ECE 4213/5213

MODULE 2

Index

1. Vector Spaces, Basis, Dimension, Completeness.....2.1 – 2.4
2. Inner Product, Norm.....2.5 – 2.7
3. Banach Spaces, L^p Spaces, ℓ^p Spaces.....2.8 – 2.14
4. More on Basis.....2.14 – 2.16
5. Systems.....2.16 – 2.17
6. Memoryless.....2.17 – 2.18
7. Causality.....2.19
8. Time Invariance.....2.20 – 2.21
9. Stability.....2.22 – 2.25
10. Linearity.....2.26 – 2.27
11. Invertibility.....2.27 – 2.28
12. LTI Systems.....2.29 – 2.30
13. Initial Relaxation.....2.31
14. ARMA Models, FIR & IIR Systems.....2.32 – 2.39
15. Discrete-Time Convolution.....2.40 – 2.63
16. Continuous-Time Convolution.....2.64 -2.75
17. LTI System Connections.....2.76 – 2.81
18. System Properties Revisited for LTI Systems.....2.82 – 2.85
19. Step Response of LTI Systems.....2.86 – 2.87
20. Right/Left-Sided Signals, Even, Odd, and Conjugate Symmetry.....2.88 – 2.95

DEF: A vector space is a collection of mathematical objects (called "vectors") along with a set of scalars (constants) that is defined so that several properties are satisfied.

→ Depending how you list them, there are about ten or twelve properties that need to be satisfied.

→ The vectors can be quite abstract, but for a concrete example think of $\mathbb{R}^2$... the set of "vectors" $\begin{bmatrix} a \\ b \end{bmatrix}$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

→ The set of scalars is called the "field." For $\mathbb{R}^2$, think of the scalar field as being $\mathbb{R}$.

→ We say that the vector space is defined "over the field."

→ There needs to be a binary operator "+" that combines two vectors and makes another vector. For $\mathbb{R}^2$, think of it as ordinary addition:

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} \in \mathbb{R}^2$$

→ There needs to be an operator "(,)" or "x" that combines a vector and a scalar to make another vector. For $\mathbb{R}^2$, think of it as ordinary scalar multiplication:

$$c\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix} \in \mathbb{R}^2.$$

→ We don't need to be too concerned about the properties, but here are some examples:

Let V be a vector space over a field $\mathcal{F}$.
Then

- if $\vec{v}, \vec{w} \in V$, $\vec{v}+\vec{w} \in V$ (closure of +)
- $\exists\, \vec{0} \in V$ such that $\vec{v}+\vec{0}=\vec{v}\ \forall\ \vec{v} \in V$
  (existence of identity element for +)
- if $\vec{v}, \vec{w} \in V$ and $c \in \mathcal{F}$, then

$$c(\vec{v}+\vec{w}) = c\vec{v} + c\vec{w}$$

(distributivity of scalar multiplication over +)

... and so on...

→ Suppose that $\mathcal{S} \subset \mathcal{V}$ (i.e., $\mathcal{S}$ is a proper subset of $\mathcal{V}$) such that

EVERY $\vec{v} \in \mathcal{V}$ can be written as a linear combination of the vectors in $\mathcal{S}$.

Then $\mathcal{S}$ is called a spanning set for $\mathcal{V}$.

EX: the set $\left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \end{bmatrix} \right\}$ is a spanning set for $\mathbb{R}^2$.

DEF: a spanning set with the fewest possible members is called a basis, also known as a "minimal spanning set".

EX for $\mathbb{R}^2$, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the so called natural basis.

EX: for $\mathbb{R}^2$, $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ is also a basis.

EX: for $\mathbb{R}^2$, $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$ is also a basis.

DEF: the dimension of a vector space is the number of vectors in a basis.

NOTE: $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \end{bmatrix}$ defines a sequence of vectors.

→ Every vector in the sequence is in $\mathbb{R}^2$.

→ It is a convergent sequence that converges to the limit $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is a vector in $\mathbb{R}^2$.

DEF: If $\mathcal{V}$ is a vector space and every convergent sequence of vectors from $\mathcal{V}$ converges to a limit that is also in $\mathcal{V}$, then $\mathcal{V}$ is called complete.

→ This means there are no "holes" in the space.

→ Sequences of guys from the space do not converge to something outside the space

Inner Product

An inner product is a function with domain $\mathcal{V} \times \mathcal{V}$ and range $\mathcal{F}$ [e.g., it gobbles up two vectors and spits out a scalar] such that, $\forall\ \vec{u}, \vec{v}, \vec{w} \in \mathcal{V}$ and $a, b \in \mathcal{F}$,

① $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle^*$ (conjugate)

② $\langle a\vec{u}, \vec{v} \rangle = a\langle \vec{u}, \vec{v} \rangle$ and

$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

③ $\langle \vec{u}, \vec{u} \rangle \geq 0$

④ $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$.

Seems abstract, but it's familiar in $\mathbb{R}^2$:

$$\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ac + bd$$

NOTE: in $\mathbb{C}^2$, $\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ac^* + bd^*$

NOTE: for complex-valued signals $x[n]$, $y[n]$,

$$\langle x[n], y[n] \rangle = \sum_{n \in \mathbb{Z}} x[n]\, y^*[n]$$

NOTE: for complex valued signals $x(t)$, $y(t)$,

$$\langle x(t), y(t) \rangle = \int_{\mathbb{R}} x(t) y^*(t)\, dt$$

DEF: A vector space $\mathcal{V}$ that is defined over a field $\mathcal{F}$ and equipped with an inner product is called an inner product space.

DEF: A norm is a mapping from $\mathcal{V}$ to $\mathcal{F}$ (e.g. it's a function that gobbles up a vector and spits out a number) such that

1. $\|\vec{u}\| \geq 0 \quad \forall\, \vec{u} \in \mathcal{V}$
2. $\forall\, \vec{u} \in \mathcal{V}$ and $c \in \mathcal{F}$, $\|c\vec{u}\| = |c| \cdot \|\vec{u}\|$
3. $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \quad \forall\, \vec{u}, \vec{v} \in \mathcal{V}$
4. $\|\vec{u}\| = 0$ iff $\vec{u} = \vec{0}$.

→ You can think of "norm" as "length"!

EX: Euclidean norm in $\mathbb{R}^2$:

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \sqrt{a^2 + b^2}$$

FACT: in any inner product space, there is a natural norm induced by the inner product. It is given by

$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle},$$

i.e., $\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle$.

→ This is the Euclidean norm in $\mathbb{R}^2$:

EX: $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2$, length = 5.

$$\langle \vec{u}, \vec{u} \rangle = \left\langle \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\rangle = 3 \cdot 3 + 4 \cdot 4$$
$$= 9 + 16$$
$$= 25$$
$$= \text{length squared}$$
$$= \|\vec{u}\|^2.$$

DEF: A Banach Space is a complete normed inner product space.

☆☆ Now lets put it all together for pages 2-1 through 2-8

→ Banach spaces are important and here's why:

- We will model systems as functions that have domains and ranges that are sets of signals that are generally Banach spaces.

→ We won't make use of any detailed theory of Banach spaces, but you should be aware of what they are and realize that our systems are functions having Banach spaces for domains and ranges.

Some Especially Important Banach Spaces:

The "El-Pee" spaces.

- For a discrete-time signal $x[n]: \mathbb{Z} \to \mathbb{C}$, the Euclidean norm is given by

$$\|x[n]\| = \sqrt{\sum_{n \in \mathbb{Z}} |x[n]|^2}$$

$$= \left[\sum_{n \in \mathbb{Z}} |x[n]|^2 \right]^{1/2}.$$

- Let's generalize this:

  let $p \in \mathbb{R}$ such that $1 \leq p < \infty$.

  The "p-norm", a.k.a. "ℓ^p-norm" of a complex valued sequence $x[n]$ is given by

$$\|x[n]\|_{\ell^p} = \left[\sum_{n \in \mathbb{Z}} |x[n]|^p \right]^{1/p}.$$

- FACT: the set of all $x[n]: \mathbb{Z} \to \mathbb{C}$ with $\|x\|_{\ell^p} < \infty$ is a Banach space over $\mathbb{C}$. It is called $\ell^p(\mathbb{Z})$ ("el-pee of $\mathbb{Z}$").

Ex: $\ell^1(\mathbb{Z})$, the set of absolutely summable sequences... a Banach space.

→ Recall from signals and systems: this is exactly the space of impulse responses of all BIBO stable LTI discrete time systems.

Ex: $\ell^2(\mathbb{Z})$, the set of square summable sequences... a Banach space.

→ Also called "the discrete Hilbert space."

→ It's important in real-world engineering because it is the space of discrete-time signals that have finite energy -- i.e., the ones that can exist in real life.

Note: consider

$$\lim_{p\to\infty} \|x[n]\|_{\ell^p} = \lim_{p\to\infty} \left[\sum_{n\in\mathbb{Z}} |x[n]|^p\right]^{1/p} = \sup_n |x[n]|$$

Intuition: when you raise the elements of the vector to the ∞ power and add, only the biggest guy matters. When you take the ∞^{th} root, you just back the biggest guy

- This is called the ℓ^∞-norm or the "sup norm", written $\|x[n]\|_{\ell^\infty}$.

→ The space of all guys $x[n]$ with a finite ℓ^∞-norm is also a Banach space, called $\ell^\infty(\mathbb{Z})$.

- The corresponding set of spaces for continuous-time signals $x(t): \mathbb{R} \to \mathbb{C}$ are also called "El-Pee" spaces, but they are written with a capital "L" as in $L^p(\mathbb{R})$.

- For a signal $x(t): \mathbb{R} \to \mathbb{C}$, the L^p-norm is (it is assumed that $p \in \mathbb{R}$ and $1 \leq p < \infty$):

$$\|x(t)\|_{L^p} = \left[\int_{\mathbb{R}} |x(t)|^p dt \right]^{1/p}$$

NOTE: $\int_{\mathbb{R}}$ means $\int_{-\infty}^{\infty}$ i.e., integrate on $\mathbb{R}$

- The space of signals $x(t)$ with a finite L^p-norm is a Banach space over $\mathbb{C}$. It is called $L^p(\mathbb{R})$ ("El-Pee of R").

EX:

$L^1(\mathbb{R})$: the space of absolutely integrable signals $x(t)$.

- corresponds to the space of impulse responses of all possible BIBO stable LTI continuous time systems
- $L^1(\mathbb{R})$ is the most natural space for defining the Fourier transform. Every $x(t)$ in $L^1(\mathbb{R})$ has a Fourier transform that converges as a plain old Riemann integral... this is not true for the other L^p spaces.

$L^2(\mathbb{R})$: the space of square integrable continuous-time signals, also called "Hilbert space."

- corresponds to the space of $x(t)$ that have finite energy... e.g., all the ones that can exist in the real world.
- Nicely related to the Parseval theorem... take a look back at it now... notice that both sides of the equation are just L^2-norms, e.g.

$$\text{Parseval: } \|x(t)\|_{L^2}^2 = \|X(\Omega)\|_{L^2}^2 \quad (\text{up to } \tfrac{1}{2\pi}).$$

Explanation: when we use "f" for frequency in Hertz, the spectral basis for signals $x(t)$ is given by $\{e^{j2\pi ft}\}_{f\in\mathbb{R}}$. The Fourier transform is then:

$$X(f) = \int_{\mathbb{R}} x(t) e^{-j2\pi ft}\, dt \qquad \text{(forward transform)}$$

$$x(t) = \int_{\mathbb{R}} X(f) e^{j2\pi ft}\, df \qquad \text{(reverse transform)}$$

→ Because this basis is orthonormal (as we will show later), there is no "$\frac{1}{2\pi}$" on the inverse transform.

- Parseval's relation then says:

$$\int_{\mathbb{R}} |x(t)|^2\, dt = \int_{\mathbb{R}} |X(f)|^2\, df$$

→ which is the same thing as

$$\int_{\mathbb{R}} x(t)\, x^*(t)\, dt = \int_{\mathbb{R}} X(f)\, X^*(f)\, df$$

→ which is the same as

$$\langle x(t), x(t) \rangle = \langle X(f), X(f) \rangle$$

- In other words,

$$\|x(t)\|_{L^2}^2 = \|X(f)\|_{L^2}^2 \qquad \star\star$$

→ The norm is <u>not</u> <u>changed</u> by the transform.

- A transformation (on a vector space) that does not change the norm is called an isomorphism.

$\Rightarrow$ Parseval's relation says that when $x(t) \in L^2(\mathbb{R})$, the Fourier transform does not change the norm... $x(t)$ and $X(f)$ have the same length.

- For this reason, people often say that "The Fourier transform is a Hilbert space isomorphism."

- Another way to think of it is: if you have a transformation that does not change the length of a vector, then it can only change the direction of the vector. In other words, it must be a rotation. Some people like to think of the Fourier transform as a rotation on $L^2(\mathbb{R})$.

Now, if you want to use $\Omega = 2\pi f$ and write the Fourier transform in terms of radian frequency (like the book does), your spectral basis is $\{e^{j\Omega t}\}_{\Omega \in \mathbb{R}}$. Because this basis is orthogonal but not orthonormal, the Fourier transform becomes

$$X(\Omega) = \int_{\mathbb{R}} x(t) e^{-j\Omega t} \, dt$$

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(\Omega) e^{j\Omega t} \, d\Omega$$

$\longrightarrow$

And Parseval's relation becomes:

$$\int_{\mathbb{R}} |x(t)|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |X(\Omega)|^2 d\Omega$$

→ In other words, if $x(t) \in L^2(\mathbb{R})$, then $x(t)$ and $X(\Omega)$ have the same norm up to a scale factor of $\frac{1}{\sqrt{2\pi}}$ ... which occurs because the basis is non-orthonormal.

⇒ This is generally true of all Fourier representations, including the DTFT and the DFT. They all preserve the norm ... up to a scale factor if the basis is non-orthonormal.

DTFT:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

DFT:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

---- End of "Parseval" Explanation - - - - -

- In practice, $L^2(\mathbb{R})$ is the basic space on which the Fourier transform is defined... although it takes some Banach space theory to do so... we'll talk about it later.

- The L^∞-norm is given by

$$\|x(t)\|_{L^\infty} = \lim_{p\to\infty} \|x(t)\|_{L^p}$$

$$= \lim_{p\to\infty} \left[\int_{\mathbb{R}} |x(t)|^p \, dt \right]^{1/p}$$

$$= \sup_{t\in\mathbb{R}} |x(t)| .$$

- The space of $x(t): \mathbb{R} \to \mathbb{C}$ with $\|x(t)\|_{L^\infty} < \infty$ is a Banach space called $L^\infty(\mathbb{R})$.

NOTE: In this class, we will not deal with vector spaces defined over weird or abstract fields. The field will always be $\mathbb{R}$ or $\mathbb{C}$. Unless otherwise specified, assume $\mathbb{C}$.

More on Basis

- Let $\mathcal{V}$ be a vector space and $\mathcal{B}$ be a basis.

DEF: If it is true that

$$\forall\ \vec{u}, \vec{v} \in \mathcal{B} \text{ such that } \vec{u} \neq \vec{v}$$
$$\langle \vec{u}, \vec{v} \rangle = 0,$$

then $\mathcal{B}$ is an orthogonal basis.

→ In other words, all the basis vectors have zero dot products with each other... they are all mutually orthogonal.

DEF: If $\mathcal{B}$ is an orthogonal basis and, moreover, $\|\vec{u}\| = 1\ \forall\ \vec{u} \in \mathcal{B}$, then $\mathcal{B}$ is called an orthonormal basis.

→ For an orthonormal basis, the basis vectors are all mutually orthogonal and they all have unit length.

EX: the set of discrete-time signals $\{\ldots, \delta[n+2], \delta[n+1], \delta[n], \delta[n-1], \ldots\}$ is an orthonormal basis for the set of signals $x[n]: \mathbb{Z} \to \mathbb{C}$.

- This can be written concisely as

$$\{\delta[n-k]\}_{k \in \mathbb{Z}}.$$

- This basis is called the "natural basis"
- It plays the same role in $\ell^p(\mathbb{Z})$ that the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ plays in $\mathbb{R}^2$.
- The dot product of a discrete time signal $x[n]$ with the basis signal (vector) $\delta[n-k]$ is given by

$$\langle x[n], \delta[n-k] \rangle = \sum_{n \in \mathbb{Z}} x[n]\delta[n-k] = x[k]$$

$k=-3$: $\langle x[n], \delta[n+3] \rangle = x[-3]$

$k=0$: $\langle x[n], \delta[n] \rangle = x[0]$

$k=7$: $\langle x[n], \delta[n-7] \rangle = x[7]$

Thus, a discrete-time signal $x[n]$ can be written as

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$= \sum_{k \in \mathbb{Z}} x[k]\delta[n-k].$$

→ This will be <u>important</u> when we talk about LTI systems and convolution.

<u>Systems</u>

- The system gobbles up an input signal $x[n]$ and spits out an output signal $y[n]$.
- Thus, the system H matches each input $x[n]$ from the space of allowable input signals to one and only one output signal $y[n]$ from the space of allowable output signals.
- Thus, H is a <u>function</u>. The domain is a space of signals and the range is a space of signals.

- The rule that tells how input signals get mapped into output signals is called the "Input/output relation" or "I/O relation" of the system.

- In math, we write $y[n] = H\{x[n]\}$.

  - or more formally, e.g.,

$$H : \ell^p(\mathbb{Z}) \to \ell^q(\mathbb{Z}) \text{ by } y[n] = H\{x[n]\}.$$

EX: $y[n] = \frac{1}{2}x[n] + \frac{1}{4}x[n-1] + \frac{1}{4}x[n-2]$

Adjectives to Describe Systems

Now we will build up a list of adjectives that can be used to describe the action of a system.

① Memoryless: The system H is memoryless if the current value of the output signal can be calculated from the current value of the input signal without needing any past or future values of the input.

EX: $y[n] = 3x[n] + 2n$

→ memoryless because you can calculate $y[n]$ if you know $x[n]$... you don't need $x[n-1]$ (past input) or $x[n+2]$ (future input) for example.

EX: centered average filter:

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

→ Not memoryless because, in order to calculate $y[n]$, you need to know values of the input signal from times other than "n".

② Causal: the system H is causal if you can calculate the value $y[n]$ of the output signal knowing only the present and past values of the input... i.e., you don't need any future values of the input signal.

EX: centered average filter:

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

→ Not causal. To calculate the current value $y[n]$ of the output signal, you have to know a future value $x[n+1]$ of the input signal.

EX: delayed average filter:

$$y[n] = \frac{1}{3}x[n-2] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n]$$

→ This is causal because you can calculate the current value $y[n]$ of the output signal without having to know any future values of the input signal.

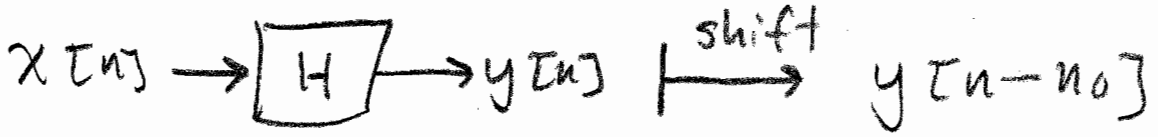
(3) Time Invariant, also known as "shift invariant" and "translation invariant":

The system H is time invariant if the action of the system commutes with time shifts.

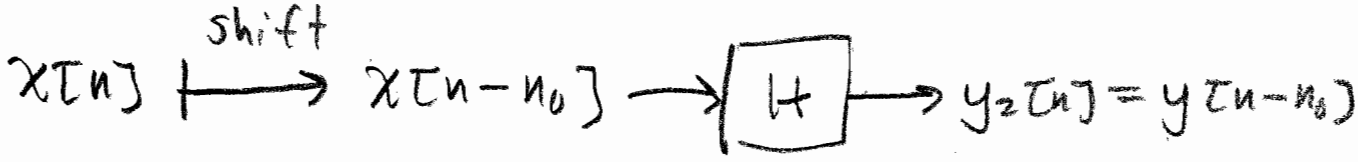
- In other words, H is time invariant if

$$H\{x[n-n_0]\} = y[n-n_0] \quad \forall\ x[n] \text{ and all } n_0 \in \mathbb{Z}.$$

- In pictures, it doesn't matter if we put $x[n]$ through the system and then shift:



or if we shift $x[n]$ first and then put it through the system:



→ For a time invariant system, the result is the same either way.

General strategy for proving that a system <u>is</u> time invariant:

(A) Let $x_1[n]$ be an arbitrary signal.

(B) Let $y_1[n] = H\{x_1[n]\}$ and find $y_1[n-n_0]$ for $n_0 \in \mathbb{Z}$ (n_0 must be arbitrary).

(C) Let $x_2[n] = x_1[n-n_0]$.

(D) Find $y_2[n] = H\{x_2[n]\}$ and show that $y_2[n] = y_1[n-n_0]$

To show that a system H <u>is not</u> time invariant, it is sufficient to construct a single signal $x[n]$ and shift amount n_0 for which $H\{x[n-n_0]\} \neq y[n-n_0]$.

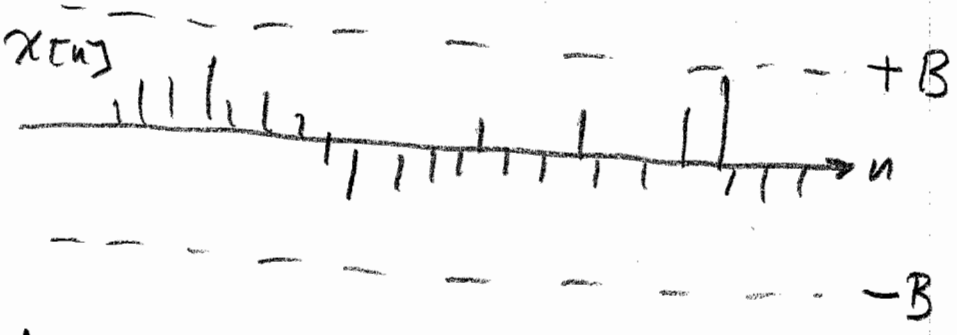
→ See the 3793 notes for examples.

DEF: a signal $x[n]$ is **bounded** if

$\exists B \in \mathbb{R}$, $B > 0$, such that

$$|x[n]| \leq B \quad \forall \; n \in \mathbb{Z}.$$

- Intuitively, this means that all the values $x[n]$ of the signal lie between $-B$ and $+B$.
- In other words, the signal $x[n]$ lives in a tunnel of radius B:



- The bound is not unique if it exists... if B is a bound, then so is $B+1$. The question is: does any such bound exist?

EX: $x[n] = \left(\frac{1}{2}\right)^n u[n]$

This is bounded... for example by $B = 1$, because:

$$\forall \; n < 0, \quad |x[n]| = 0 < 1,$$

$$\forall \; n \geq 0, \quad |x[n]| = \left|\left(\tfrac{1}{2}\right)^n\right| = \left(\tfrac{1}{2}\right)^n \leq 1.$$

$\rightarrow$ So $|x[n]| \leq 1 \; \forall \; n \in \mathbb{Z}$.

EX: $x[n] = 2^n u[n]$.

Not Bounded.

Proof: suppose that $B \in \mathbb{R}$, $B > 0$, and that $x[n]$ is bounded by B. Then $B \geq 2$, because $|x[1]| = 2$.

Now consider $n' = \lceil \log_2 B \rceil + 1$.
This implies $n' \geq 2$, because $B \geq 2$ and
$\lceil \log_2 2 \rceil + 1 = \lceil 1 \rceil + 1 = 1 + 1 = 2$.

Now, $|x[n']| = 2^{n'} = 2^{\lceil \log_2 B \rceil + 1}$

$$= 2 \cdot 2^{\lceil \log_2 B \rceil}$$
$$\geq 2 \cdot 2^{\log_2 B}$$
$$\geq 2B$$
$$> B.$$

This contradicts the fact that $x[n]$ is bounded by B. Therefore, no such bound exists and $x[n]$ is not bounded. QED.

(4) BIBO stable; e.g. "bounded-input bounded-output" stable.

The system H is BIBO stable if every bounded input signal produces a bounded output signal.

NOTE: This does not say anything about what happens when the input is unbounded.

NOTE: The bounds on the input and output do not have to be the same... for every bounded input, it just has to be that there exists some bound on the output.

EX: $y[n] = 2x[n] + 3x[n-1]$.

This system is BIBO stable. If the input is bounded by B, then the output is bounded by 5B.

Proof: Suppose $x[n]$ is a bounded input signal. Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|x[n]| \leq B \ \forall \ n \in \mathbb{Z}$.

Now, $|y[n]| = |2x[n] + 3x[n-1]|$

$\leq 2|x[n]| + 3|x[n-1]|$

$\leq 2B + 3B = 5B.$ QED.

EX: $y[n] = \begin{cases} 0, & n<0 \\ \sum_{k=0}^{n} x[k], & n \geq 0. \end{cases}$

This system is NOT BIBO stable.

Proof: Let $x[n] = u[n]$. Then $x[n]$ is a bounded input signal, because $|x[n]| \leq 1$ $\forall\, n \in \mathbb{Z}$.

Now, suppose $B \in \mathbb{R}$, $B>0$, and $y[n]$ is bounded by B. We will show that such a B cannot exist. Let $m = \lceil B \rceil + 1$. Since $B>0$, this implies $m \geq 1$.

So $|y[m]| = \sum_{k=0}^{m} x[k] = \sum_{k=0}^{m} 1 = m+1$

$= \lceil B \rceil + 2 > B.$

This contradicts the fact that $y[n]$ is bounded by B. Therefore, no such B exists and $y[n]$ is unbounded. Since a bounded input signal produced an unbounded output signal, the system is not BIBO Stable. QED.

(5) **Linear:** The system H is linear if the action of the system commutes with linear combinations.

- In other words, it's linear if $\forall$ signals $x_1[n]$ and $x_2[n]$ and all constants α, β,

$$H\{\alpha x_1[n] + \beta x_2[n]\} = \alpha H\{x_1[n]\} + \beta H\{x_2[n]\}.$$

NOTE: if the input and output signals come from vector spaces defined over the field $\mathbb{R}$, i.e. if it's a real-valued system, then $\alpha, \beta \in \mathbb{R}$. If instead the input and output vector spaces are defined over the field $\mathbb{C}$, then $\alpha, \beta \in \mathbb{C}$.

NOTE: Linear is sometimes broken into two properties: "homogeneous" and "additive".

(5A) **Homogeneous:** the system H is homogeneous if the action of the system commutes with scalar multiplication, i.e., if

$$H\{\alpha x[n]\} = \alpha H\{x[n]\} \quad \forall \text{ signals } x[n]$$

and all constants α.

(5B) Additive: the system H is additive if the action of the system commutes with sums, i.e., if

$$H\{x_1[n] + x_2[n]\} = H\{x_1[n]\} + H\{x_2[n]\}$$

$\forall$ signals $x_1[n]$ and $x_2[n]$.

NOTE: Homogeneity plus additivity is equivalent to linear.

→ See the 3793 notes for examples of linear and nonlinear systems.

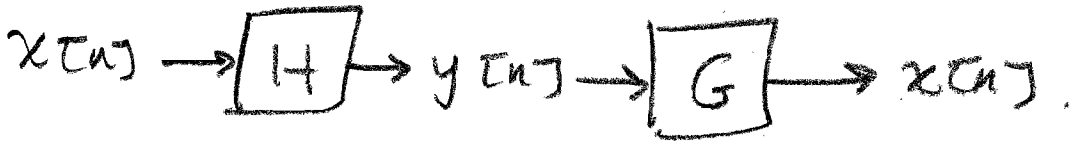
(6) Invertible: The system H is invertible if the input signal $x[n]$ can be determined from knowledge of the output signal $y[n]$ and the I/O relation.

→ In other words, if the mapping H is one-to-one, so that it has an inverse $G = H^{-1}$ that is also a function.

→

→ Intuitively, this means that any two distinct input signals $x_1[n]$ and $x_2[n]$ will produce output signals $y_1[n] = H\{x_1[n]\}$ and $y_2[n] = H\{x_2[n]\}$ that are also distinct, so that the input signal can always be determined from the output signal $y[n]$.

→ In other words, it means that there exists an inverse system G that "undues" the action of H:



→ As you know, for LTI systems the question of whether or not the system is invertible is best addressed in the frequency domain.

LTI systems

- Systems that are both linear and time invariant are called LTI systems.
- This is an extremely important class of systems.
- Why?

(A) it is always possible to solve for the output signal in terms of the input signal... for arbitrary input signals.

(B) The I/O relations of many LTI systems are constant coefficient linear difference equations. Such equations describe a huge variety of problems that are important in real-world engineering and science applications.

- This means that a linear combination of shifts of the output signal is equal to a linear combination of shifts of the input signal.

For example,

$$y[n] + \frac{1}{2}y[n-1] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$$

- For a causal LTI system, this means that a linear combination of the past and present outputs is equal to a linear combination of the past and present inputs.

- The general form of the I/O relation for a causal LTI system is

$$\sum_{k=0}^{M} \alpha_k y[n-k] = \sum_{\ell=0}^{N} \beta_\ell x[n-\ell], \quad (*)$$

where the α_k and β_ℓ are constants.

- Given the difference equation (*) and the input signal $x[n]$, there are time-domain techniques for solving for the signal $y[n]$, but we won't discuss them here... at least not for now.

- This brings us to our final "system property" or adjective ⟶

(6) Initial Relaxation: We may be given the difference equation (*), the input signal $x[n]$, and specification of some number of "initial conditions" on the output, for example $y[-1] = 1$ and $y[-2] = \frac{1}{2}$. In such problems, the solution for $y[n]$ for $n \geq 0$ generally depends on the input for $n \geq 0$ (the "forced response") and the initial conditions (the "homogeneous" response).

→ If the initial conditions are all zero, then the system is called initially relaxed.

- The causal LTI system I/O relation

$$\sum_{k=0}^{M} \alpha_k y[n-k] = \sum_{\ell=0}^{N} \beta_\ell x[n-\ell]$$

is also sometimes called an "ARMA model".

  - The terms on the left side corresponding to the coefficients α_k are called the "autoregressive terms" or "AR terms".
  - The terms on the right side corresponding to the coefficients β_k are called the "moving average terms" or "MA terms".

- If M and N are both finite and all the α_k terms are zero <u>except</u> α_0, then the LTI system will have an impulse response of finite length. Such systems are called "Finite Impulse Response systems" or "FIR systems".

- If some of the α_k besides α_0 are also nonzero, the impulse response will have infinite length and the system is called "IIR" for "Infinite Impulse Response."

Example: consider a discrete-time LTI system H with input $x[n]$ and output $y[n] = H\{x[n]\}$ related by the linear constant coefficients difference equation (I/O relation)

$$y[n] = \frac{1}{2}x[n] + \frac{3}{10}x[n-1] + \frac{1}{5}x[n-2] \quad (*)$$

→ with respect to page Z-32, we see that $a_0 = 1$ and the rest of the A_k coefficients a_k are zero for $k \neq 0$.

→ Moreover, we have a finite number of nonzero MA coefficients b_k.

⇒ So this is an FIR system.

FACT: the impulse response of this system is

$$h[n] = \frac{1}{2}\delta[n] + \frac{3}{10}\delta[n-1] + \frac{1}{5}\delta[n-2],$$

which can be established by comparing (*) above to the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Example: Now consider an LTI discrete-time system with input $x[n]$ and output $y[n]$ related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] \qquad (*)$$

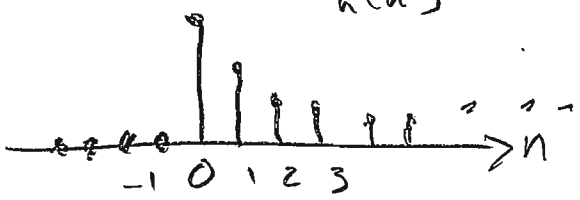
$\Rightarrow$ In this case, we have $a_0 = 1$ and $a_1 = -\frac{1}{2}$. Since a_0 is NOT the only nonzero AR coefficient, this is an IIR system.

FACT: the impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$= \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2] + \cdots + \left(\frac{1}{2}\right)^k u[n-k] + \cdots$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k]$$



Show This:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k]\, x[n-k] = \longrightarrow$$

$$\cdots y[n] = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k x[n-k]$$

$$= \underbrace{\left(\tfrac{1}{2}\right)^0 x[n-0]}_{k=0 \text{ term}} + \underbrace{\sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^k x[n-k]}_{\text{rest}}$$

let $m = k-1$, $k = m+1$

$$= x[n] + \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^{m+1} x[n-(m+1)]$$

$$= x[n] + \tfrac{1}{2}\left\{\sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^m x[(n-1)-m]\right\}$$

$$= x[n] + \tfrac{1}{2} y[n-1].$$

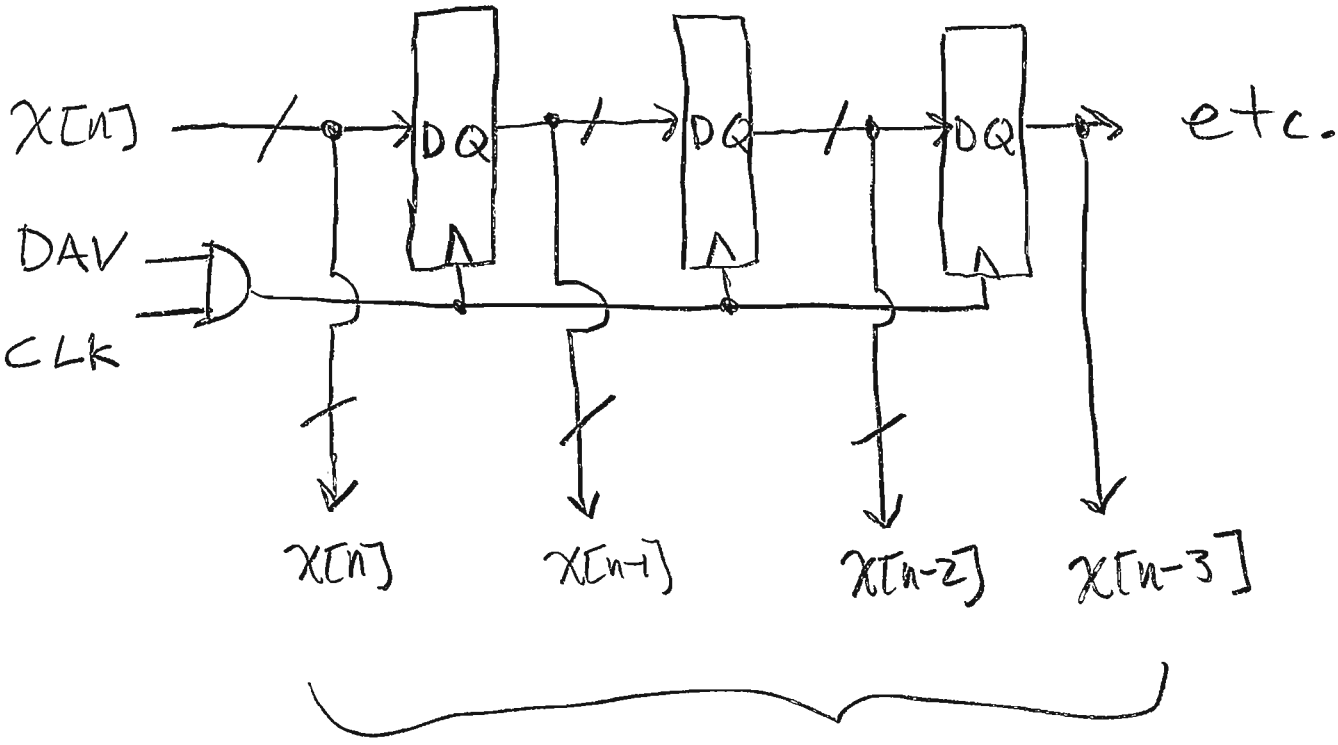
So $y[n] - \frac{1}{2} y[n-1] = x[n]$ ✓

⇒ An IIR system is also called "recursive," whereas FIR systems are often called "nonrecursive."

⇒ To really understand this, let's consider building digital hardware to realize the FIR system $y[n] = \frac{1}{2}x[n] + \frac{3}{10}x[n-1] + \frac{1}{5}x[n-2]$ on page 2-33 and the IIR system $y[n] = x[n] - \frac{1}{2}y[n-1]$ on page 2-34.

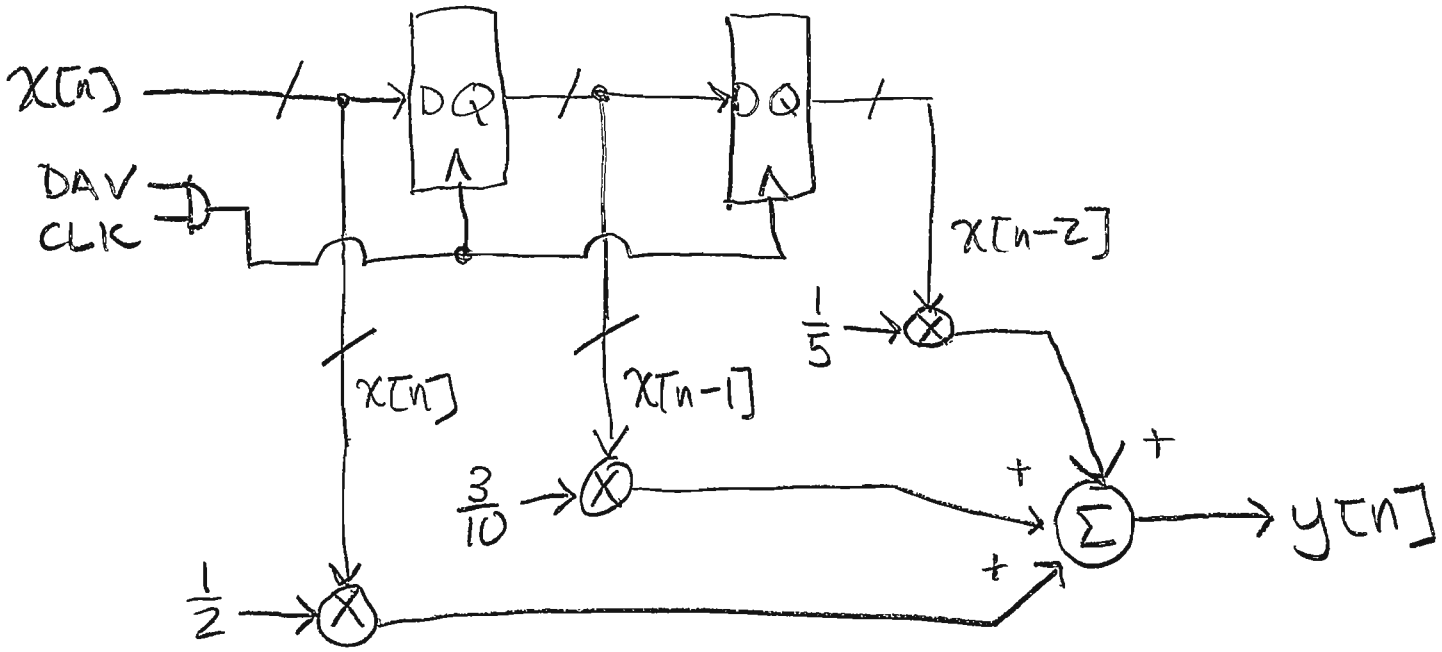
A <u>Delay Line</u> is a sequence of registers (e.g., multi-bit D flip-flops) that are hooked up in series to generate delayed samples of a signal... such as $x[n]$, $x[n-1]$, $x[n-2]$, etc.

- Let DAV = "data available" be a signal that is logic 1 when $x[n]$ becomes available and zero otherwise.



These are called "pickoff nodes"

- So the FIR system on page 2-33 with $y[n] = \frac{1}{2}x[n] + \frac{3}{10}x[n-1] + \frac{1}{5}x[n-2]$ can be realized in digital hardware like this:



NOTE: Registers, e.g. digital memory elements, are required to implement a discrete-time system that has memory.

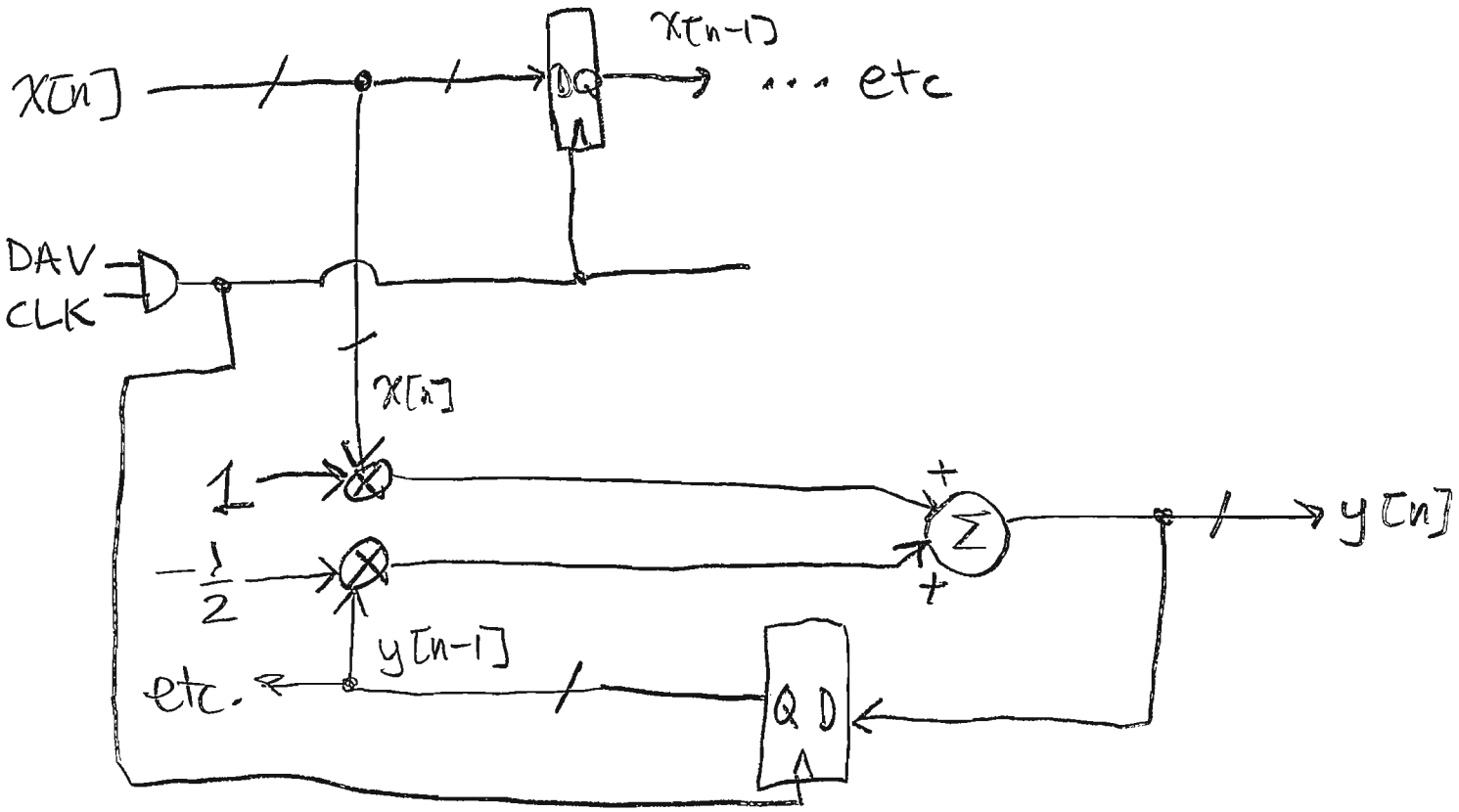
For a memoryless system, registers are not needed.

For the IIR system $y[n] = x[n] - \frac{1}{2}y[n-1]$ on page 2-34, we saw that the impulse response is

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \delta[n-k].$$

⇒ So a delay line of infinite length would be needed to realize the system like we just did on page 2-37.

⇒ However, a <u>recursive</u> solution can be realized by implementing a <u>second</u> delay line to retain the shifts of $y[n]$:



- This recursive solution works because an infinite set of MA terms can generally be traded for a finite number of AR terms:

$$y[n] = x[n] * h[n]$$

$$= x[n] + \tfrac{1}{2}x[n-1] + \tfrac{1}{4}x[n-2]$$

$$+ \cdots + \left(\tfrac{1}{2}\right)^k x[n-k] + \cdots$$

$$\Updownarrow$$

$$y[n] = x[n] - \underbrace{\tfrac{1}{2}y[n-1]}$$

one AR term can be used to replace an infinite set of MA terms... "recursion"

$\Rightarrow$ Another fact that is usefull in the analysis of ARMA systems is:

- You can also generally trade an infinite set of AR terms for a finite set of MA terms.

Discrete Time Convolution

- Let H be a discrete-time LTI system:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = H\{x[n]\}$$

DEF: the "impulse response" $h[n]$ is what comes out when the input is $x[n] = \delta[n]$:

$$\delta[n] \longrightarrow \boxed{H} \longrightarrow h[n] = H\{\delta[n]\}$$

$\Rightarrow$ Because the system is time invariant, if you put in a shifted delta $\delta[n-k]$, $k \in \mathbb{Z}$, then you get out a shifted impulse response $h[n-k]$:

$$H\{\delta[n-2]\} = h[n-2]$$
$$H\{\delta[n+3]\} = h[n+3]$$

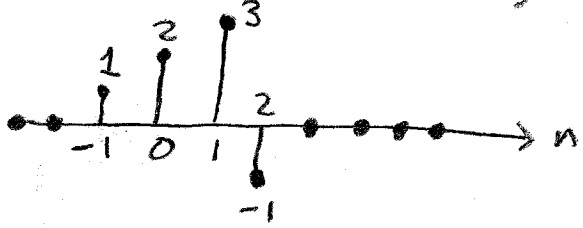
$\Rightarrow$ This is only true for a time invariant system. For a time varying system, totally different signals might come out for $\delta[n]$ and $\delta[n-2]$.

- Because the system is LTI, it's easy to find the output if the input is written as a sum of shifted Kronecker deltas:

$$5\delta[n-2] + 7\delta[n+3] \longrightarrow \boxed{H} \longrightarrow 5h[n-2] + 7h[n+3]$$

- This is why it is so useful to write an arbitrary input $x[n]$ as a linear combination of the natural basis $\{\delta[n-k]\}_{k\in\mathbb{Z}}$.

EX: $x[n] = 1\delta[n+1] + 2\delta[n] + 3\delta[n-1] - 1\delta[n-2]$



- For an LTI system H,

$$y[n] = H\{x[n]\}$$

$$= H\{\delta[n+1] + 2\delta[n] + 3\delta[n-1] - \delta[n-2]\}$$

$$\left(\begin{matrix}\text{b/c } H\\ \text{is linear}\end{matrix}\right) = H\{\delta[n+1]\} + 2H\{\delta[n]\} + 3H\{\delta[n-1]\} - H\{\delta[n-2]\}$$

$$\left(\begin{matrix}\text{b/c } H \text{ is}\\ \text{time invariant}\end{matrix}\right) = h[n+1] + 2h[n] + 3h[n-1] - h[n-2].$$

- $h[n+1]$: response due to $1\delta[n+1]$
- $2h[n]$: response due to $2\delta[n]$
- $3h[n-1]$: response due to $3\delta[n-1]$
- $-h[n-2]$: response due to $-1\delta[n-2]$

EASY!

- This is called linear convolution.

- More generally, suppose $x[n]$ is an arbitrary input signal and let H be an LTI system with impulse response $h[n]$.

- Write $x[n]$ as a sum of the natural basis:

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

- On the next page, we'll use time domain analysis to find the output $y[n]$.

  - We'll write this in two columns to help overcome "fear of the capital sigma!"

    - In the left column, we'll write everything out with "+" and "...".

    - In the right column, we'll write exactly the same thing, but using the shorthand "capital-Σ" notation to make "do loops!"

      $\longrightarrow$ This saves a ton of writing.

$x[n] \rightarrow \boxed{H} \rightarrow y[n]$

Basis:

$\{\ldots, \delta[n-1], \delta[n], \delta[n+1], \ldots\}$

Input:

$$x[n] = \ldots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Output:

$$y[n] = H\{x[n]\}$$

$$= H\{\ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots\}$$

$$= \ldots + H\{x[-1]\delta[n+1]\} + H\{x[0]\delta[n]\} + H\{x[1]\delta[n-1]\} + \cdots$$

$$= \ldots + x[-1]H\{\delta[n+1]\} + x[0]H\{\delta[n]\} + x[1]H\{\delta[n-1]\} + \cdots$$

$$= \ldots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \cdots$$

Basis:

$$\{\delta[n-k]\}_{k \in \mathbb{Z}}$$

Input:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Output:

$$y[n] = H\{x[n]\}$$

$$= H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} H\{x[k]\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- This is called convolution and written

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \equiv x[n] * h[n].$$

NOTE: in the Lab Manual, Mitra writes this as

$$y[n] = x[n] \circledast h[n],$$

which is not standard.

NOTE: this is also called "linear convolution" to distinguish it from another kind of operation called "circular convolution" that we'll talk about later.

- The fact that the output of an LTI system H is given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

follows directly from the facts that the system is linear and time invariant when you write the input as a linear combination of the natural basis.

FACT: Convolution is commutative.

- In math: $x[n] * h[n] = h[n] * x[n]$

or $\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

- In words: When you compute the signal

$y[n] = x[n] * h[n]$,

the convolution of $x[n]$ and $h[n]$, it doesn't matter which signal gets the "k" and which signal gets the "n-k" in the convolution sum.

$\Rightarrow$ For every "n", $y[n]$ is the exact same number either way you do it.

Proof: Let $y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$.

Let $y_2[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$.

Then $y_1[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$m = n-k$, $k = n-m$ $\Big\}$ $= \sum_{m=+\infty}^{-\infty} x[n-m]\, h[m]$

(order of summation doesn't matter) $= \sum_{m=-\infty}^{\infty} x[n-m]\, h[m]$

$= \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$

(write "k" instead of "m") $= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = y_2[n]$ ✓

[redacted]

- When we say that "H is a system with impulse response $h[n]$", it implies that H is an LTI system.

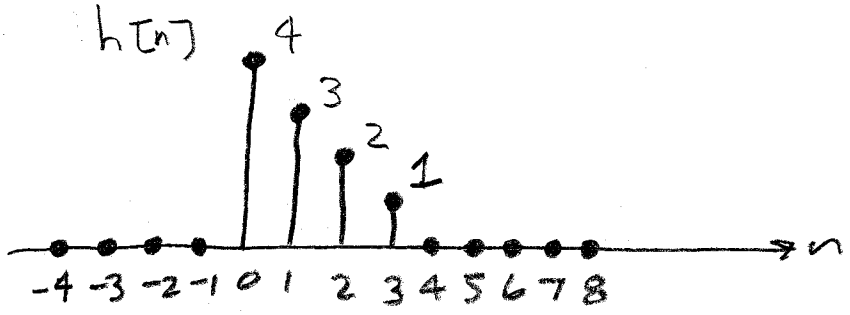
- If H is not LTI, then "impulse response" has no meaning.

- We have used math to show analytically that the output of any LTI system is the convolution of the input with the impulse response.

- Now let's develop some intuition about what it really means.

- Let H be an LTI system with impulse response

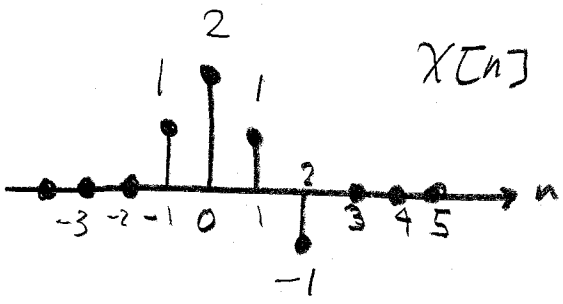
$$h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + 1\delta[n-3]$$



2-47

- Let the input be

$$x[n] = 1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]$$

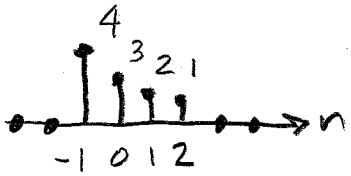


- We have that

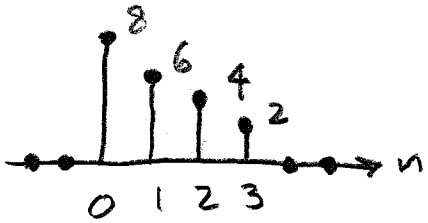
$$y[n] = H\{x[n]\}$$

$$= H\{1\delta[n+1]\} + H\{2\delta[n]\} + H\{1\delta[n-1]\} + H\{-1\delta[n-2]\}$$

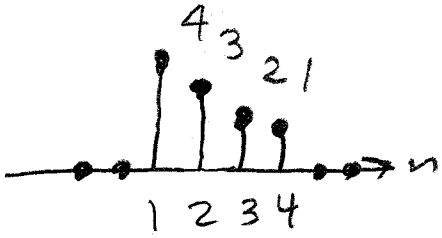
- The input $1\delta[n+1]$ at $n=-1$ causes a response $1h[n+1]$:



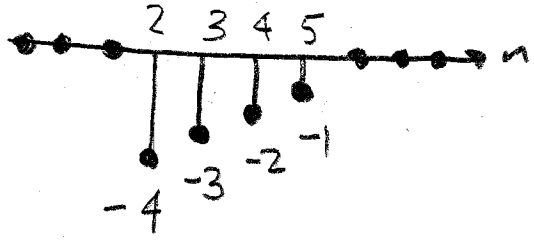
- The input $2\delta[n]$ at $n=0$ causes a response $2h[n]$:



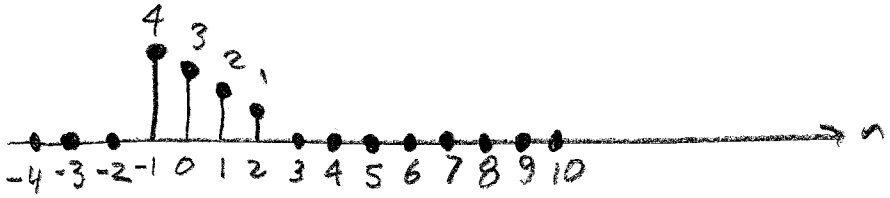
- The input $1\delta[n-1]$ at $n=1$ causes a response $1h[n-1]$:

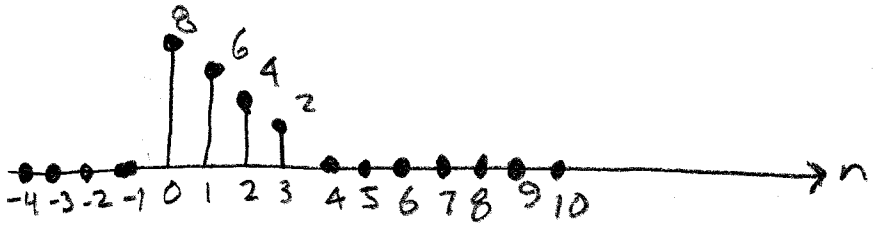


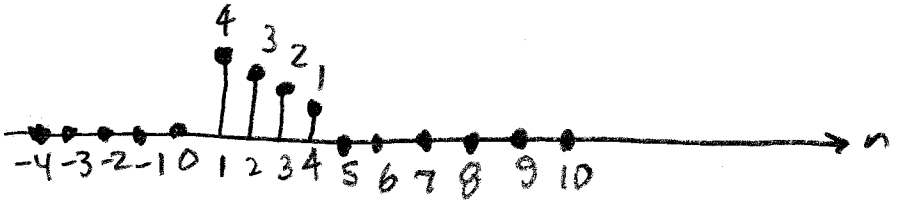
- The input $-1\delta[n-2]$ at $n=2$ causes a response $-1h[n-2]$:

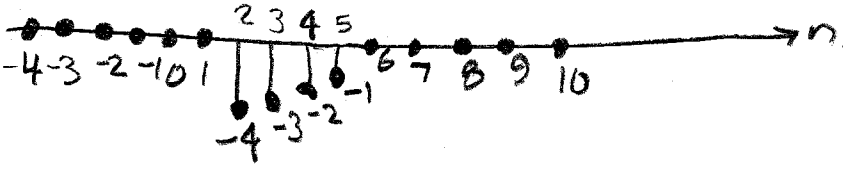


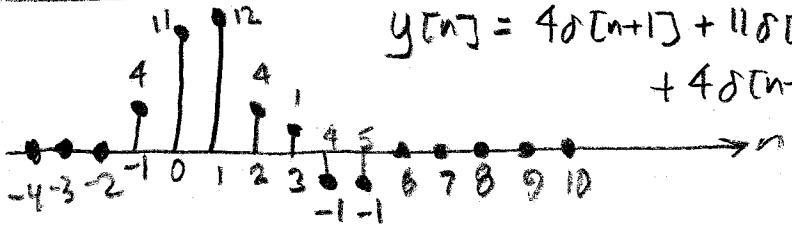
- The total response $y[n]$ is the sum of these individual responses:











$$y[n] = 4\delta[n+1] + 11\delta[n] + 12\delta[n-1] + 4\delta[n-2] + 1\delta[n-3] - 1\delta[n-4] - 1\delta[n-5]$$

- One way to think of this:

Input at $n=-1$ is $1\delta[n+1]$ → Causes $1h[n]$ to start coming out at $n=-1$.

Input at $n=0$ is $2\delta[n]$ → Causes $2h[n]$ to start coming out at $n=0$.

Input at $n=1$ is $1\delta[n-1]$ → Causes $1h[n]$ to start coming out at $n=1$.

Input at $n=2$ is $-1\delta[n-2]$ → Causes $-h[n]$ to start coming out at $n=2$.

- To find the value of the output signal $y[n]$ at some particular time like $n=2$, you have to add up what is coming out at $n=2$ from <u>each</u> of the input terms.

- We can tabulate this ...

Output

Input	n=-1	n=0	n=1	n=2	n=3	n=4	n=5
n=-1 : x[-1]=1	x[-1]h[0]=4	x[-1]h[1]=3	x[-1]h[2]=2	x[-1]h[3]=1	0	0	0
n=0 : x[0]=2	0	x[0]h[0]=8	x[0]h[1]=6	x[0]h[2]=4	x[0]h[3]=2	0	0
n=1 : x[1]=1	0	0	x[1]h[0]=4	x[1]h[1]=3	x[1]h[2]=2	x[1]h[3]=1	0
n=2 : x[2]=-1	0	0	0	x[2]h[0]=-4	x[2]h[1]=-3	x[2]h[2]=-2	x[2]h[3]=-1

At $n=2$,

$x[-1]h[3]$ is still coming out because of $x[-1]$
$x[0]h[2]$ is still coming out because of $x[0]$
$x[1]h[1]$ is still coming out because of $x[1]$
$x[2]h[0]$ is coming out because of $x[2]$.

NOTE: indices of x go up, indices of h go down

$$y[2] = x[-1]h[3] + x[0]h[2] + x[1]h[1] + x[2]h[0]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$

2-50

- For any particular time n, the number $y[n]$ is the sum of what's coming out at time n due to <u>each</u> of the input terms.
- To find the output <u>signal</u> $y[n]$, we generally have to do this calculation <u>for every n</u>.
- Doing this graphically is called "graphical convolution."
- On the test, we hope that we don't have to actually do the calculation for every n.
  - → We hope that, instead, we can just do it a few times, and that each time we do it will cover a <u>whole batch</u> of <u>ns</u>.

☆☆☆ How to work the problems on a test without making mistakes:

① Choose which signal gets "k" and which one gets "n−k", i.e.,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \quad \sim \text{or} \sim \quad y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

→ you can always get the right answer either way, but one way might be a little easier to workout.

→ Usually, it's best to pick the one with the simpler expression to get the "n−k".

→ For the following steps, assume we picked the first way... i.e., $x[k]$ and $h[n-k]$.

(2) Make a graph of $x[k]$.

(3) Make a graph of $h[n-k]$ (as a function of k).

→ To avoid making mistakes, it's best to do this in <u>three steps</u>.

(3A) Graph $h[k]$.

(3B) Shift the graph in (3A) to the right by $-n$ to obtain the graph of $h[k-(-n)] = h[n+k]$.

(3C) Flip the graph in (3B) with respect to the k-axis to obtain the graph of $h[n-k]$.

(4) To find $y[n]$, you must multiply the graphs in (2) and (3C), and then add up the product graph from $k=-\infty$ to $k=+\infty$.

⇒ The product graph generally depends on "n"!

- So, in general you have to do this for each n.

- But you hope that by carefully examining the graphs in (2) and (3C), you can find just a few expressions that will "cover all of the ns."

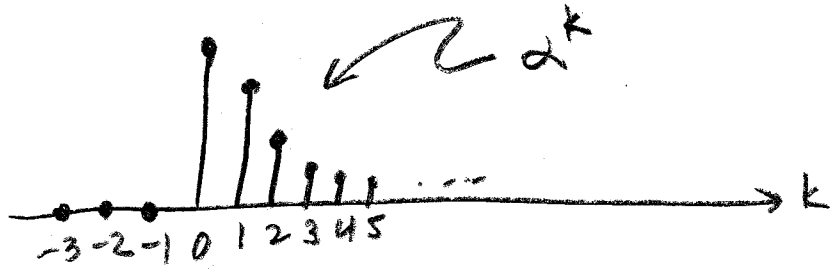
  - These "few expressions" are often called "regions."

EX: $x[n] = \alpha^n u[n]$, $0 < \alpha < 1$

$h[n] = u[n]$.

→ Since $x[n]$ has the more complicated expression, we will put the "k" on x and the "$n-k$" on h.

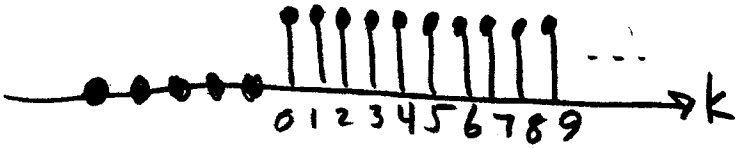
- Graph $x[k]$:



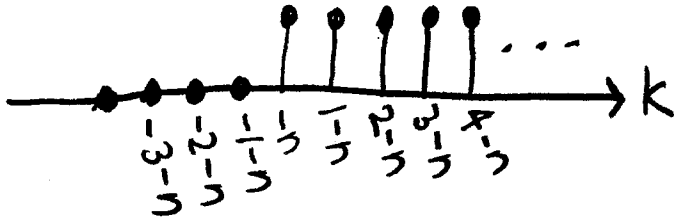
2-55

-Graph $h[n-k]$ in three steps:

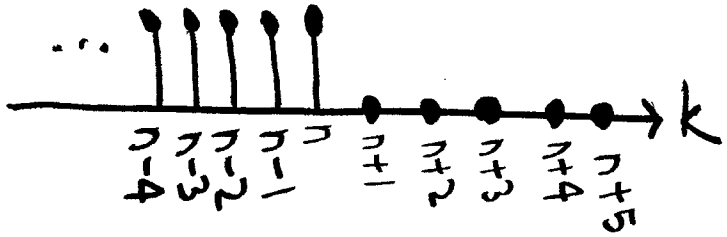
$h[k]$



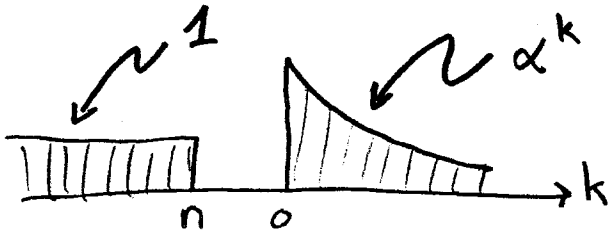
$h[k-(-n)] = h[n+k]$ (shift right by $-n$)



$h[n-k]$ (flip)



- When $n<0$, we have



NOTE: these graphs are just intended to show where $x[k]$ and $h[n-k]$ are nonzero. It's easier and clearer to draw them like continuous-time functions even though they are really discrete.

→ From the graphs above, we see that, when $n<0$, for <u>every</u> k either $x[k]=0$ or $h[n-k]=0$.

→ So the product $x[k]h[n-k]=0 \quad \forall k$ when $n<0$.

→ So $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} 0 = 0$

when $n<0$.

- When $n \geq 0$, we have

→ In this case the product $x[k]h[n-k]$ will be nonzero from $k=0$ to $k=n$, but zero for all other k.

→ So the sum $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ has nonzero terms from $k=0$ to $k=n$.

→ In this range of k, we have (see graphs on page ~~2.21, 2.22~~ 2.54, 2.55)

$x[k] = \alpha^k$ and $h[n-k] = 1$.

→ So, for $n \geq 0$, we have

$$y[n] = \sum_{k=0}^{n} \alpha^k \cdot 1 = \sum_{k=0}^{n} \alpha^k$$

Applying the sum formula $\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$, $\alpha \neq 1$

We have $y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}$

- For this problem, these two cases cover all of the "n"s.

(2-58)

- Putting it all together:

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & n \geq 0 \end{cases}$$

- This can also be written as

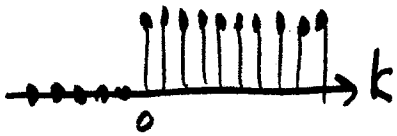
$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}\, u[n].$$

Let's try working this one the "other way."

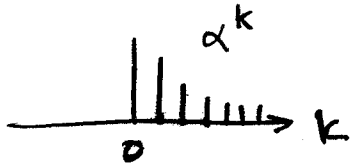
$$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$$
$$h[n] = u[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

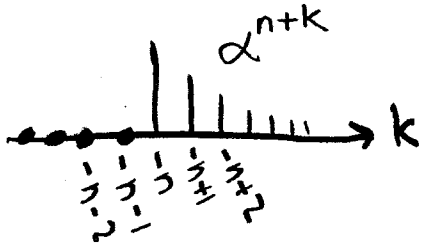
$h[k]$



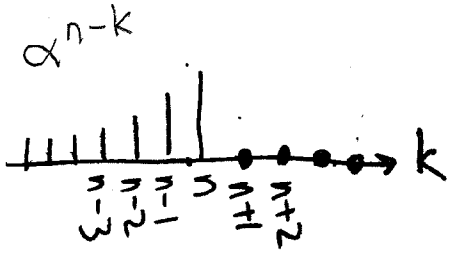
$x[k]$



$x[k--n] = x[n+k]$

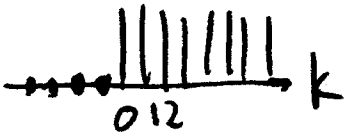


$x[n-k]$

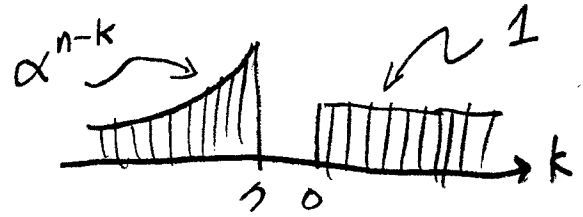


Recall from last page:

$h[k]$

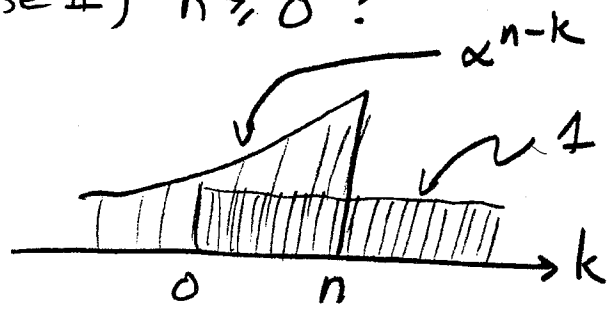


Case I) $n < 0$:



No overlap: $y[n] = 0$.

Case II) $n \geq 0$:



$$y[n] = \sum_{k=0}^{n} \alpha^{n-k} \cdot 1 = \sum_{k=0}^{n} \alpha^n \alpha^{-k}$$

$$= \alpha^n \sum_{k=0}^{n} \alpha^{-k} = \alpha^n \sum_{k=0}^{n} \left(\frac{1}{\alpha}\right)^k$$

$$= \alpha^n \frac{\left(\frac{1}{\alpha}\right)^0 - \left(\frac{1}{\alpha}\right)^{n+1}}{1 - \frac{1}{\alpha}}$$

$$= \alpha^n \frac{1 - \alpha^{-n-1}}{1 - \alpha^{-1}} \cdot \frac{\alpha}{\alpha}$$

$$= \alpha^n \frac{\alpha - \alpha^{-n}}{\alpha - 1} = \frac{\alpha^{n+1} - 1}{\alpha - 1}$$

$$= \frac{1 - \alpha^{n+1}}{1 - \alpha} \checkmark$$

All Together:

$$y[n] = \begin{cases} 0, & n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & n\geq 0 \end{cases}$$

$\Rightarrow$ Note that this agrees with the solution we obtained on page 2.58.

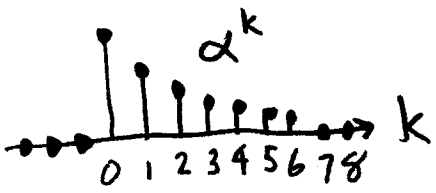
EX: $x[n] = \begin{cases} 1, & 0\leq n\leq 4 \\ 0, & \text{otherwise} \end{cases} = u[n]-u[n-5]$

$$h[n] = \begin{cases} \alpha^n, & 0\leq n\leq 6 \\ 0, & \text{otherwise} \end{cases} = \alpha^n\{u[n]-u[n-7]\},$$

$0<\alpha<1$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

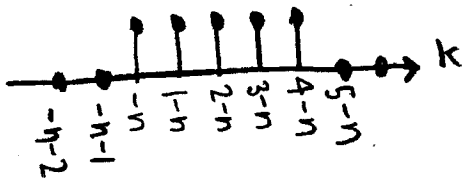
$h[k]$



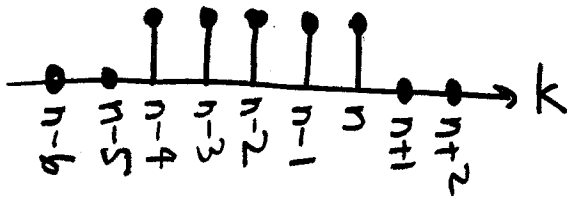
$x[k]$

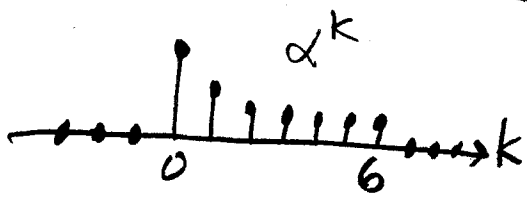
$k \to -n$

$x[-n+k] = x[n+k]$

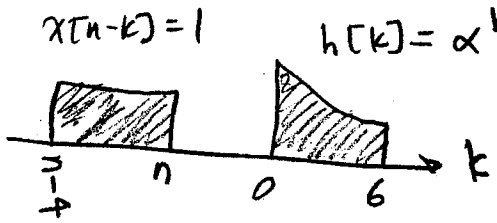


$x[n-k]$ $h[k]$ (2-61)



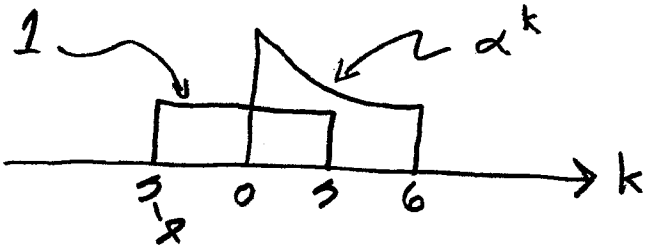


Case I) $n<0$:



$y[n]=0$

Case II) $n \geq 0$ and $n-4<0$: $0 \leq n < 4$:



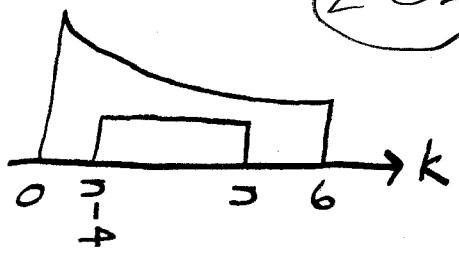
In this case, $h[k]$ "turns the sum on" at $k=0$ and $x[n-k]$ "turns it off" at $k=n$.

$$y[n] = \sum_{k=0}^{n} h[k]\, x[n-k] = \sum_{k=0}^{n} \alpha^k \cdot 1 = \sum_{k=0}^{n} \alpha^k$$

$$= \frac{\alpha^0 - \alpha^{n+1}}{1-\alpha} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Case III) $n-4 \geq 0$ and $n < 6$: $4 \leq n < 6$:

In this case $x[n-k]$ turns the sum on at $k = n-4$ and also turns it off at $k = n$.

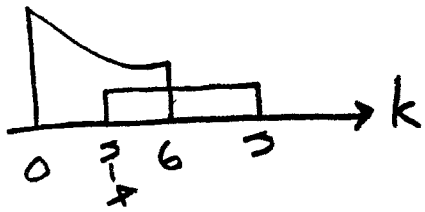


$$y[n] = \sum_{k=n-4}^{n} h[k]x[n-k]$$

$$= \sum_{k=n-4}^{n} \alpha^k = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$$

Case IV) $n \geq 6$ and $n-4 < 7$: $6 \leq n < 11$:

$x[n-k]$ turns sum on at $k = n-4$ and $h[k]$ turns it off at $k = 6$.

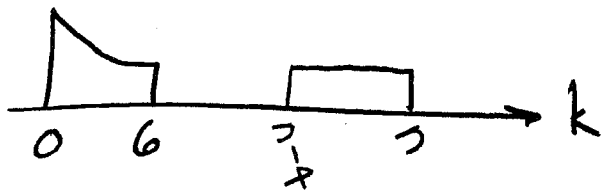


$$y[n] = \sum_{k=n-4}^{6} h[k]x[n-k]$$

$$= \sum_{k=n-4}^{6} \alpha^k = \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}$$

→

Case II) $n \geq 11$:



$y[n] = 0.$

(2-63)

All Together:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \leq n < 4 \\ \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}, & 4 \leq n < 6 \\ \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}, & 6 \leq n < 11 \\ 0, & n \geq 11 \end{cases}$$

CONTINUOUS TIME CONVOLUTION

- For continuous-time signals $x(t)$, the natural basis is given by the translates of the Dirac delta:

$$\{\delta(t-\tau)\}_{\tau \in \mathbb{R}}.$$

- Rigorous treatment of $\delta(t)$ requires 20th century math. We'll talk about it more later.

→ For now, continue to think about $\delta(t)$ the same way you did in your undergrad signals and systems course, i.e.

$$\delta(t) = 0, \quad t \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-t_0)\, x(t)\, dt = x(t_0)$$

→ with $x(t) = 1$, this implies $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$.

- Dot product for continuous time signals:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \text{a number}$$

- To write $x(t)$ as a sum of the natural basis, we need to:

  - find dot product of $x(t)$ with each basis signal
  - Add up dot products times the basis signals.

- Dot product of $x(t)$ with the τ'th basis signal $\delta(t-\tau)$:

$$\langle x(t), \delta(t-\tau) \rangle = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(\tau)$$

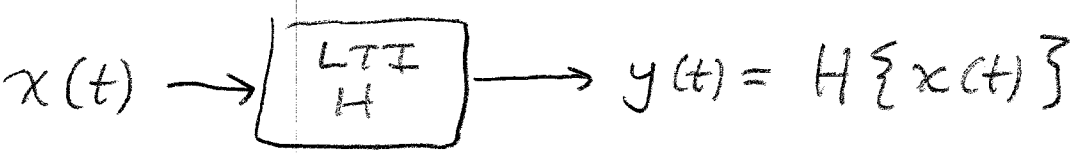
(a number)

- Add up dot products times basis vectors:

$$x(t) = \sum_{\substack{\text{basis's} \\ \tau}} \left(\tau\text{'th dot product} \right) \times \left(\tau\text{'th basis signal} \right)$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

- Now let H be a continuous-time LTI system:



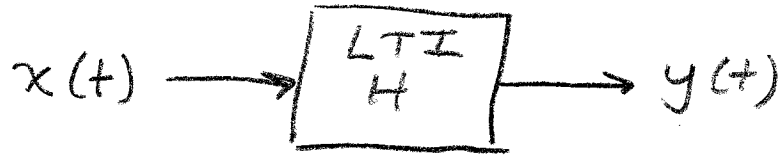
- When the input is $\delta(t)$, the output is called the "impulse response" $h(t)$:

- Because H is time invariant, we have

$$H\{\delta(t-t_0)\} = h(t-t_0).$$

- Because H is linear (and TI), we have $H\{a\delta(t-t_0)+b\delta(t-t_1)\} = ah(t-t_0)+bh(t-t_1)$.

- So, when we write the input $x(t)$ as a weighted sum of shifted Dirac deltas, it is clear that the output must be a weighted sum of shifted impulse responses:



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} H\{x(\tau)\delta(t-\tau)\}\,d\tau \quad \text{"additivity" part of linear}$$

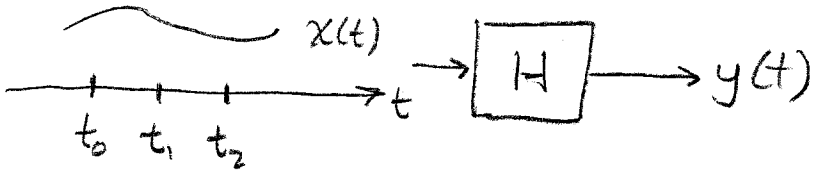
$$= \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}\,d\tau \quad \text{"homogeneity" part of linear}$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{because } H \text{ is time invariant}$$

$$\equiv x(t) * h(t)$$

"convolution of $x(t)$ and $h(t)$."

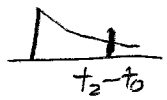
- Intuition:



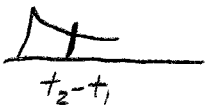
- The input term $x(t_0)\delta(t-t_0)$ arrives at time t_0 and makes $x(t_0)h(t-t_0)$ come out.
- The input term $x(t_1)\delta(t-t_1)$ arrives at time t_1 and makes $x(t_1)h(t-t_1)$ come out.
- " " " $x(t_2)\delta(t-t_2)$ " " " t_2 " " $x(t_2)h(t-t_2)$ come out.

→ And lots of other input terms arrive between these.

- To find the value of the output signal $y(t)$ at some particular time like $t=t_2$, we have to add up what is coming out at t_2 due to <u>all</u> the different input terms.
- This will include:



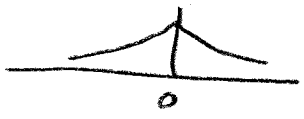
$x(t_0)h(t_2-t_0)$ due to input at $t=t_0$



$x(t_1)h(t_2-t_1)$ due to input at $t=t_1$

$x(t_2)h(0)$ due to input at $t=t_2$

→

- If H is non-causal, then $h(t)$ might be two-sided 

, and there could also be something coming out at $t=t_2$ due to <u>future</u> input terms that didn't even arrive yet.

- To find the number $y(t_2)$, we need to add up what is coming out at $t=t_2$ due to <u>all</u> the input terms, including past, present, and future input terms:

$$y(t_2) = \int_{-\infty}^{\infty} x(\theta)\, h(t_2-\theta)\, d\theta$$

- But we have not assumed anything special about t_2. The same reasoning can be applied to find the number $y(t)$ for any time t, and thus we have a function (signal)

$$y(t) = \int_{-\infty}^{\infty} x(\theta)\, h(t-\theta)\, d\theta$$

"continuous-time convolution"

- Like in discrete-time, continuous-time convolution is commutative:
  - It doesn't matter which signal gets the "τ" and which signal gets the "$t-\tau$"...
  - For every t, $y(t)$ is the same number either way.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

- You can always get the right answer either way.

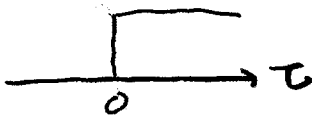
- Usually, it will be slightly less work if you put the "$t-\tau$" on the signal with the less complicated expression.

EX: $x(t) = e^{-at}u(t)$, $a>0$.

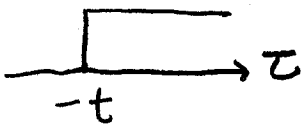
$h(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

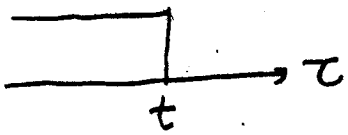
$h(\tau)$



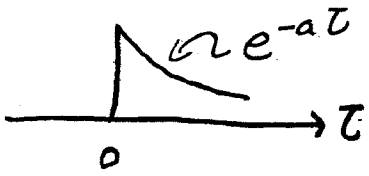
$h(\tau-(-t)) = h(t+\tau)$



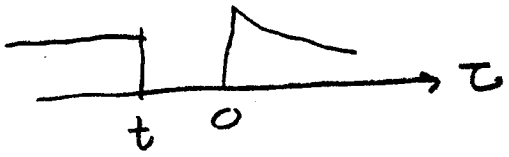
$h(t-\tau)$



$x(\tau)$

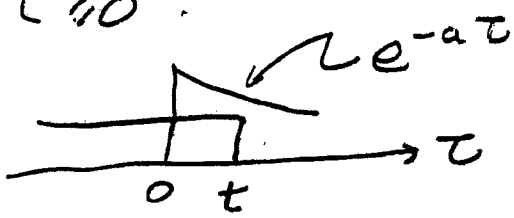


case I) $t<0$:



$$y(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0.$$

case II) $t \geq 0$:



$$y(t) = \int_0^t e^{-a\tau}\,d\tau = -\frac{1}{a}\left[e^{-a\tau}\right]_{\tau=0}^{t}$$

$$= -\frac{1}{a}\left[e^{-at}-1\right] = \frac{1}{a}\left[1-e^{-at}\right]$$

ALL TOGETHER:

$$y(t) = \begin{cases} 0, & t<0 \\ \frac{1}{a}\left[1-e^{-at}\right], & t\geq 0 \end{cases}$$

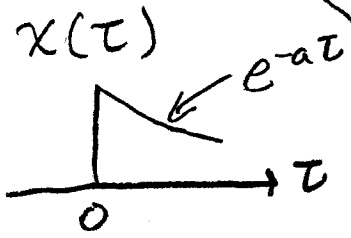
$$= \frac{1}{a}\left[1-e^{-at}\right]u(t).$$

EX: Same problem the "other way".

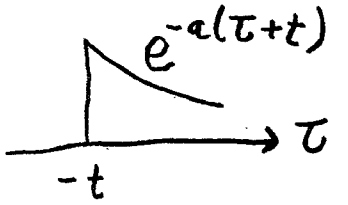
$x(t) = e^{-at}u(t)$, $a>0$

$h(t) = u(t)$

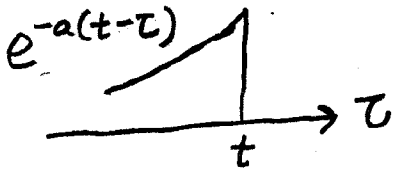
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$



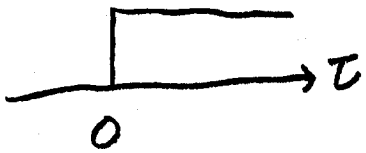
$x(\tau - -t) = x(\tau + t)$



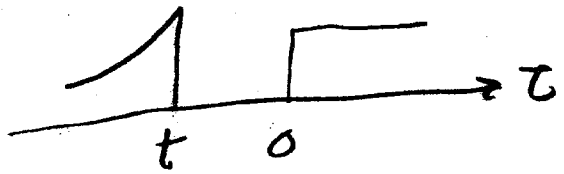
$x(t-\tau)$



$h(\tau)$

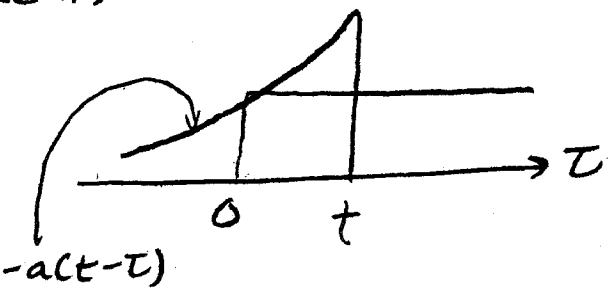


Case I) $t<0$:



$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0.$$

Case II) $t \geq 0$:



$$y(t) = \int_0^t e^{-a(t-\tau)}\, d\tau = \int_0^t e^{-at}e^{a\tau}\, d\tau$$

$$= e^{-at}\int_0^t e^{a\tau}\, d\tau$$

$$\frac{1}{a}e^{-at}\left[e^{a\tau}\right]_{\tau=0}^{t} = \frac{1}{a}e^{-at}\left[e^{at}-1\right]$$

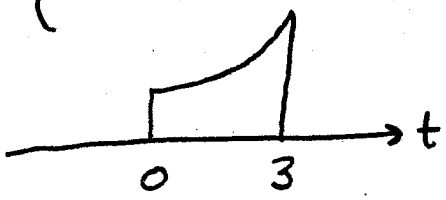
$$= \frac{1}{a}\left[1-e^{-at}\right].$$

All Together: $y(t) = \frac{1}{a}\left[1-e^{-at}\right]u(t)$.

EX: $x(t) = \begin{cases} e^{3t}, & 0 \le t \le 3 \\ 0, & \text{other} \end{cases}$

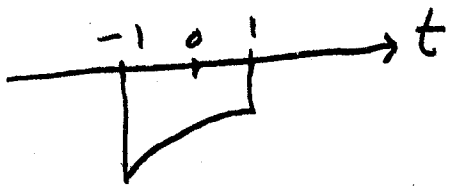
$= e^{3t}\{u(t) - u(t-3)\}$

$=$



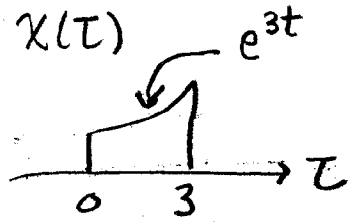
$h(t) = \begin{cases} -e^{-2t}, & -1 \le t \le 1 \\ 0, & \text{other} \end{cases} = -e^{-2t}\{u(t+1) - u(t-1)\}$

$=$

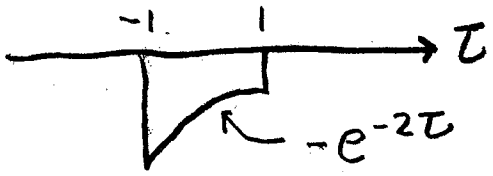


$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

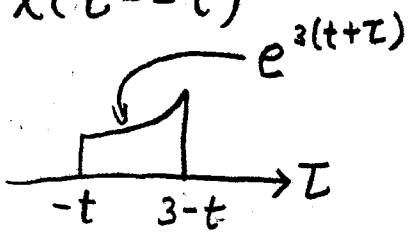
$x(\tau)$



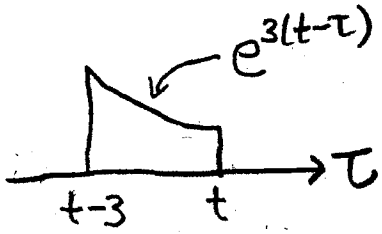
$h(\tau)$



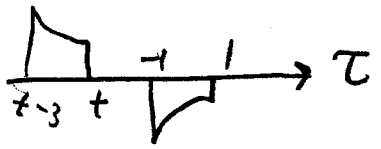
$x(\tau - -t)$



$x(t-\tau)$

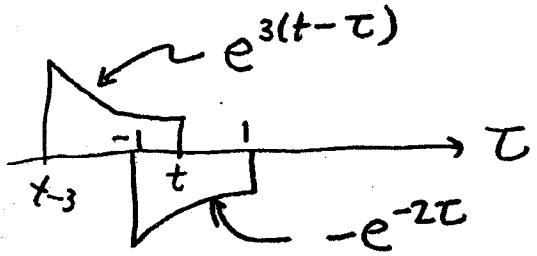


Case I) $t < -1$:



$$y(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0.$$

Case II) $t \geq -1$ and $t < 1$: $\quad -1 \leq t < 1$:



$$y(t) = \int_{-1}^{t} e^{3(t-\tau)} \left[-e^{-2\tau}\right] d\tau = -\int_{-1}^{t} e^{3t} e^{-3\tau} e^{-2\tau} \, d\tau$$

$$= -e^{3t} \int_{-1}^{t} e^{-5\tau} \, d\tau = \frac{1}{5} e^{3t} \left[e^{-5\tau}\right]_{\tau=-1}^{t}$$

$$= \frac{1}{5} e^{3t} \left[e^{-5t} - e^{5}\right] = \frac{1}{5}\left[e^{-2t} - e^{3t+5}\right]$$

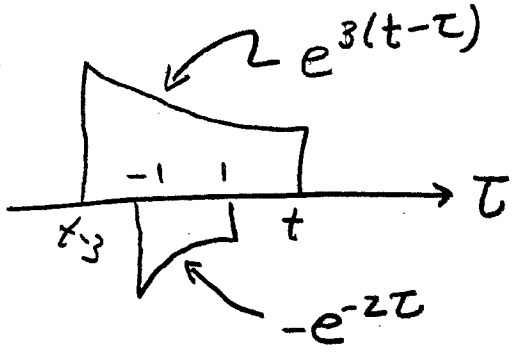
$$= \frac{1}{5} e^{-2t} - \frac{1}{5} e^{3t+5}$$

→

Case III) $t \geq 1$ and $t-3 < -1$

$t \geq 1$ and $t < 2$

$1 \leq t < 2$:



$$y(t) = \int_{-1}^{1} e^{3(t-\tau)}\left[-e^{-2\tau}\right] d\tau$$

$$= -e^{3t}\int_{-1}^{1} e^{-3\tau} e^{-2\tau} d\tau = -e^{3t}\int_{-1}^{1} e^{-5\tau} d\tau$$

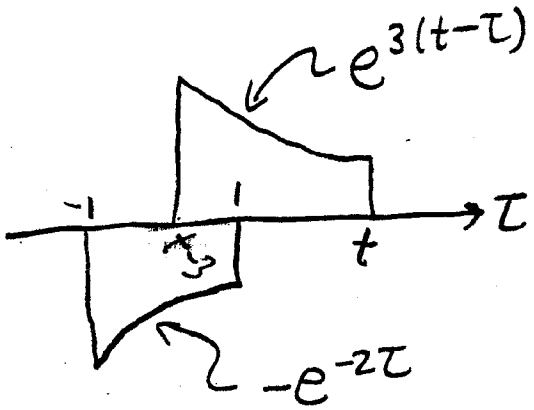
$$= \frac{1}{5}e^{3t}\left[e^{-5\tau}\right]_{\tau=-1}^{1} = \frac{1}{5}e^{3t}\left[e^{-5} - e^{5}\right]$$

$$= \frac{1}{5}e^{3t-5} - \frac{1}{5}e^{3t+5}$$

Case IV) $t-3 \geq -1$ and $t-3 < 1$

$t \geq 2$ and $t < 4$

$2 \leq t < 4$:



$$y(t) = \int_{t-3}^{1} e^{3(t-\tau)}\left[-e^{-2\tau}\right] d\tau$$

$$= -e^{3t}\int_{t-3}^{1} e^{-3\tau} e^{-2\tau} d\tau \longrightarrow$$

2-74

Case IV) ...

$$= -e^{3t}\int_{t-3}^{1} e^{-5\tau}\,d\tau = \frac{1}{5}e^{3t}\left[e^{-5\tau}\right]_{\tau=t-3}^{1}$$

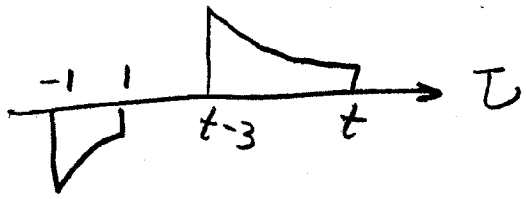
$$= \frac{1}{5}e^{3t}\left[e^{-5} - e^{-5(t-3)}\right]$$

$$= \frac{1}{5}e^{3t}\left[e^{-5} - e^{-5t}e^{15}\right]$$

$$= \frac{1}{5}\left[e^{3t}e^{-5} - e^{3t}e^{-5t}e^{15}\right]$$

$$= \frac{1}{5}\left[e^{3t-5} - e^{-2t}e^{15}\right] = \frac{1}{5}e^{3t-5} - \frac{1}{5}e^{-2t+15}$$

Case V) $t-3 \geq 1 \;:\; t \geq 4$:



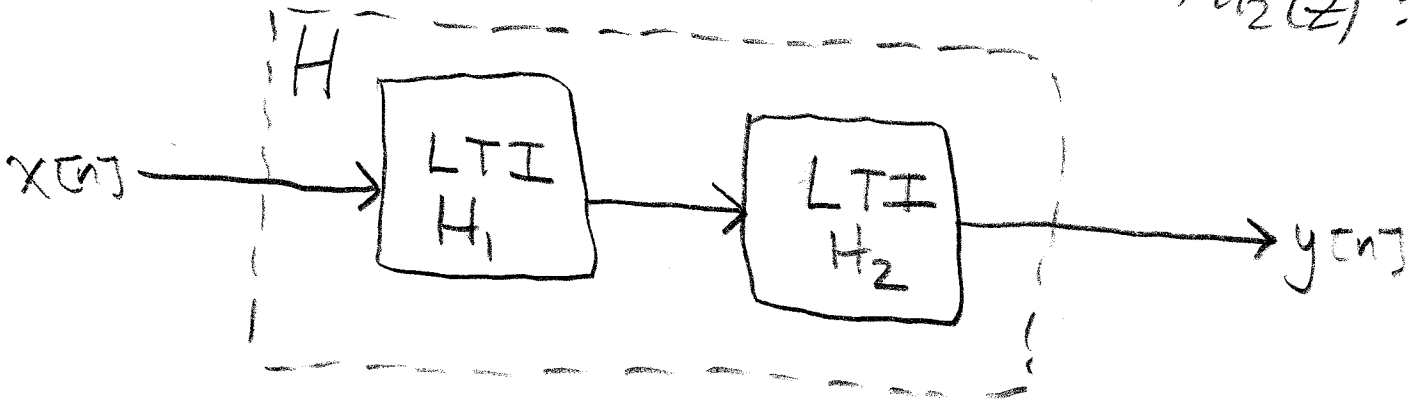
$y(t) = 0.$

All Together:

$$y(t) = \begin{cases} 0, & t < -1 \\ \frac{1}{5}e^{-2t} - \frac{1}{5}e^{3t+5}, & -1 \leq t < 1 \\ \frac{1}{5}e^{3t-5} - \frac{1}{5}e^{3t+5}, & 1 \leq t < 2 \\ \frac{1}{5}e^{3t-5} - \frac{1}{5}e^{-2t+15}, & 2 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

- From your undergraduate signals & systems course, you should already know and be able to prove that:

→ The "series" or "cascade" connection of two discrete-time LTI systems H_1 and H_2 is a new system H that is LTI with impulse response $h_1[n] * h_2[n]$, frequency response $H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$, and transfer function $H(z) = H_1(z) H_2(z)$:



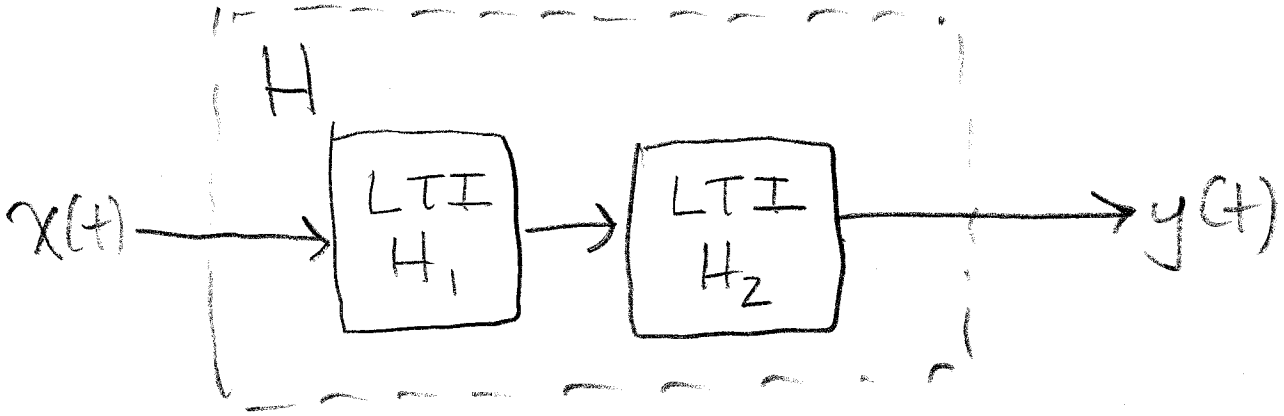
H is LTI.

$h[n] = h_1[n] * h_2[n]$

$H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$

$H(z) = H_1(z) H_2(z)$

→ The same is true for the series (e.g. cascade) connection of two continuous-time LTI systems:



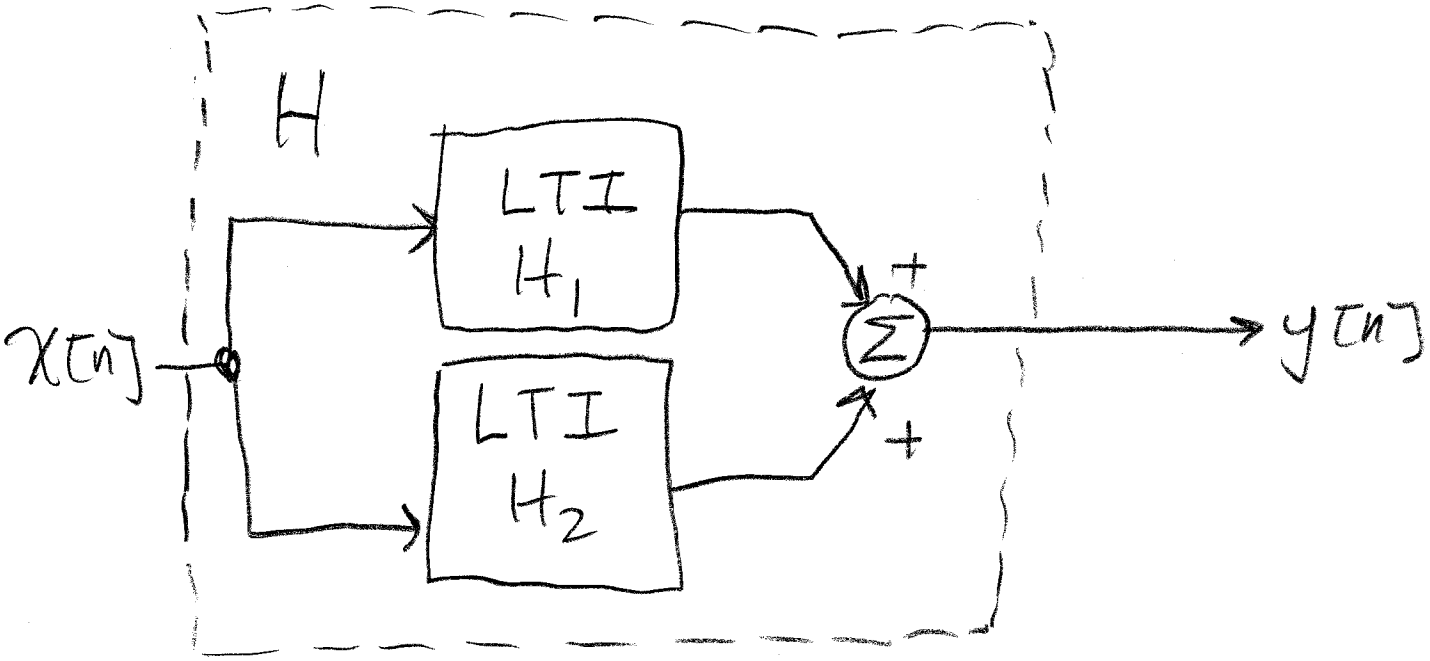
H is LTI.

$$h(t) = h_1(t) * h_2(t)$$

$$H(\Omega) = H_1(\Omega) H_2(\Omega)$$

$$H(s) = H_1(s) H_2(s).$$

→ The parallel connection of two discrete-time LTI systems is also LTI :



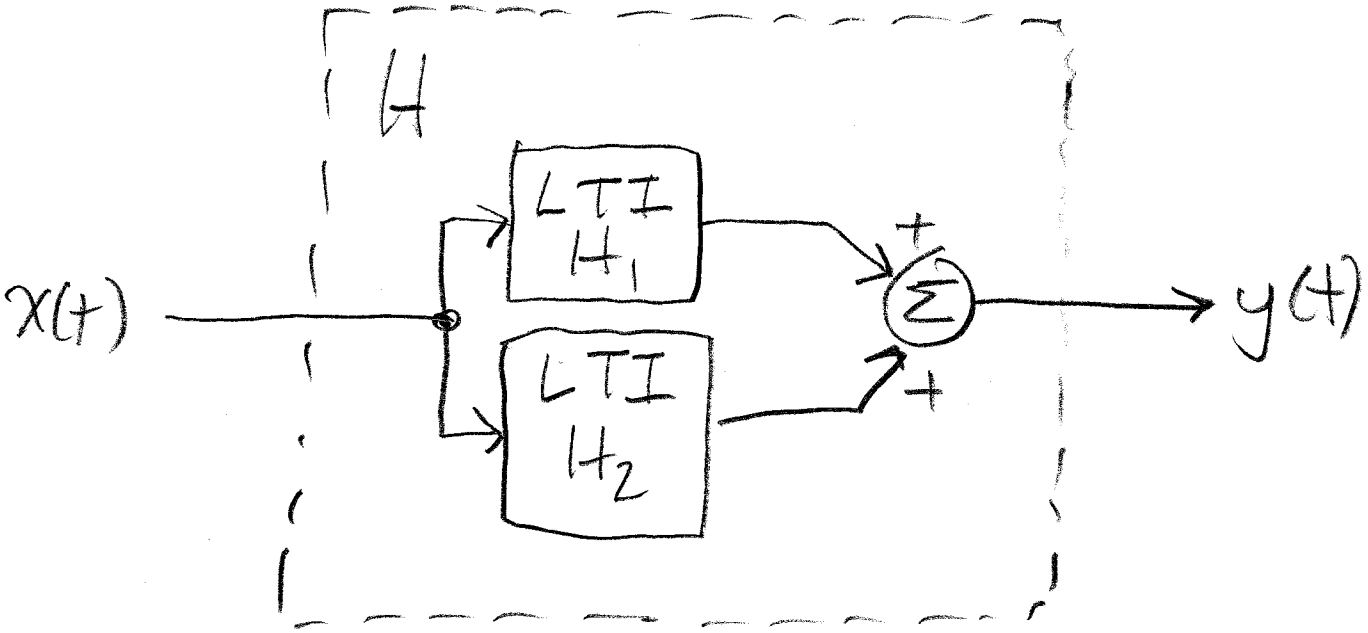
with

impulse response $h[n] = h_1[n] + h_2[n]$

freq. response $H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$

transfer fcn $H(z) = H_1(z) + H_2(z)$

→ And the same is true for the parallel connection of two continuous-time LTI systems:



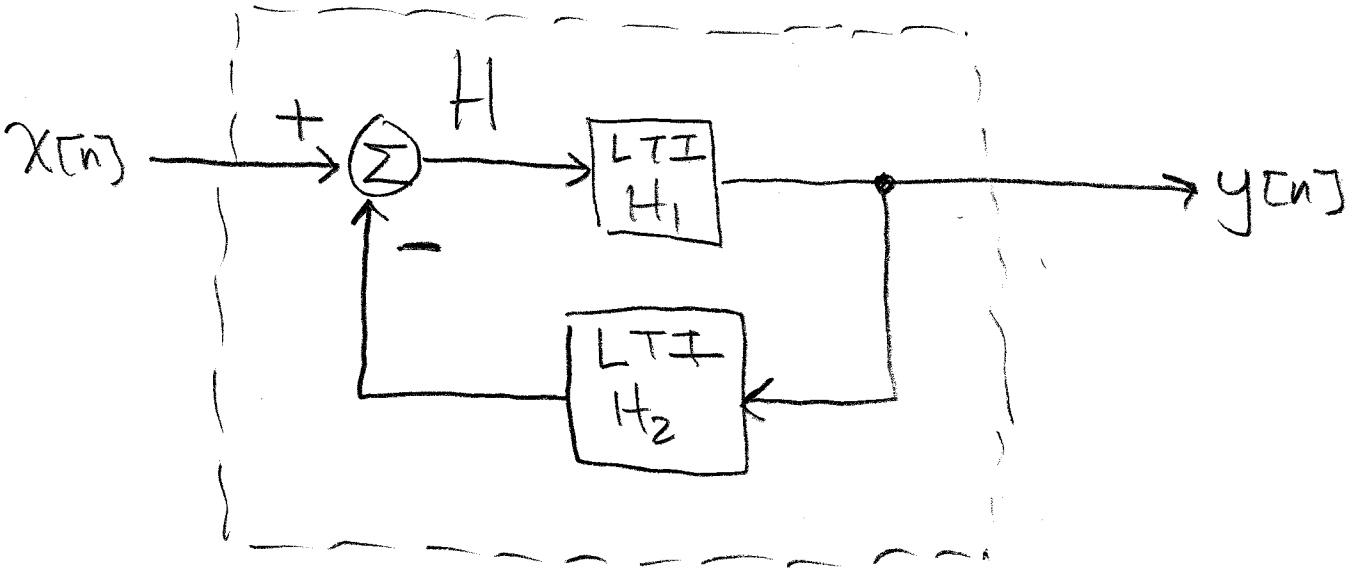
H is LTI.

Impulse response: $h(t) = h_1(t) + h_2(t)$

freq. response: $H(\Omega) = H_1(\Omega) + H_2(\Omega)$

transfer fcn: $H(s) = H_1(s) + H_2(s)$

- Again from your undergrad signals & systems course, you should know and be able to prove that the negative feedback connection of two discrete-time LTI systems H_1 & H_2 is LTI:

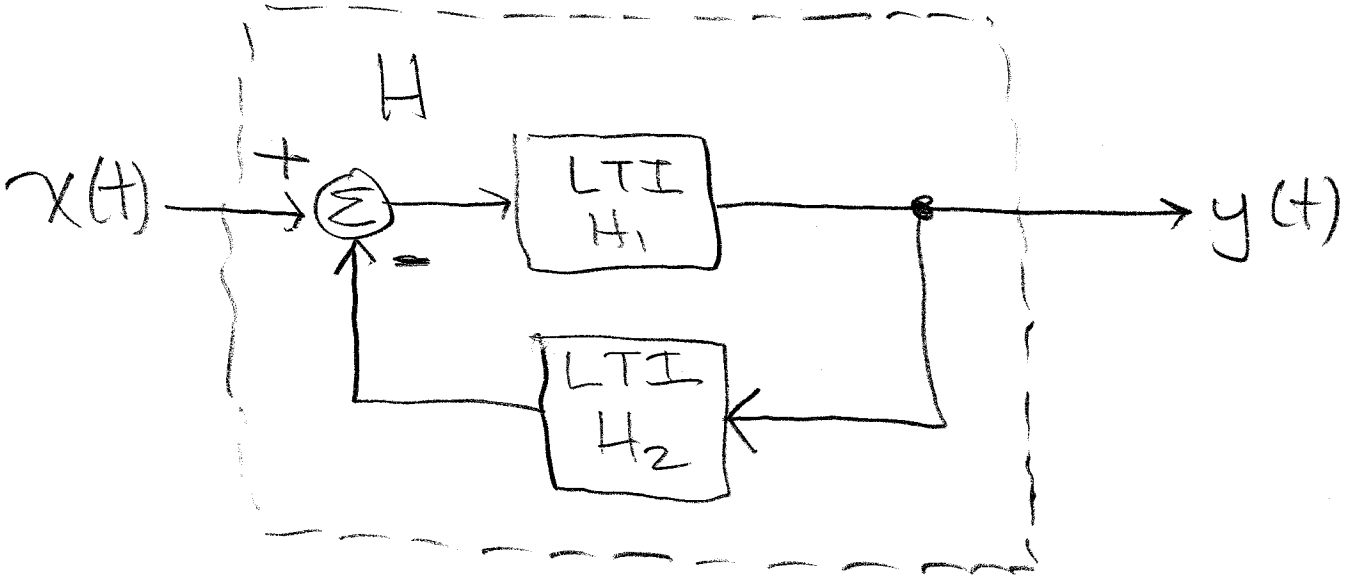


Impulse response: no general time domain solution... though you can find it given any particular H_1 & H_2 by using frequency domain techniques.

Freq. Response: $$H(e^{j\omega}) = \frac{H_1(e^{j\omega})}{1 + H_1(e^{j\omega})H_2(e^{j\omega})}$$

Transfer Fcn: $$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

- And the same is true in continuous time:



H is LTI.

impulse resp: no general time domain solution.

freq Response: $$H(\Omega) = \frac{H_1(\Omega)}{1 + H_1(\Omega) H_2(\Omega)}$$

Transfer Fcn: $$H(s) = \frac{H_1(s)}{1 + H_1(s) H_2(s)}$$

System properties revisited for LTI systems:

→ If you are given (or have proved) that a system is LTI, then there are alternate methods for proving the other properties:

- memoryless
- causal
- stable

in terms of the impulse response.

→ For an LTI system, these alternate methods are usually quicker and easier than the general methods we have already discussed.

- You should already know and be able to prove these things from your undergrad signals & systems course.

Memoryless:

- A discrete-time LTI system H is memoryless iff the impulse response is a constant times $\delta[n]$, i.e.

  iff $h[n] = K\delta[n]$ for some constant K.

- A continuous-time LTI system H is memoryless iff the impulse response is a constant times $\delta(t)$, i.e.

  iff $h(t) = K\delta(t)$ for some constant K.

Causal?

- A discrete-time LTI system H is causal iff the impulse response is zero for all the negative integers, i.e.

  iff $h[n] = 0 \quad \forall \; n < 0$.

- A continuous-time LTI system H is causal iff the impulse response is zero on the negative half-line, i.e.

  iff $h(t) = 0 \quad \forall \; t < 0$.

Stability:

- A discrete-time LTI system H is BIBO stable iff the impulse response $h[n] \in \ell^1(\mathbb{Z})$, i.e.

$$\text{iff} \quad \sum_{n=-\infty}^{\infty} |h[n]| < \infty. \quad \left(\begin{array}{c}\text{absolutely}\\ \text{summable}\end{array}\right)$$

- A continuous-time LTI system H is BIBO stable iff the impulse response $h(t) \in L^1(\mathbb{R})$, i.e.

$$\text{iff} \quad \int_{-\infty}^{\infty} |h(t)|\, dt < \infty. \quad \left(\begin{array}{c}\text{absolutely}\\ \text{integrable}\end{array}\right)$$

Unit Step Response

- We have been characterizing ~~LSI~~ LTI systems in terms of their unit impulse responses.
- It is also possible to study ~~LSI~~ LTI systems in terms of their response to the unit step. This response is called the "unit step response".

→ For a discrete-time ~~LSI~~ LTI system, the unit step response $s[n]$ is the output when the input is $u[n]$.

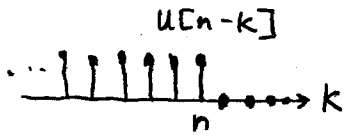
→ For a continuous-time ~~LSI~~ LTI system, the unit step response is $s(t)$ is the output when the input is $u(t)$.

- Thus,

$$s[n] = u[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

$$= \sum_{k=-\infty}^{n} h[k]$$



→ So, $s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k]$

$$= h[n] + \sum_{k=-\infty}^{n-1} \left(h[k] - h[k] \right)$$

$$= h[n]$$

→ Then the relationship between $h[n]$ and $s[n]$ is

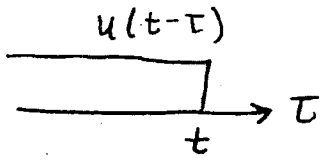
$$h[n] = s[n] - s[n-1].$$

- Likewise, for a continuous-time ~~LSI~~ LTI system,

$$s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} h(\tau)\, d\tau.$$



- Applying the fundamental theorem of calculus, we see that the relationship between $s(t)$ and $h(t)$ is

$$h(t) = \frac{d}{dt} s(t).$$

~~We are done with Chapter 2!!~~

Some More Adjectives to Describe Signals.

- Here we consider a discrete time signal $x[n] \in \ell^p(\mathbb{Z})$ that has domain $\mathbb{Z}$ and range $\mathbb{R}$ or $\mathbb{C}$.

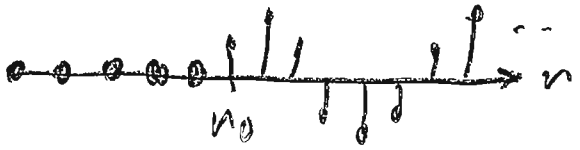
- and a continuous-time signal $x(t) \in L^p(\mathbb{R})$ that has domain $\mathbb{R}$ and range $\mathbb{R}$ or $\mathbb{C}$.

$\Rightarrow$ The definitions given here require modification for a finite length signal $x[n]$ that has a finite domain $[0, N-1]$ for $N \in \mathbb{N} = \mathbb{Z}^+$.

Right sided:

- $x[n]$ is right sided if $\exists\, n_0 \in \mathbb{Z}$ s.t. $x[n] = 0 \;\forall\; n < n_0$.

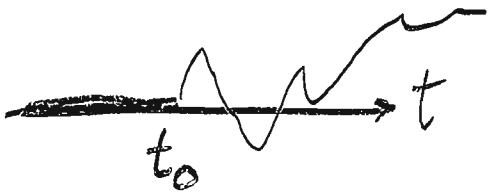
Ex:



- if $n_0 = 0$, then $x[n]$ is called causal.

- $x(t)$ is right sided if $\exists t_0 \in \mathbb{R}$ s.t. $x(t) = 0 \ \forall \ t < t_0$.

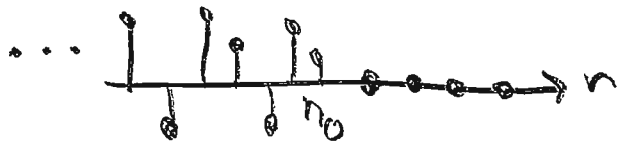
EX:



- if $t_0 = 0$, then $x(t)$ is called causal.

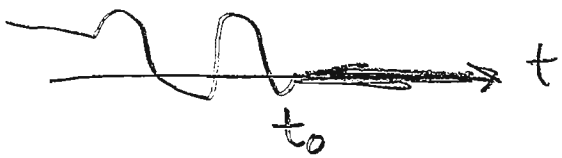
Left sided:

- $x[n]$ is left sided if $\exists n_0 \in \mathbb{Z}$ such that $x[n] = 0 \ \forall \ n > n_0$.



- if $n_0 = 0$, $x[n]$ is called anticausal.

- $x(t)$ is left sided if $\exists t_0 \in \mathbb{R}$ s.t. $x(t) = 0$ $\forall t > t_0$.

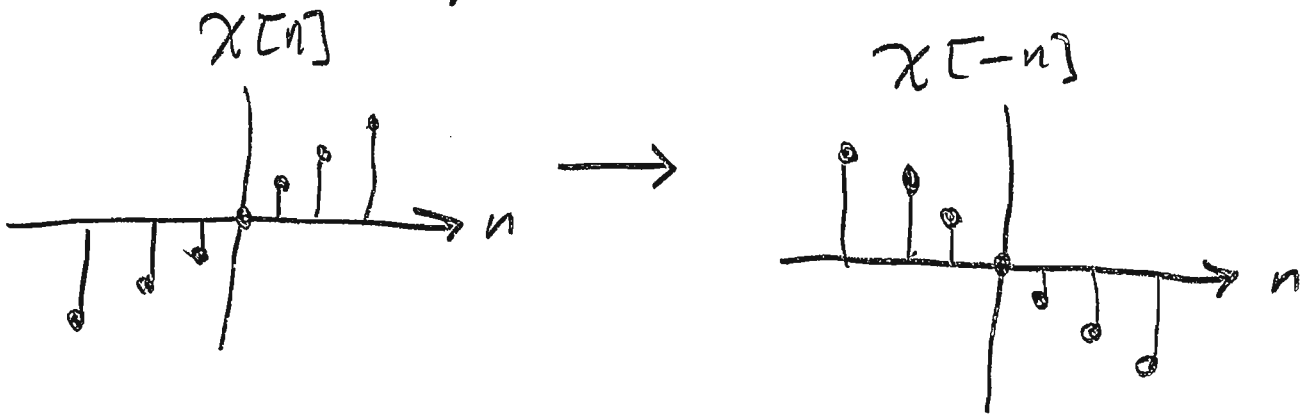


- if $t_0 = 0$, $x(t)$ is called anticausal.

- A signal that is <u>neither</u> left sided nor right sided is called "<u>two sided</u>."

  → A two sided signal is generally nonzero for at least some t or n "all the way" in both directions on the t or n axis.

- The signal $x[-n]$ is called the <u>reflection</u> of $x[n]$. The graph of $x[-n]$ is obtained by flipping the graph of $x[n]$ about the vertical or "y-axis":



- In continuous time, the signal $x(-t)$ is called the <u>reflection</u> of $x(t)$.

Symmetry

- if $x[n] = x[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called <u>even</u>.

- if $x(t) = x(-t)$ $\forall t \in \mathbb{R}$, then $x(t)$ is called <u>even</u>.

- if $x[n] = -x[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called <u>odd</u>. This implies that $x[0] = 0$.

- if $x(t) = -x(-t)$ $\forall t \in \mathbb{R}$, then $x(t)$ is called <u>odd</u>. This implies that $x(0) = 0$.

- if $x[n] = x^*[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called <u>conjugate symmetric</u>. This implies that $x[0]$ is <u>real</u>.

- if $x(t) = x^*(-t)$ $\forall t \in \mathbb{R}$, then $x(t)$ is called <u>conjugate symmetric</u>. This implies $x(0) \in \mathbb{R}$.

- if $x[n] = -x^*[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called conjugate antisymmetric. This implies that $x[0]$ is purely imaginary.
- if $x(t) = -x^*(-t)$ $\forall t \in \mathbb{R}$, then $x(t)$ is called conjugate antisymmetric. This implies that $x(0)$ is purely imaginary.

FACT: any real signal $x[n]$ can be written uniquely as

$$x[n] = x_{ev}[n] + x_{od}[n]$$

where

$$x_{ev}[n] = \frac{1}{2}\{x[n] + x[-n]\}$$

is even

and where

$$x_{od}[n] = \frac{1}{2}\{x[n] - x[-n]\}$$

is odd.

Note:

$$\text{DTFT}\{x_{ev}[n]\} = \text{Re}\left[\text{DTFT}\{x[n]\}\right]$$

$$\text{DTFT}\{x_{od}[n]\} = j\,\text{Im}\left[\text{DTFT}\{x[n]\}\right]$$

$\Rightarrow$ Recall: $X(e^{j\omega})$ is conjugate symmetric if $x[n]$ is real.

$X(e^{j\omega})$ is real & even if $x[n]$ is real & even.

FACT: any real signal $x(t)$ can be written uniquely as

$$x(t) = x_{ev}(t) + x_{od}(t)$$

where

$$x_{ev}(t) = \frac{1}{2}\{x(t) + x(-t)\}$$

is even and where

$$x_{od}(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

is odd.

Note: $\mathcal{F}\{x_{ev}(t)\} = \text{Re}\left[\mathcal{F}\{x(t)\}\right]$

$$\mathcal{F}\{x_{od}(t)\} = j\,\text{Im}\left[\mathcal{F}\{x(t)\}\right]$$

FACT: Any signal $x[n]$ can be written uniquely as

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$x_{cs}[n] = \frac{1}{2}\left\{x[n] + x^*[-n]\right\}$$

is conjugate symmetric and where

$$x_{ca}[n] = \frac{1}{2}\left\{x[n] - x^*[-n]\right\}$$

is conjugate antisymmetric.

$\Rightarrow$ The DTFT of $x_{cs}[n]$ is the real part of the DTFT of $x[n]$.

The DTFT of $x_{ca}[n]$ is j times the imaginary part of the DTFT of $x[n]$.

Fact: Any signal $x(t)$ can be written uniquely as

$$x(t) = x_{cs}(t) + x_{ca}(t)$$

where

$$x_{cs}(t) = \frac{1}{2}\left\{x(t) + x^*(-t)\right\}$$

is conjugate symmetric and where

$$x_{ca}(t) = \frac{1}{2}\left\{x(t) - x^*(-t)\right\}$$

is conjugate antisymmetric.

$\Rightarrow$ The FT of $x_{cs}(t)$ is the real part of the FT of $x(t)$.

The FT of $x_{ca}(t)$ is j times the imaginary part of the FT of $x(t)$.