end{bmatrix} + 24 \begin{bmatrix}0\\3\end{bmatrix}$$

basis vectors too long by 3.

dot products too big by 3

$$= \begin{bmatrix}45\\0\end{bmatrix} + \begin{bmatrix}0\\72\end{bmatrix} = \begin{bmatrix}45\\72\end{bmatrix} = 9\begin{bmatrix}5\\8\end{bmatrix} = 9\vec{x}.$$

length of basis vector squared.

$\Rightarrow$ If your basis vectors are orthogonal, but not orthonormal, then you have to divide the representation by the squared length of a basis vector.

- Now consider the set of continuous-time signals $x(t)$.

  ⇒ when <u>only</u> continuous-time signals are under consideration, it is common to use ω for the continuous-time frequency index, just like we just did for the DTFT.

  ⇒ However, in a discussion that may involve <u>both</u> continuous-time signals and discrete-time signals, different frequency indices need to be used for the two.

    → It is customary to use Ω for the continuous-time frequency index and ω for the discrete-time frequency index.

  <u>NOTE</u>: This is customary, but it is <u>backwards</u> from Chapter 7 of the ECE 3793 textbook.

⇒ In this class, we will use Ω for continuous-time frequency and ω for discrete time frequency.

- Back to the set of continuous-time signals $x(t)$:

<u>FACT</u>: The set $\{e^{j\Omega t}\}_{\Omega \in \mathbb{R}}$ is a basis for this set of signals.

- As in discrete time, this "spectral" basis is orthogonal but not orthonormal.
- Each basis signal has length $\sqrt{2\pi}$ ... so we will have to divide by 2π in the representation;

  e.g., we will have to multiply the coefficients (dot products) by $\frac{1}{2\pi}$.

<u>Step 1</u>: for each basis signal, use the dot product to find the required coefficient:

$$c_\Omega = \langle x(t), e^{j\Omega t} \rangle$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt$$

$$\equiv X(\Omega)$$

$\Rightarrow$ This is called the <u>Fourier transform</u> of $x(t)$.

<u>Step 2</u> : Add up all the coefficients (dot products) times the corresponding basis signals and divide by 2π to get back our $x(t)$:

$$x(t) = \frac{1}{2\pi} \underset{\Omega \in \mathbb{R}}{\text{Add}} \left\{ (\text{coeff})(\text{basis signal}) \right\}$$

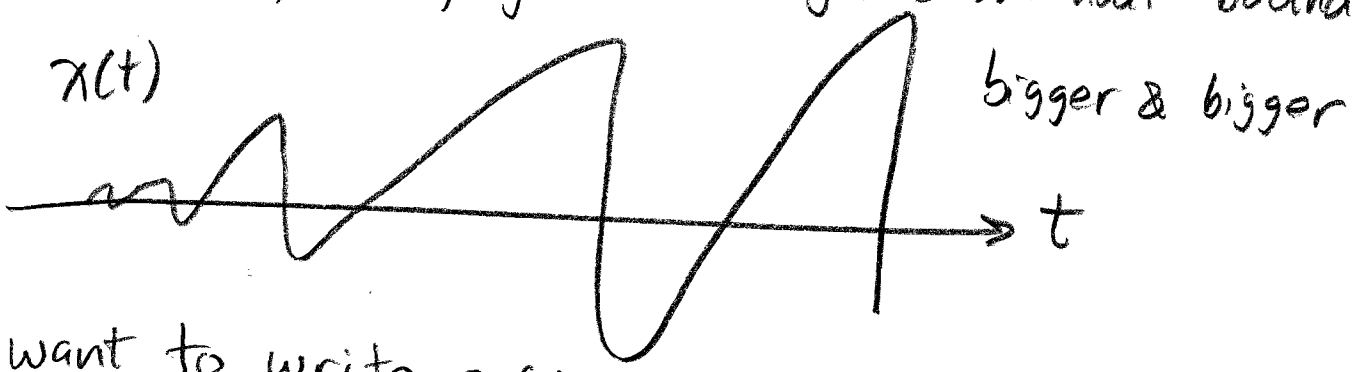
$$= \frac{1}{2\pi} \underset{\Omega \in \mathbb{R}}{\text{Add}} \left\{ c_\Omega e^{j\Omega t} \right\}$$

$$= \frac{1}{2\pi} \underset{\Omega \in \mathbb{R}}{\text{Add}} \left\{ X(\Omega) e^{j\Omega t} \right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} \, d\Omega$$

- This is called the <u>Inverse Fourier Transform</u>.

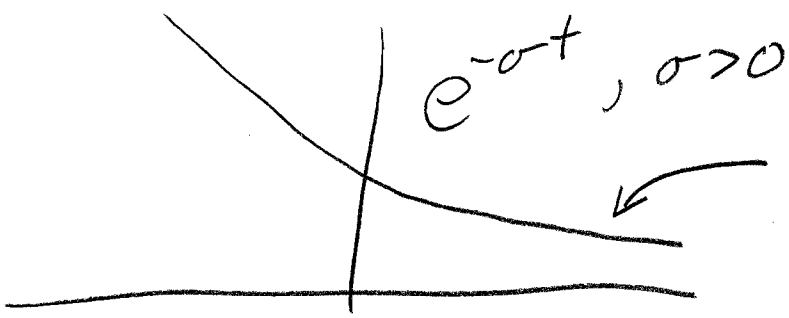
- Now suppose we have an $x(t)$ who is a "bad guy."
  - → As $t \to \infty$, $x(t)$ grows and grows without bound:



- we want to write $x(t)$ as a linear composition of the basis $\{e^{j\Omega t}\}_{\Omega \in \mathbb{R}}$.
- we try to compute the coefficients $X(\Omega)$ using the dot product

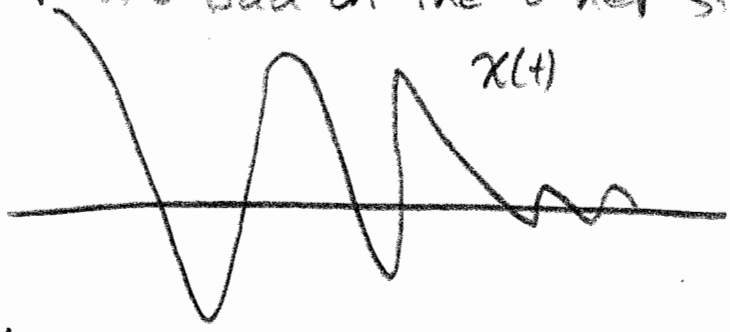
$$X(\Omega) = \langle x(t), e^{j\Omega t} \rangle = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

  - → But this integral blows up because $x(t)$ is so bad.
- Maybe we can "fix things up" if we multiply $x(t)$ by a decaying exponential $e^{-\sigma t}$, where $\sigma \in \mathbb{R}$ and $\sigma > 0$:

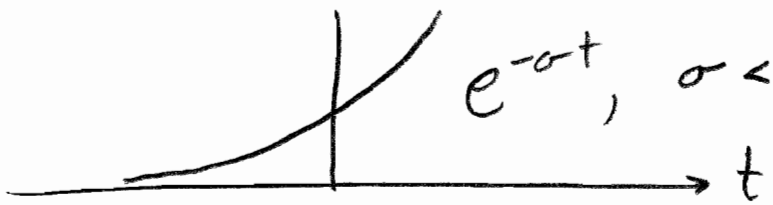


going to zero faster than any finite order polynomial.

- For a signal that's bad on the other side



we will have to use an exponential that vanishes to the left (and grows to the right)... a growing exponential $e^{-\sigma t}$ with $\sigma < 0$:



- In general, for "bad guys" $x(t)$ we choose an appropriate "fixer upper parameter" σ and then write the "fixed up" signal $e^{-\sigma t} x(t)$ as a linear combination of the basis $\{e^{j\Omega t}\}_{\Omega \in \mathbb{R}}$.
- The "rate" σ of the exponential is a real parameter.

→ For now, call the coefficients (dot products) $X(\sigma, \Omega)$:

$$X(\sigma, \Omega) = \langle e^{-\sigma t} x(t), e^{j\Omega t} \rangle$$
$$= \int_{-\infty}^{\infty} e^{-\sigma t} x(t) e^{-j\Omega t} dt$$
$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\Omega)} dt$$

- By defining a complex variable $s = \sigma + j\Omega$, we can write the σ and Ω parameters together to get

$$X(s) = \langle e^{-\sigma t} x(t), e^{j\Omega t} \rangle = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

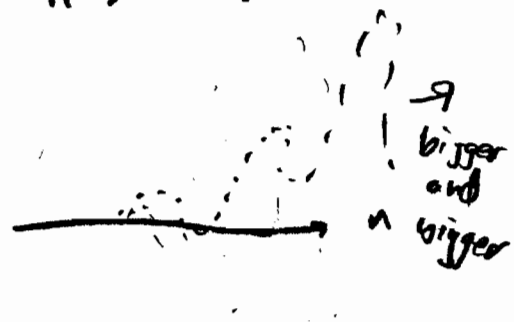
- This is called the <u>Laplace</u> <u>transform</u> of $x(t)$.

- The procedure for adding up the coefficients times the basis functions to get back to $x(t)$ is called the <u>inverse</u> <u>Laplace</u> <u>transform</u>.

- It is similar to what we have already done with the discrete-time Fourier transform and the Fourier transform, but a little more complicated because of the fixer-upper function $e^{-\sigma t}$.

- Now suppose we have a "bad guy" $x[n]$ who grows out of control as $n \to \infty$ or as $n \to -\infty$:

or



- When we try to use the dot product to write $x[n]$ as a linear combination of the basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi]}$, the coefficients

$$X(e^{j\omega}) = \langle x[n], e^{j\omega n} \rangle$$
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

blow up.

- Like before, we can fix this up by multiplying $x[n]$ times an appropriate exponential α^{-n}, $\alpha \in \mathbb{R}$, $\alpha > 0$, $n \in \mathbb{Z}$

→ if $\alpha > 1$, then α^{-n} is a <u>decaying</u> exponential:

→ if $0 < \alpha < 1$, then α^{-n} is a growing exponential:

- Now we will write the fixed up signal $\alpha^{-n}x[n]$ as a linear combination of the basis $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi]}$ using the dot product.

- The required coefficients are

$$X(\alpha, e^{j\omega}) = \langle \alpha^{-n}x[n], e^{j\omega n}\rangle$$

$$= \sum_{n=-\infty}^{\infty} \alpha^{-n}x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\alpha^{-n}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(\alpha e^{j\omega}\right)^{-n}$$

- But for any pair of reals α, ω, $\alpha e^{j\omega}$ is just a complex number in polar form.

- So define $z = \alpha e^{j\omega}$. Then we have

$$X(z) = \langle \alpha^{-n}x[n], e^{j\omega n}\rangle$$

$$= \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- This is called the <u>Z-transform</u> of $x[n]$.

- When we add up the coefficients $X(z)$ times the basis signals to get $x[n]$, it is called the <u>inverse</u> <u>Z-transform</u>.

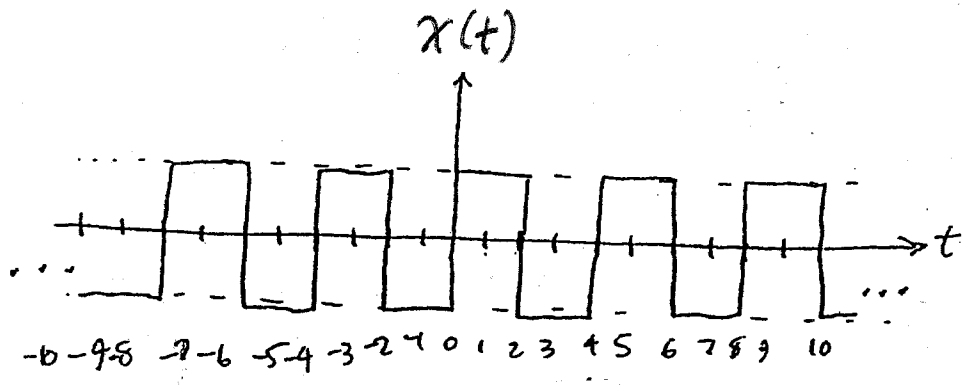
PERIODIC SIGNALS

Continuous time: if $\exists\, T > 0 \in \mathbb{R}$ s.t.

$x(t+T) = x(t) \;\; \forall\, t \in \mathbb{R}$, then $x(t)$ is periodic with period T.

- intuitively: if you "go ahead by T", then you get the same number.
- This has to be true <u>for all</u> t, not just one or a few.
- It means that $x(t) = x(t+T)$ <u>and also</u> $x(t+T) = x(t+2T)$. So, if $x(t)$ is periodic with period T, then it is also periodic with period $2T$ and with period kT for any $k \in \mathbb{N}$.

<u>EX</u>: periodic with period $T=4$ (and $T=8, 12, 16 \ldots$)



<u>DEF</u>: If there is a unique <u>smallest</u> $T_0 > 0$ s.t. $x(t+T_0) = x(t) \;\; \forall\, t \in \mathbb{R}$, then T_0 is called the <u>fundamental</u> <u>period</u> of $x(t)$.

Discrete time: if $\exists\ N > 0 \in \mathbb{N}$ s.t.

$x[n+N] = x[n]\ \forall n \in \mathbb{Z}$, then $x[n]$ is periodic with period N.

- This implies that $x[n]$ is also periodic with period kN for any $k \in \mathbb{N}$.

<u>DEF</u>: if N_0 is the smallest positive integer such that $x[n+N_0] = x[n]\ \forall n \in \mathbb{Z}$, then N_0 is called the <u>fundamental</u> <u>period</u> of $x[n]$.

→ A signal that is <u>not</u> periodic is called "aperiodic."

<u>Note</u>: a constant discrete-time signal like $x[n] = 5$ is periodic with <u>any</u> period $N \in \mathbb{N}$. In this case, the fundamental period is $N_0 = 1$.

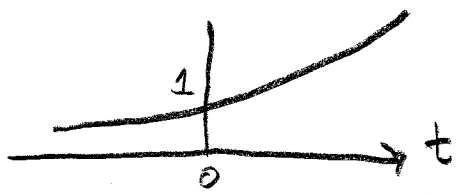
<u>Note</u>: a constant continuous-time signal like $x(t) = 5$ is periodic with <u>any</u> period $T \in \mathbb{R}^+$. In this case, the fundamental period is <u>undefined</u>, because there is <u>no</u> <u>unique</u> <u>smallest</u> T_0 such that $x(t+T_0) = x(t)\ \forall t \in \mathbb{R}$.

CONTINUOUS-TIME EXPONENTIAL SIGNALS

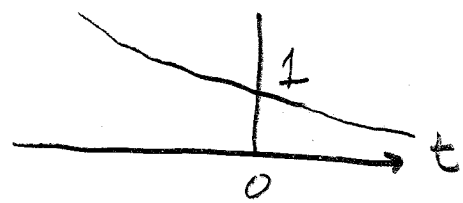
- The general form is: $x(t) = e^{at}$.

Case I: a <u>real</u>. There are two possible behaviors:

- $a > 0$: growing exponential
- $a < 0$: decaying exponential



growing



decaying

Case II: a <u>pure imaginary</u>. Then $\exists\, \Omega_0 \in \mathbb{R}$ s.t. $a = j\Omega_0$ and

$$x(t) = e^{at} = e^{j\Omega_0 t} = \cos\Omega_0 t + j\sin\Omega_0 t$$

- $\mathrm{Re}\{x(t)\}$ is a cosine of frequency Ω_0.
- $\mathrm{Im}\{x(t)\}$ is a sine of frequency Ω_0.

$\rightarrow$ Since $\sin\theta = \cos(\theta - \frac{\pi}{2})$, the real and imaginary parts are the same up to a phase shift by $\frac{\pi}{2}$ radians.

- Since the fundamental period of sine and cosine is 2π radians, the fundamental period of $\cos\Omega_0 t$ and $\sin\Omega_0 t$ is $T_0 = \left|\frac{2\pi}{\Omega_0}\right|$.

- Another way to look at the fundantal period:

  T_0 is the smallest positive number such that

  $$e^{j\Omega_0 t} = e^{j\Omega_0(t+T_0)} = e^{j\Omega_0 t}e^{j\Omega_0 T_0} \quad \forall t \in \mathbb{R}.$$

→ So T_0 is the smallest positive number such that

$$e^{j\Omega_0 T_0} = 1 = \cos\Omega_0 T_0 + j\sin\Omega_0 T_0.$$

$$\Rightarrow \quad \cos\Omega_0 T_0 = 1$$

$$\sin\Omega_0 T_0 = 0$$

$$\Rightarrow \text{ So } \Omega_0 T_0 = k2\pi \text{ for any } k \in \mathbb{Z}.$$

→ The smallest positive period is when $k=1$. Then we have $\Omega_0 T_0 = 2\pi$, so

$$T_0 = \left|\frac{2\pi}{\Omega_0}\right|.$$

- The signal $x(t) = e^{j\Omega_0 t}$ is called a <u>complex exponential</u> or a <u>complex sinusoid</u>.

→ These signals are conjugate symmetric.

<u>Note</u>: for any complex number $z \in \mathbb{C}$, $z = a + jb$,

$$z + z^* = (a+jb) + (a-jb) = 2a$$

$$\rightarrow \text{So } \mathrm{Re}[z] = \frac{z+z^*}{2}$$

$$z - z^* = (a+jb) - (a-jb) = 2jb$$

$$\rightarrow \text{So } \mathrm{Im}[z] = \frac{z-z^*}{2j}$$

- Applying these formulas to the signal $x(t) = e^{j\Omega_0 t}$, we have

$$\left.\begin{aligned} \cos\Omega_0 t &= \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} \\ \sin\Omega_0 t &= \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j} \end{aligned}\right\} \text{Another form of Euler's formula.}$$

<u>Note</u>: it is standard to use ω & Ω for radian frequency and f for Hertzian frequency.

- Since 2π rad $= 1$ Hz,

$$\omega = 2\pi f \qquad f = \omega/2\pi$$
$$\Omega = 2\pi f \qquad f = \Omega/2\pi$$

- If we multiply $x(t) = e^{j\Omega_0 t}$ by a complex constant $C = re^{j\theta}$, we get

$$Cx(t) = re^{j\theta}e^{j\Omega_0 t} = re^{j(\Omega_0 t + \theta)}$$

$$= r\left[\cos(\Omega_0 t + \theta) + j\sin(\Omega_0 t + \theta)\right]$$

→ The amplitude is <u>scaled</u> by $|C| = r$ and the phase is <u>shifted</u> by $\arg C = \theta$.

Case <u>III</u>: a <u>complex</u>. Then $\exists\ \sigma, \Omega_0 \in \mathbb{R}$ s.t.

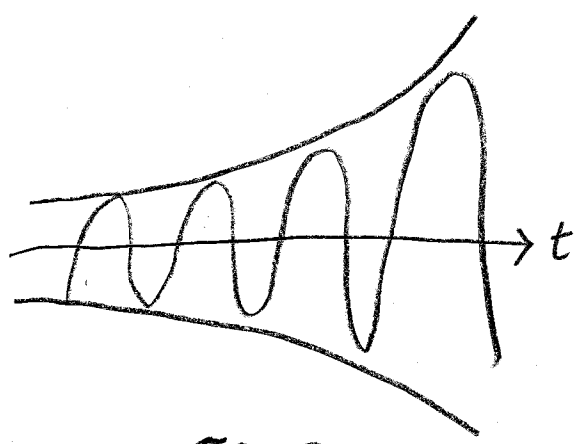
$a = \sigma + j\Omega_0$, and

$$x(t) = e^{at} = e^{(\sigma + j\Omega_0)t} = e^{\sigma t}e^{j\Omega_0 t}$$

$$= e^{\sigma t}\left[\cos\Omega_0 t + j\sin\Omega_0 t\right]$$

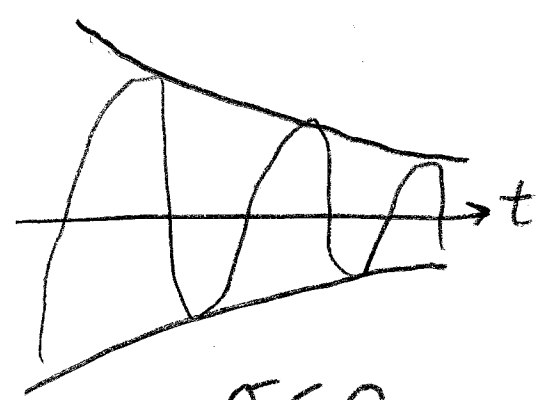
→ a real exponential times a complex sinusoid.

- This is called a "damped sinusoid."



$\sigma > 0$

"growing"



$\sigma < 0$

"decaying"

- If we multiply $x(t) = e^{(\sigma + j\Omega_0)t}$ by a complex constant $C = re^{j\theta}$, then we get

$$Cx(t) = re^{j\theta} e^{(\sigma + j\Omega_0)t}$$

$$= re^{\sigma t}\left[\cos(\Omega_0 t + \theta) + j\sin(\Omega_0 t + \theta)\right]$$

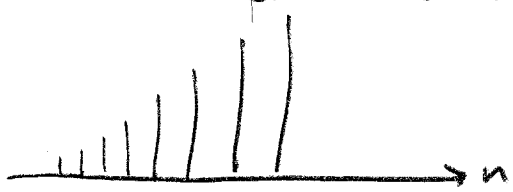
→ like before, the amplitude is scaled by $|C| = r$ and the phase is shifted by $\arg C = \theta$.

DISCRETE-TIME EXPONENTIAL SIGNALS

- General form: $x[n] = \alpha^n$.

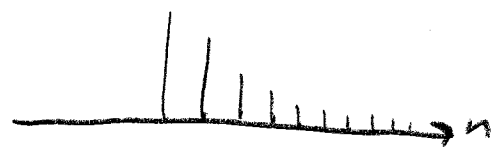
Case I: α <u>real</u>. There are 4 possible behaviors:

$\alpha > 1$: growing, non-alternating



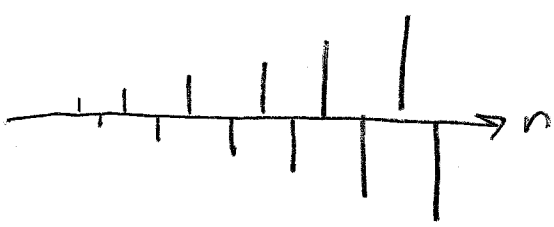
<u>EX:</u> $x[n] = 2^n$

$0 < \alpha < 1$: decaying, non-alternating



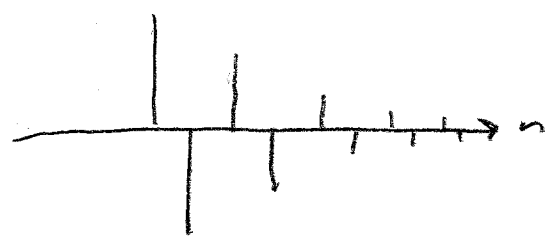
<u>EX:</u> $x[n] = \left(\frac{1}{2}\right)^n$

$\alpha < -1$: growing, alternating



<u>EX:</u> $x[n] = (-2)^n$

$-1 < \alpha < 0$: decaying, alternating



<u>EX:</u> $x[n] = \left(-\frac{1}{2}\right)^n$

Case II: $\alpha = e^{j\omega_0}$, $\omega_0 \in \mathbb{R}$.

$$x[n] = \alpha^n = e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

"discrete-time complex sinusoid"

- more on this in a minute.
- If we multiply $x[n] = e^{j\omega_0 n}$ by a complex constant $C = re^{j\theta}$, we get

$$Cx[n] = re^{j\theta}e^{j\omega_0 n} = re^{j(\omega_0 n + \theta)}$$

$$= r\left[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)\right]$$

$\rightarrow$ amplitude is scaled by $|C| = r$ and phase is shifted by $\arg C = \theta$.

Case III: α <u>complex</u>. Then $\exists\ r, \omega_0 \in \mathbb{R}$ with $r \geq 0$ s.t. $\alpha = re^{j\omega_0}$.

$$x[n] = \alpha^n = (re^{j\omega_0})^n = r^n e^{j\omega_0 n}$$

$$= r^n\left[\cos\omega_0 n + j\sin\omega_0 n\right]$$

"damped sinusoid"

- a real exponential times a complex sinusoid.

<u>Note:</u> since $r > 0$, the real exponential will <u>not</u> alternate.

- If we multiply $x[n] = (re^{j\omega_0})^n$ by a complex constant C, we get

$$Cx[n] = |C| e^{j \arg C} r^n e^{j\omega_0 n}$$

$$= |C| r^n e^{j(\omega_0 n + \arg C)}$$

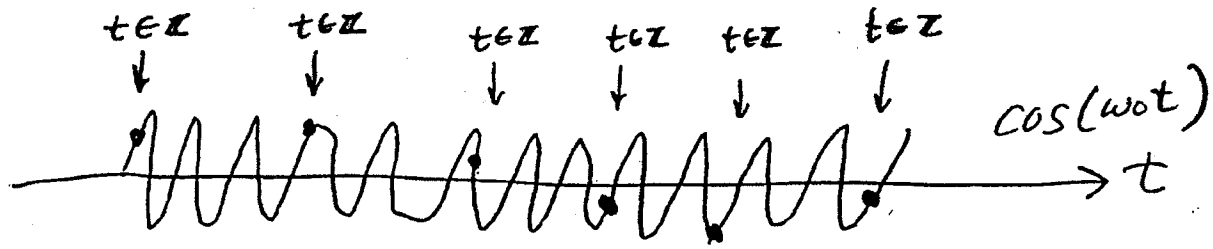
$$= |C| r^n \left[\cos(\omega_0 n + \arg C) + j \sin(\omega_0 n + \arg C)\right]$$

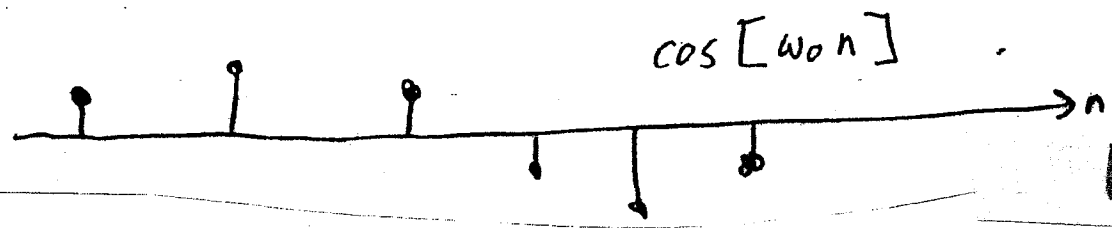
$\rightarrow$ amplitude is scaled by $|C|$ and phase is shifted by $\arg C$.

IMPORTANT DIFFERENCES BETWEEN DISCRETE AND CONTINUOUS SINUSOIDS

- To understand the differences, consider
  - → a continuous time signal $x(t) = \cos \omega_0 t$
  - → a discrete time signal $x[n] = \cos \omega_0 n$, obtained by <u>sampling</u> $x(t)$.

- As ω_0 gets bigger and bigger, $x(t)$ oscillates faster and faster.
- $x[n]$ also oscillates faster <u>up to a point</u>, but then, as ω_0 gets bigger, $x[n]$ oscillates slower!!
- Why?
  - If ω_0 is big, $\cos \omega_0 t$ may go through several cycles between places where t is an integer.
  - Then the graph of $\cos \omega_0 n$ doesn't look much like the graph of $\cos \omega_0 t$:



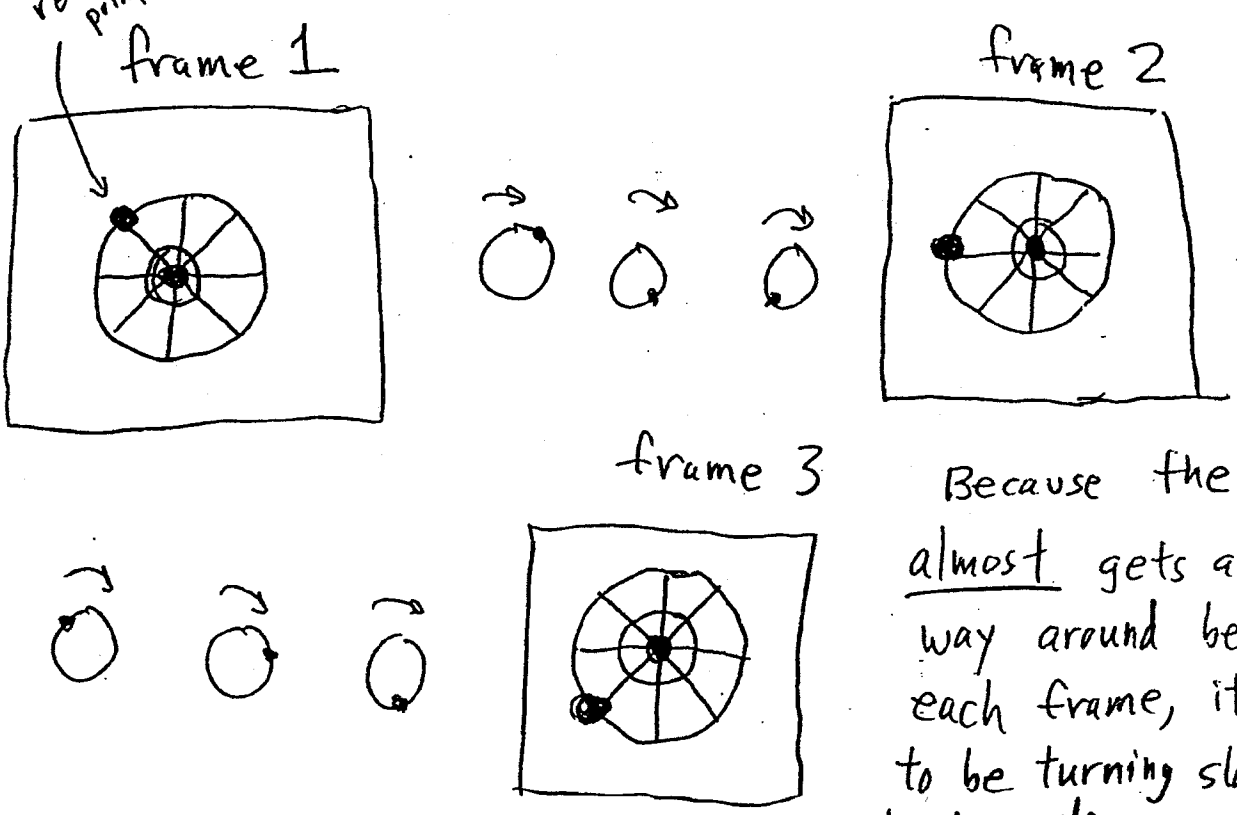


- Notice how the high frequency continuous-time sinusoid turned into a slowly varrying discrete-time waveform.

- This phenomenon is called "<u>ALIASING</u>".

- It is the same effect that makes wagon wheels look like they are spinning slowly backwards on film.

→ Standard video has a frame rate of 30 frames per second.



Because the wheel <u>almost</u> gets all the way around between each frame, it appears to be turning slowly backwards.

- Another difference: all of the unique discrete cosines can be generated with frequencies ω that are between 0 and π.

- The reason is that, for any $k \in \mathbb{Z}$,

$$\cos \omega_0 n = \cos\left[(\omega_0 + 2\pi k) n\right]$$

$\rightarrow$ They have the same graph.

- If ω_0 is <u>outside</u> $[-\pi, \pi]$, then we can choose a $k \in \mathbb{Z}$ such that $\omega_1 = \omega_0 + 2\pi k$ <u>and</u> $-\pi \leq \omega_1 \leq \pi$.

$\rightarrow$ Then $\cos \omega_0 n = \cos \omega_1 n \quad \forall n \in \mathbb{Z}$.

$\rightarrow$ Next, set $\omega_2 = |\omega_1|$. Since cosine is <u>even</u>, we have

$$\cos \omega_2 n = \cos \omega_1 n = \cos \omega_0 n,$$

and $0 \leq \omega_2 \leq \pi$.

$\Rightarrow$ so $\cos \omega_2 n$ has the same graph as $\cos \omega_0 n$, but ω_2 is between 0 and π.

- In other words, if we have $x[n] = \cos \omega_0 n$ and $\omega_0 \notin [0, \pi]$, then there is another choice $\omega_2 \in [0, \pi]$ such that $x[n] = \cos \omega_2 n$.

- Similarly, for any $k \in \mathbb{Z}$,

$$\sin \omega_0 n = \sin\left[(\omega_0 + 2\pi k) n\right]$$

- So, if we have $x[n] = \sin \omega_0 n$ and ω_0 is <u>outside</u> $[-\pi, \pi]$, then we can choose a $k \in \mathbb{Z}$ such that $\omega_1 = \omega_0 + 2\pi k$ <u>and</u> $-\pi \leq \omega_1 \leq \pi$.

  → Then $\sin \omega_1 n = \sin \omega_0 n \quad \forall n \in \mathbb{Z}$ ... they have the same graph.

  → However, since sine is <u>odd</u>, the second simplification is not possible...

- So: we can generate all of the possible unique discrete sines with frequencies ω that are between $-\pi$ and π.

- Since $e^{j\omega_0 n} = \cos \omega_0 n + j \sin \omega_0 n$, we can also generate all of the possible unique discrete complex sinusoids with frequencies ω that are between $-\pi$ and π.

- For this reason, people sometimes say that "discrete frequencies greater than π do not exist."

→ Not exactly true. They do exist. But they give us signals with graphs that are the same as those that can be obtained with another frequency that is between $-\pi$ and π.

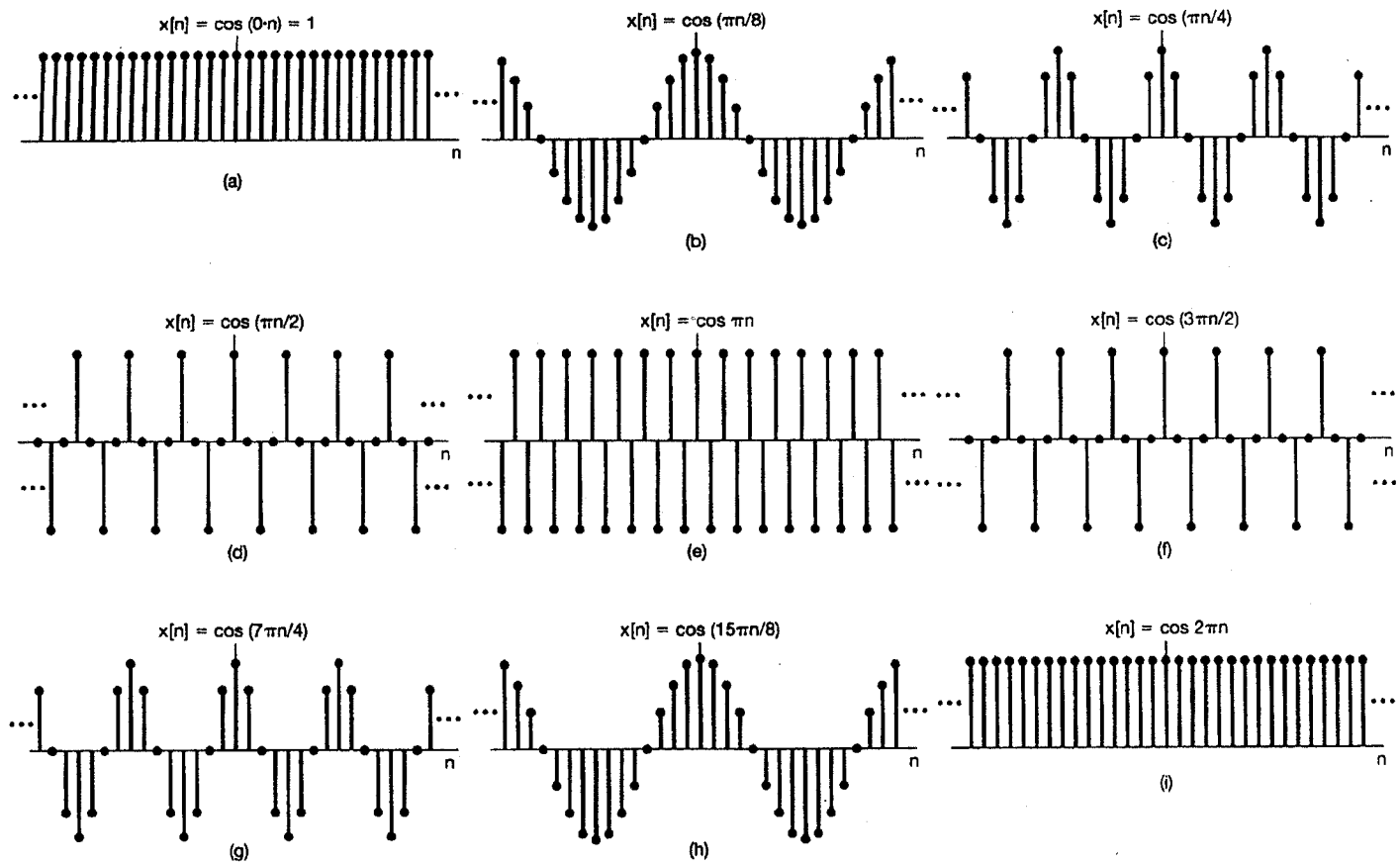


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

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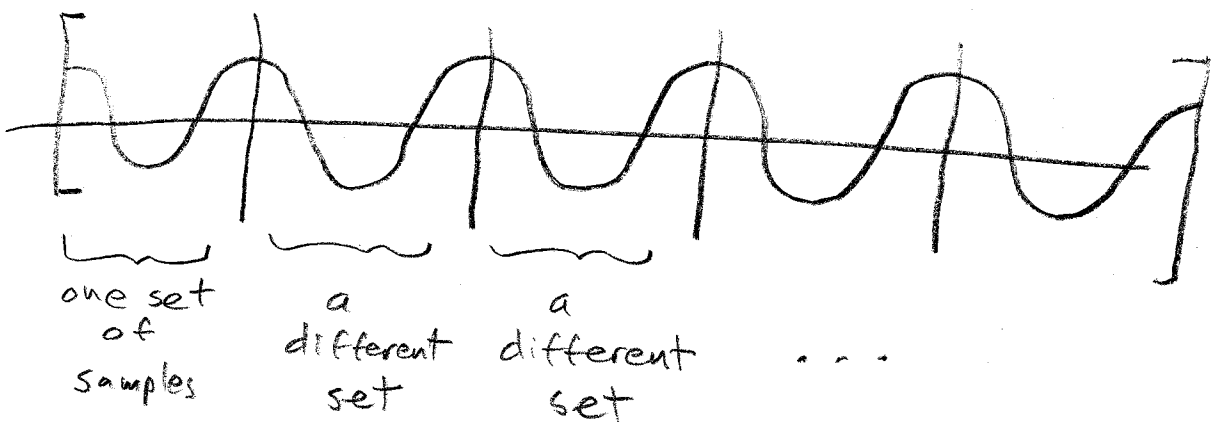
- Another difference: the discrete-time signals $\cos \omega_0 n$, $\sin \omega_0 n$, and $e^{j\omega_0 n}$ are periodic <u>only</u> if $\frac{\omega_0}{2\pi} \in \mathbb{Q}$. ($\mathbb{Q}$ = "the rationals"... the set of numbers $\frac{p}{q}$ where $p, q \in \mathbb{Z}$).

- Otherwise, the samples $\cos \omega_0 n$ fall at different places in each period of $\cos \omega_0 t$.



→ unless $\frac{\omega_0}{2\pi} \in \mathbb{Q}$, the samples don't "line up" from period to period...

- So no two "periods" of $\cos \omega_0 n$ get the same samples and the discrete signal is <u>not</u> periodic.

- If $\frac{\omega_0}{2\pi} \in \mathbb{Q}$, then the discrete-time signals $\cos\omega_0 n$, $\sin\omega_0 n$, and $e^{j\omega_0 n}$ <u>are</u> periodic.

→ Then write $\frac{\omega_0}{2\pi} = \frac{m}{N}$, where $\frac{m}{N}$ is in "reduced form" so that m and N have no common factors.

→ N is the fundamental period

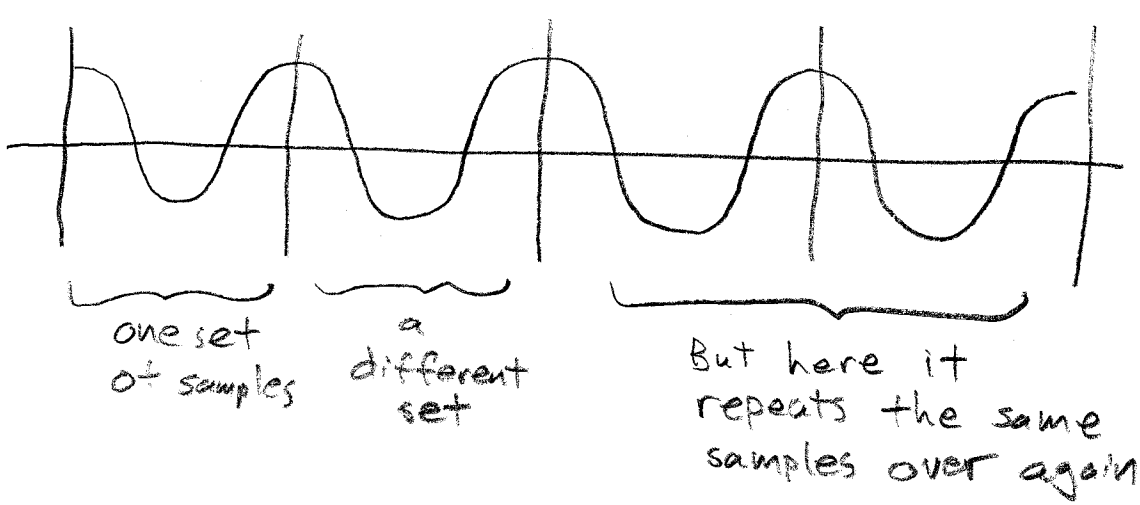
→ It takes m periods of $\cos\omega_0 t$ to make one period of $\cos\omega_0 n$.

<u>EX</u>: $x[n] = \cos\left(\frac{4\pi}{9} n\right)$

$$\omega_0 = \frac{4\pi}{9}$$

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi} \cdot \frac{4\pi}{9} = \frac{2}{9} = \frac{m}{N}$$

- Fundamental period $= N = 9$
- Each period of $\cos\omega_0 n$ looks like <u>two</u> periods of $\cos\omega_0 t$.



EX: $x[n] = \cos\left(\frac{\pi}{6} n\right)$ $\qquad \omega_0 = \frac{\pi}{6}$

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi} \cdot \frac{\pi}{6} = \frac{1}{12} = \frac{m}{N}$$

- Fundamental period $= N = 12$
- In this case, since $m=1$ the samples <u>do</u> line up in each period of $\cos\omega_0 t$, and so one period of $\cos\omega_0 n$ looks like one period of $\cos\omega_0 t$.

EX: $\omega_0 = \frac{8\pi}{31}$; $\quad \frac{\omega_0}{2\pi} = \frac{8\pi}{2\pi \cdot 31} = \frac{4}{31} = \frac{m}{N}$

- Fundamental period $= 31$
- It takes $m=4$ periods of $\cos\omega_0 t$ to make one period of $\cos\omega_0 n$.

EX: $x[n] = \cos\frac{n}{6} \longrightarrow \omega_0 = \frac{1}{6}$

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi \cdot 6} = \frac{1}{12\pi} \notin \mathbb{Q}$$

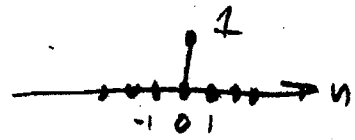
$\Rightarrow$ NOT PERIODIC

$\rightarrow$ The samples <u>never</u> line up in <u>any</u> two periods of $\cos\omega_0 t$.

Discrete-Time Unit Impulse and Unit Step

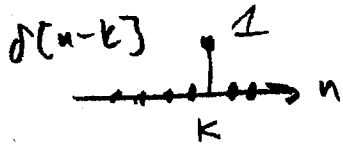
- we have already seen the "Kronecker delta" or "discrete time unit impulse":

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{other} \end{cases}$$

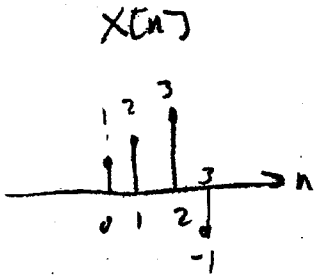


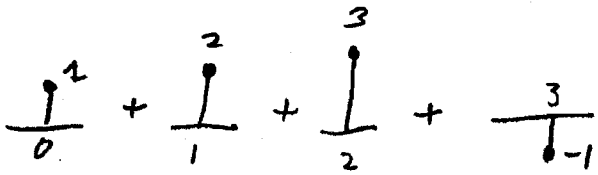
"turned on at zero"

- The translate $\delta[n-k]$ is "turned on" at $n=k$:



- Clearly, any discrete-time signal $x[n]$ can be written as a linear combination (weighted sum) of the set (basis) $\{\delta[n-k]\}_{k\in\mathbb{Z}}$.





$$= \delta[n] + 2\delta[n-1] + 3\delta[n-2] - \delta[n-3]$$

1.48

- You can do this by taking dot products:

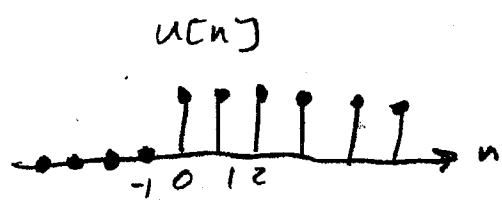
$$c_k = \langle x[n], \delta[n-k] \rangle = \sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = x[k]$$

- When you work on the time graph of a discrete-time signal, you are representing the signal in terms of the basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$

- This is called discrete-time time domain analysis.

- <u>Discrete-time unit step</u>

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



NOTE: $u[n] = \sum_{k=-\infty}^{n} \delta[k]$

Also, from graph using dot products: $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

- NOTE: $\delta[n] = u[n] - u[n-1]$

Two ways to see:

① graphical:

②

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$u[n-1] = \sum_{k=-\infty}^{n-1} \delta[k]$$

$$u[n] - u[n-1] = \sum_{k=-\infty}^{n} \delta[k] - \sum_{k=-\infty}^{n-1} \delta[k]$$

$$= \delta[n] + \sum_{k=-\infty}^{n-1} \delta[k] - \sum_{k=-\infty}^{n-1} \delta[k]$$

$$= \delta[n].$$

Also, $\delta[n-k] = u[n-k] - u[n-k-1]$

on at k on at k+1