ECE 4213/5213

MODULE 6

Index

z - Transform Review

- you should already be familiar with the z-transform, its relationship to the DTFT, and its relationship to the Laplace transform from your undergraduate signals and systems course.

- The z-transform is treated in Chapter 3 of the Oppenheim & Schafer text.

- Here we'll just do a brief review.

Eigenvalue Interpretation

FACT: for any fixed complex number z, the signal z^n is an eigenfunction of any discrete time LTI system.

Proof: Let H be an LTI system with impulse response $h[n]$. Let $z \in \mathbb{C}$ be a constant and let the system input be z^n.

$\longrightarrow$

Then the system output is given by

$$y[n] = H\{z^n\} = z^n * h[n]$$

$$= \sum_{k\in\mathbb{Z}} h[k]\, z^{n-k} = \sum_{k\in\mathbb{Z}} h[k]\, z^n z^{-k}$$

$$= \underbrace{z^n}_{\text{The input}} \underbrace{\sum_{k\in\mathbb{Z}} h[k]\, z^{-k}}_{\text{A number that depends on the value of the constant } z \text{ and on the system}} = H(z)\, z^n .$$

QED.

- So we see that, for the set of eigenfunctions $\{z^n\}_{z\in\mathbb{C}}$, the eigenvalues are given by the z-transform of the impulse response.

- $H(z)$ is called the "transfer function" of the system.

The "Fixer-Upper" Interpretation

- Another way to think of the z-transform that is probably more intuitive for most people.

- We already talked about this in the notes for chapter 1.

- Suppose we want to consider signals $x[n]$ that may have divergent "bad behaviour" on the right side (as $n \to \infty$) or on the left side (as $n \to -\infty$).

EX: $x[n] = 3^n$ or $x[n] = 3^n u[n]$: bad on the right.

$x[n] = \left(\frac{1}{2}\right)^n$ or $x[n] = \left(\frac{1}{2}\right)^n u[-n]$; bad on the left.

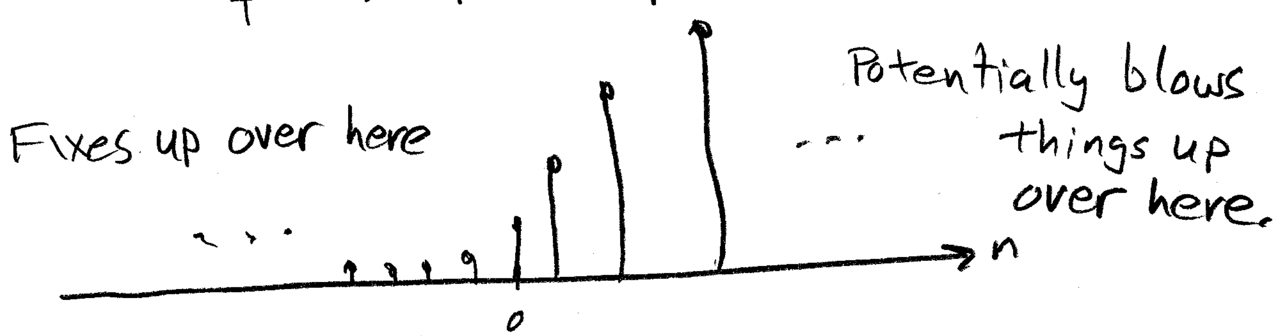
- The DTFT fails to converge for signals of this type.

- If we want to be able to use transform analysis to analyze how such signals interact with LTI systems, we will need a transform that is more powerful than the DTFT.

- The strategy is to multiply the "bad" signal $x[n]$ times a "fixer-upper" signal r^{-n}, where $r \in \mathbb{R}$ and $r \geq 0$.
- The fixer-upper corrects the bad behaviour in $x[n]$ so that we can take the DTFT of the "fixed up" signal $x[n]r^{-n}$.

→ if $0 < r < 1$, then the fixer-upper r^{-n} fixes up on the <u>left</u>.

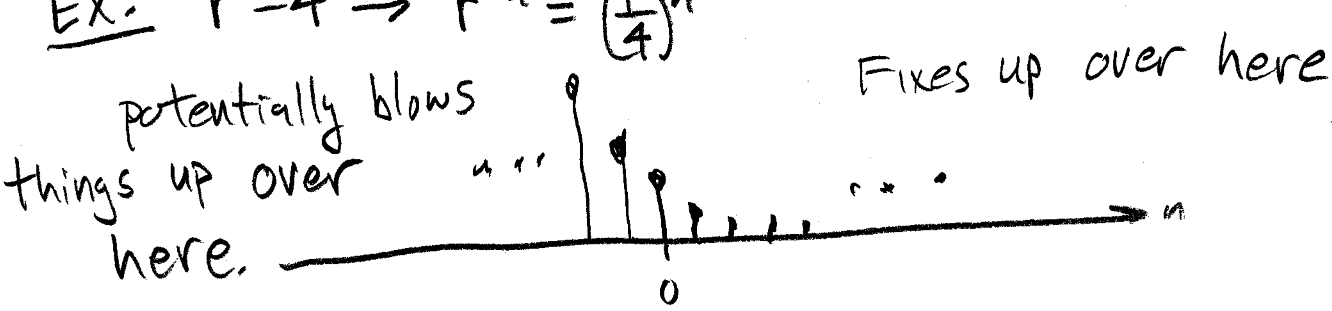
<u>EX</u>: $r = \frac{1}{4} \rightarrow r^{-n} = 4^n$:



→ If $r = 1$, then $r^{-n} = 1$ and the fixer-upper does nothing: the fixed up signal is $x[n]r^{-n} = x[n]$.

→ If $r > 1$, then r^{-n} fixes up on the <u>right</u>.

<u>EX:</u> $r = 4 \rightarrow r^{-n} = \left(\frac{1}{4}\right)^n$



- The z-transform of $x[n]$ is the DTFT of the fixed-up guy $x[n]r^{-n}$:

$$X(z) = \text{DTFT}\{x[n]r^{-n}\}$$

$$= \sum_{n\in\mathbb{Z}} x[n]r^{-n}e^{-j\omega n}$$

$$= \sum_{n\in\mathbb{Z}} x[n]\underbrace{(re^{j\omega})}_{\substack{\text{a complex number in polar form,}\\ \text{call it } z\text{, with magnitude}\\ r=|z| \text{ and angle } \omega=\arg z.}}{}^{-n}$$

$$= \sum_{n\in\mathbb{Z}} x[n]z^{-n}.$$

- You can think of the graph of $X(z)$ as a sheet or surface that exists over the complex z-plane.
  - Actually, since $X(z)$ is complex valued in general, it takes "two graphs" to show $X(z)$:
    - one for the real part and one for the imaginary part,
    - or one for the magnitude and one for the phase [e.g., the angle].

- There is a nice example of one of these type graphs later in this file on pages 6.14-A through 6.14-C.

- The set of complex numbers z for which $X(z)$ converges is called the "Region of Convergence" or "ROC".

- Complex numbers z for which $X(z)$ blows up (diverges) are not in the ROC.
    - These z are "singularities" of $X(z)$.
    - Singularities that occur at isolated, single points in the z-plane are called "poles".

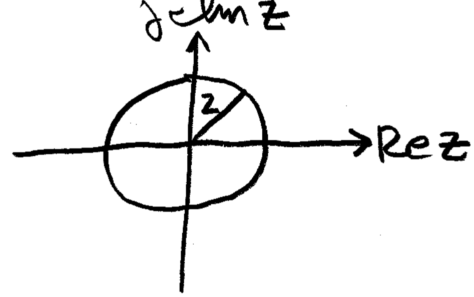
- For each real $r \geq 0$, the DTFT of the fixed up guy $x[n]r^{-n}$ either converges or it does not.
- If it converges, then the circle of radius r in the z-plane is included in the ROC of $X(z)$,
    → And the graph of the DTFT of $x[n]r^{-n}$ is part of the graph of $X(z)$.

→

EX: Suppose that

$$X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n} = \sum_{n \in \mathbb{Z}} x[n] r^{-n} e^{-j\omega n}$$

converges on the circle $|z| = 2$.

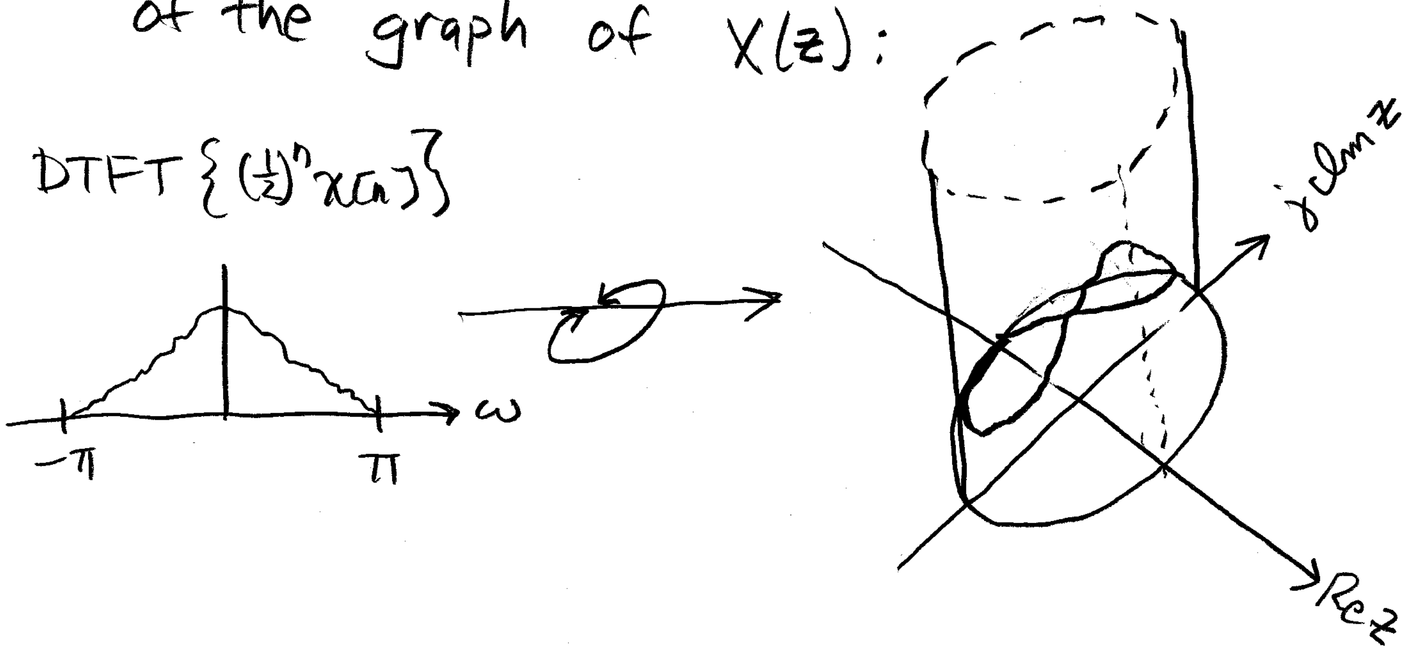


- Then the graph of the DTFT of the fixed up guy

$$x[n] 2^{-n} = \left(\tfrac{1}{2}\right)^n x[n]$$

is wrapped around this circle as part of the graph of $X(z)$:

$\text{DTFT}\left\{ \left(\tfrac{1}{2}\right)^n x[n] \right\}$



- So you can think of the ROC of $X(z)$ as a collection of circles of radius r where the DTFT of $x[n] r^{-n}$ converges.

- If $x[n]$ is zero for all the positive n's, then

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \cdots + x[-2]z^2 + x[-1]z^1 + x[0]z^0 + 0\frac{1}{z} + 0\frac{1}{z^2} + \cdots$$

and the ROC will generally include the circle of radius 0, e.g. the point $z=0$.

- If $x[n]$ is zero for all the negative n's, then the ROC will generally include the circle of radius ∞, e.g. the set $|z| = \infty$.

- You can think of the z-transform $X(z)$ as a collection of DTFT's of fixed-up guys $x[n]r^{-n}$ wrapped around concentric circles of radius r in the z-plane.

- For $r=1$, the fixed up guy is $x[n]1^n = x[n]$. So, above the unit circle in the z-plane $X(z)$ is exactly identical to $X(e^{j\omega})$.

- If $x[n]$ has a DTFT, then the unit circle is included in the ROC of $X(z)$.
- If $x[n]$ does <u>not</u> have a DTFT, then the unit circle is <u>not</u> included in the ROC of $X(z)$.

<u>NOTE:</u> when $r = 1$, then $z = re^{j\omega} = e^{j\omega}$,

so $X(z) = X(e^{j\omega})$.

- In order to specify a z-transform $X(z)$, it's necessary to specify both the function $X(z)$ <u>and</u> the ROC which tells which z's $X(z)$ converges for.

<u>EX</u>:

$$x_1[n] = \left(\tfrac{1}{2}\right)^n u[n].$$

$$X_1(z) = \sum_{n\in\mathbb{Z}} \left(\tfrac{1}{2}\right)^n u[n]\, z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$$

<u>NOTE</u>: for $|z| \le \frac{1}{2}$, the z-transform $X_1(z)$ is <u>NOT</u> equal to $\frac{1}{1-\frac{1}{2}z^{-1}}$. For $|z| \le \frac{1}{2}$, $X_1(z)$ diverges to ∞.

$\longrightarrow$

EX ... Now let

$$x_2[n] = -\left(\tfrac{1}{2}\right)^n u[-n-1].$$

Then
$$X_2(z) = \sum_{n\in\mathbb{Z}} -\left(\tfrac{1}{2}\right)^n u[-n-1]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\tfrac{1}{2}\right)^n z^{-n} = -\sum_{n=1}^{\infty} 2^n z^n$$

$$= 0 - \sum_{n=1}^{\infty} 2^n z^n = 1 - 1 - \sum_{n=1}^{\infty} 2^n z^n$$

$$= 1 - 2^0 z^0 - \sum_{n=1}^{\infty} 2^n z^n$$

$$= 1 - \sum_{n=0}^{\infty} 2^n z^n = 1 - \frac{1}{1-2z}, \quad |z| < \tfrac{1}{2}$$

$$= \frac{1-2z}{1-2z} - \frac{1}{1-2z}, \quad |z| < \tfrac{1}{2}$$

$$= \frac{1-2z-1}{1-2z}, \quad |z| < \tfrac{1}{2}$$

$$= \frac{2z}{2z-1} \cdot \frac{1/2}{1/2} = \frac{z}{z-\frac{1}{2}} \cdot \frac{z^{-1}}{z^{-1}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < 1/2.$$

$\longrightarrow$

EX ... So we have

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

and

$$X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}.$$

- Are $X_1(z)$ and $X_2(z)$ equal?

→ NO !!!!

→ There is no number $z \in \mathbb{C}$ for which $X_1(z) = X_2(z)$!!!

→ For any z that $X_1(z)$ converges to the number $\frac{1}{1-\frac{1}{2}z^{-1}}$, $X_2(z)$ diverges!

→ For any z that $X_2(z)$ converges to the number $\frac{1}{1-\frac{1}{2}z^{-1}}$, $X_1(z)$ diverges!

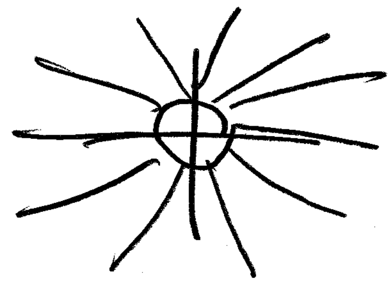
⇒ The ROC is important.

- If $x[n]$ is right sided, then any bad behavior could only be on the right side.
- If there is <u>any</u> r_0 that can fix things up for this $x[n]$, say $r_0 = 2$, then $\left(\frac{1}{2}\right)^n x[n]$ has a convergent DTFT.
- For any $r > r_0$, like $r=3$ in this case, the guy $x[n]$ is fixed up <u>even</u> <u>more</u>. The DTFT of $\left(\frac{1}{3}\right)^n x[n]$ certainly converges, because $\left(\frac{1}{3}\right)^n$ fixes up bad behavior on the right <u>even</u> <u>more</u> than $\left(\frac{1}{2}\right)^n$.

$\Rightarrow$ A right-sided $x[n]$ has an <u>exterior</u> <u>ROC</u>. It is generally the exterior of a circle that passes through the largest pole of $X(z)$.

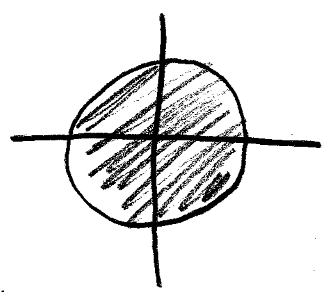
$\Rightarrow$ A left-sided $x[n]$ has an <u>interior</u> <u>ROC</u>. It is generally the interior of a circle that passes through the smallest pole of $X(z)$. $\longrightarrow$

- Any two sided $x[n]$ can be broken into the sum of a right sided part plus a left sided part,
- The z-transform of the right sided part has a ROC that is the exterior of a circle in the z-plane,



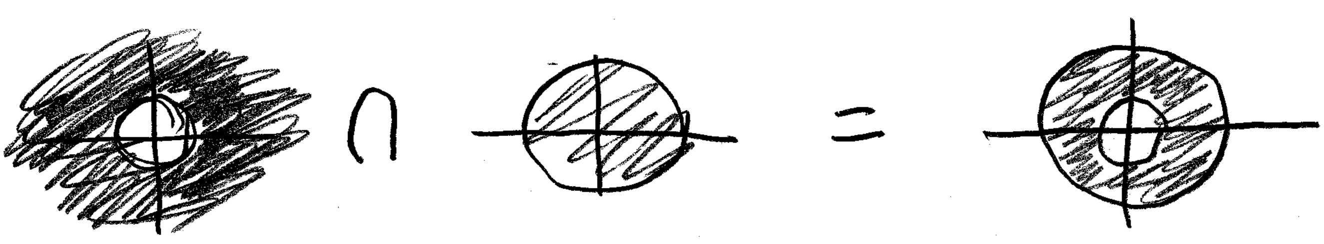
R_r

- The z-transform of the left sided part has a ROC that is the interior of a circle in the z-plane.



R_l

- The ROC of $X(z)$, the z-transform of the whole signal $x[n]$, is exactly the set of z's for which <u>both</u> the z-transform of the right sided part and the z-transform of the left sided part converge.
- The ROC of $X(z)$ is an annulus.



$\Rightarrow$ A two-sided $x[n]$ has an annular ROC.

$\Rightarrow$ An $x[n]$ with finite support (finite length) has a ROC that is generally the entire z-plane, except possibly the points $z=0$ and $z=\infty$.

$\rightarrow$ These two points must be checked individually for each $X(z)$.

$\rightarrow$ $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$$= \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0 + x[1] \frac{1}{z} + x[2] \frac{1}{z^2} + \cdots$$

$\rightarrow$ If $x[n]=0 \;\; \forall n>0$, then the point $z=0$ is included in the ROC of $X(z)$.

$\rightarrow$ If $x[n]=0 \;\; \forall n<0$, then the point $z=\infty$ is included in the ROC of $X(z)$.

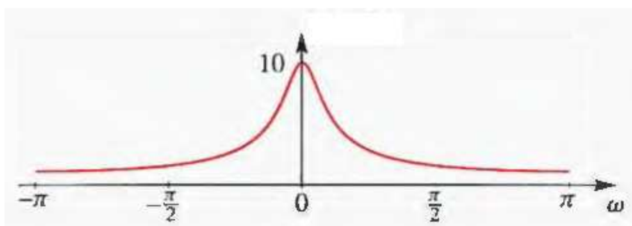
- EX: suppose H is a causal, stable discrete-time LTI system (filter) with impulse response

$$h[n] = \left(\frac{4}{5}\right)^n u[n].$$

- Using the DTFT table from the formula sheet on the course web site, we can write down the frequency response:

$$H\left(e^{j\omega}\right) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}$$

- Here is a plot of the filter magnitude response $\left|H\left(e^{j\omega}\right)\right|$:

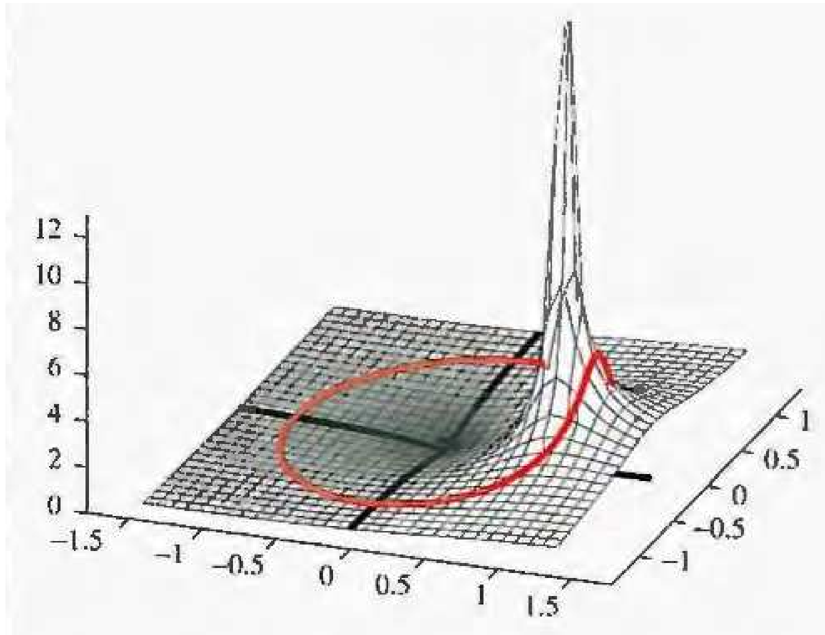


- From this plot, we see that H is a low-pass filter.
- Using the z-transform table from the formula sheet on the course web site, we can write down the transfer function:

$$H(z) = \frac{1}{1 - \frac{4}{5}z^{-1}}, \qquad ROC: |z| > \frac{4}{5}$$

- From the denominator, we see immediately that there is one pole at $z = \frac{4}{5}$.
- Notice that this pole is *inside* the unit circle, the ROC is *exterior*, and the ROC *includes* the unit circle.

- Here is a 3D plot of $|H(z)|$:



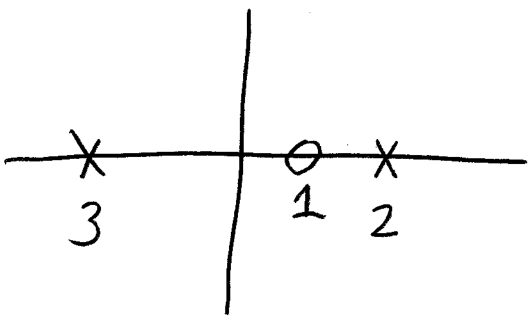
- Above the unit circle of the z-plane, $H(z) = H\left(e^{j\omega}\right)$.
- This is shown by the orange curve in the figure. This orange curve is the same graph of $\left|H\left(e^{j\omega}\right)\right|$ that we just saw on page 6.14-A, but now it is wrapped around the unit circle of the z-plane as the part of the graph of $H(z)$ where $|z|$ = r = 1.
- Notice how the pole at $z = \frac{4}{5}$, just inside the unit circle, pulls up the whole surface $H(z)$ and thus *shapes* $H\left(e^{j\omega}\right)$.
- You may have wondered why we need to have both a discrete-time Fourier transform (DTFT) and a z-transform.
- Here is one reason: a filter designer *designs* the poles and zeros of $H(z)$ to shape the frequency response $H\left(e^{j\omega}\right)$.

- The poles pull the surface $H(z)$ *up* towards ∞; the zeros pull the surface $H(z)$ *down* to zero.
- But if we only had a DTFT $H\left(e^{j\omega}\right)$, and *not* a z-transform $H(z)$, the designer could not use any poles to shape $H\left(e^{j\omega}\right)$…
    - because designing a pole directly into $H\left(e^{j\omega}\right)$ would make the frequency response fail to converge.
    - The filter would then be *unstable*.
    - So instead, we design the poles into $H(z)$, in the z-plane but off the unit circle.
    - ***Recall:*** for the filter to be both causal and stable, the poles must be placed strictly *inside* the unit circle of the z-plane.
    - There is no such restriction on the zeros – the designer is free to place zeros anywhere in the z-plane, including inside the unit circle, outside the unit circle, and even on the unit circle.

EX:

Let $X(z) = \dfrac{z-1}{(z-2)(z+3)}$

P-Z plot



- if $x[n]$ is right-sided, the ROC must be $|z|>3$. $X(e^{j\omega})$ does not exist in this case, since the ROC does not include the unit circle.

- if $x[n]$ is left-sided, the ROC must be $|z|<2$. $X(e^{j\omega})$ does exist in this case, since the ROC does include the unit circle.

- if $x[n]$ is two-sided, the ROC must be $2<|z|<3$. $X(e^{j\omega})$ does not exist in this case, since the ROC does not include the unit circle.

- In some applications, you may be interested only in causal signals $x[n]$ for which $x[n]=0$ $\forall n<0$ and causal systems H for which $h[n]=0$ $\forall n<0$.

- In such cases, you can use the unilateral z-transform

$$\mathcal{X}_u(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

- With the unilateral z-transform, you don't have to specify the ROC.

  → It's always the exterior of the circle that passes through the largest pole.

NOTE: the z-transform that is implemented in Matlab is the unilateral z-transform.

NOTE: the more general z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

is called the bilateral z-transform when there is a need to distinguish it from the unilateral z-transform.

- When you are asked to compute the z-transform $X(z) = \mathcal{Z}\{x[n]\}$,

→ Your first hope is to find $x[n]$ on a table of z-transforms like the one on the course web site or the one in Table 3.1 of the text.

→ If $x[n]$ is not in your table, then your second hope is that you can use z-transform properties to make the one you've got look like one that's in the table.

→ Tables of z-transform properties can be found on the course web site and in Table 3.2 of the text.

→ If both of the above fail, then you will have to try to calculate $X(z)$ from the definition.

NOTE: if $x[n] \in \ell^1(\mathbb{Z})$, then $X(e^{j\omega})$ exists.

- This means that the ROC of $X(z)$ includes the unit circle.

- This means that all of the transform pairs in your DTFT table can be converted into z-transform pairs by making the substitution $z = e^{j\omega}$.

Inversion

- Theoretically, $x[n]$ can be recovered from $X(z)$ by $x[n]r^{-n} = \text{DTFT}^{-1}\{X(re^{j\omega})\}$, which requires integrating around the circle of radius r (from $\omega = -\pi$ to π) for any r that is in the ROC of $X(z)$.
- But this involves solving a complex line integral in the complex plane, which can be involved.
- Your first hope is to find your $X(z)$ in a Table of z-transforms... then you just write down the answer.
- If the $X(z)$ you've got isn't in the table, then try to use z-transform properties to make the one you've got look like a combination of ones that are in the table.
- If both of the above fail, you can try to manipulate $X(z)$ into the form $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. Then you can just "pick off" the values of $x[n]$.

EX: $X(z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$,

all z except $z=0$ and $z=\infty$.

$$X(z) = (1-\tfrac{1}{2}z^{-1})(z+1)(z-1)$$
$$= (1-\tfrac{1}{2}z^{-1})(z^2-1)$$
$$= z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}$$
$$= \underbrace{1}_{x[-2]}z^2 \underbrace{-\tfrac{1}{2}}_{x[1]}z^1 \underbrace{-1}_{x[0]}z^0 + \underbrace{\tfrac{1}{2}}_{x[1]}z^{-1}$$

$$\Rightarrow x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

EX: $X(z) = \log(1+az^{-1})$, $|z| > |a|$.

Expanding in a Taylor series,

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n}{n} z^{-n}$$

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1 \\ 0, & n < 1. \end{cases}$$

- If all of the above fail and $X(z)$ is rational in z^{-1} (equivalent: rational in z), then you can try to invert by long division.
  - → See the text and/or the 3793 lecture notes for examples.

- If all of the above fail, you will have to try to invert from the definition by performing complex line integration,
  - → since there is no fundamental theorem of calculus for complex valued functions of a complex variable, this will generally involve an application of the <u>residue</u> <u>theorem</u> and/or one or more Cauchy integral theorems.

- Suppose that H is an LTI system that is both stable and causal.
  - $\rightarrow$ Since H is BIBO stable, $h[n] \in \ell^1(\mathbb{Z})$.
    - $\rightarrow$ This means $H(e^{j\omega})$ exists.
    - $\rightarrow$ This means the unit circle is in the ROC of $H(z)$.
  - $\rightarrow$ Since H is causal, $h[n] = 0 \;\; \forall n < 0$.
    - $\rightarrow$ This means $h[n]$ is right sided or finite support.
    - $\rightarrow$ This means the ROC of $X(z)$ is exterior or else is the whole z-plane (e.g., the exterior of the circle of radius zero).
- $\rightarrow$ So the ROC of $H(z)$ includes the unit circle and is exterior.

$\Rightarrow$ This means that all the poles have to be <u>inside</u> the unit circle for a causal, stable LTI system.