ECE 4213/5213

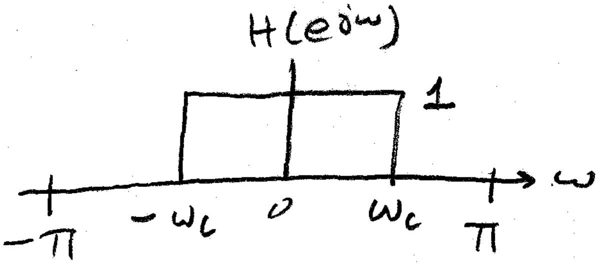
MODULE 7

Index

Chapter 7: LTI Discrete-Time Systems in the Transform Domain.

IDEAL FILTERS:

① Ideal Lowpass Filter:



Passband: $|\omega| \leq \omega_c$

Stopband: $\omega_c < |\omega| \leq \pi$

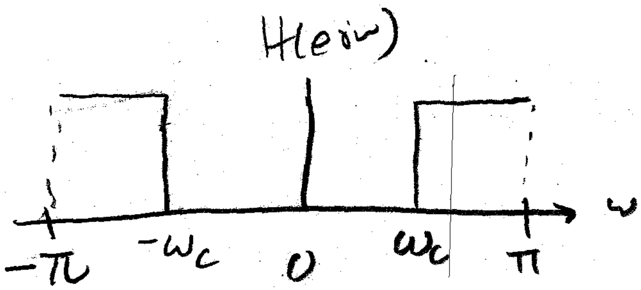
Impulse Response: $h[n] = \dfrac{\sin \omega_c n}{\pi n}$

→ This filter is not realizable.

→ Not causal, because $h[n] \neq 0$ for some $n < 0$.

→ Not BIBO stable, because $h[n]$ is <u>not</u> absolutely summable.

② Ideal Highpass Filter:



Passband: $|\omega| \geq \omega_c$

Stopband: $|\omega| < \omega_c$

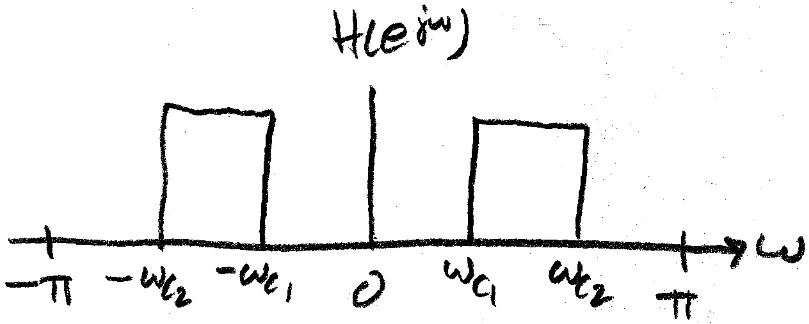
Impulse Response:

$$h[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}$$

→ Like the ideal lowpass filter, this filter is not realizable.

→ Not causal. → Not BIBO stable.

③ Ideal Bandpass Filter:



Passband: $\omega_{c1} \leq |\omega| \leq \omega_{c2}$

Stobbands: $|\omega| < \omega_{c1}$

$|\omega| > \omega_{c2}$

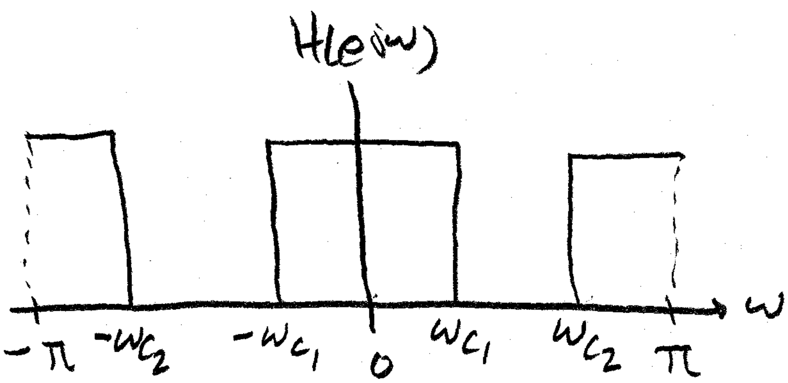
Impulse Response: $h[n] = \dfrac{\sin \omega_{c2} n}{\pi n} - \dfrac{\sin \omega_{c1} n}{\pi n}$

→ Not realizable

→ Not causal

→ Not BIBO stable

④ Ideal Bandstop Filter:



Stopband: $\omega_{c1} < |\omega| < \omega_{c2}$

passbands:

$|\omega| \leq \omega_{c1}$

$|\omega| \geqslant \omega_{c2}$

Impulse Response: $h[n] = \delta[n] + \dfrac{\sin \omega_{c1} n}{\pi n} - \dfrac{\sin \omega_{c2} n}{\pi n}$

→ Not realizable

→ Not causal

→ Not BIBO stable.

DEF: if H is a causal, stable discrete LTI system with a <u>real</u> impulse response $h[n]$ and if $|H(e^{j\omega})| \leq 1 \; \forall \omega$, then $H(z)$ is called a <u>bounded real transfer function</u>.

- This implies that $|Y(e^{j\omega})| \leq |X(e^{j\omega})| \; \forall \omega$.
- By Parseval's formula, this implies that

$$\sum_{n \in \mathbb{Z}} y^2[n] \leq \sum_{n \in \mathbb{Z}} x^2[n]$$

i.e., $\| y[n] \|_{\ell^2} \leq \| x[n] \|_{\ell^2}$.

→ Such a system is called <u>passive</u>.

→ Any BIBO stable LTI discrete-time system can be made passive by multiplying the impulse response by $\dfrac{1}{\max_{\omega} |H(e^{j\omega})|}$.

- If $|H(e^{j\omega})| = 1$, then H is called <u>lossless</u>.
  → This implies $|Y(e^{j\omega})| = |X(e^{j\omega})| \; \forall \omega$ and $\sum_{n \in \mathbb{Z}} x^2[n] = \sum_{n \in \mathbb{Z}} y^2[n]$.

ALLPASS SECTIONS

- Let $D_M(z)$ be an M^{th} order polynomial in z^{-1}:

$$D_M(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \ldots + d_{M-1} z^{-(M-1)} + d_M z^{-M}.$$

- Then

$$D_M(z^{-1}) = 1 + d_1 z + d_2 z^2 + \ldots + d_{M-1} z^{M-1} + d_M z^M$$

and $$z^{-M} D_M(z^{-1}) = z^{-M} + d_1 z^{-(M-1)} + d_2 z^{-(M-2)} + \ldots + d_{M-1} z^{-1} + d_M.$$

- If all the d_k are real, then $D_M(z)$ and $z^{-M} D_M(z^{-1})$ are called <u>mirror-image polynomials</u>... more on this in a minute.

- Let $$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}.$$

- Then $$A_M(z) A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \cdot \frac{z^{M} D_M(z)}{D_M(z^{-1})} = 1 \quad \forall z \in \mathbb{C}.$$

- If all the d_k are real, then

$$\begin{aligned} |A_M(e^{j\omega})|^2 &= A_M(e^{j\omega}) A_M^*(e^{j\omega}) \\ &= A_M(e^{j\omega}) A_M(e^{-j\omega}) \\ &= A_M(z) A_M(z^{-1}) \Big|_{z=e^{j\omega}} = 1 \quad \forall \omega \in \mathbb{R}. \end{aligned}$$

$\Rightarrow$ In other words, $|A_M(e^{j\omega})| = 1 \quad \forall \omega \in \mathbb{R}$.

$\rightarrow$ This is called an "allpass filter."

- All pass filters don't change the magnitude spectrum of the input.
- But they do change the phase.
- We will see later that it is difficult or impossible to design IIR filters with linear phase.
  - → An IIR filter with undesirable phase can be cascaded with an allpass filter to make the overall phase approximately linear in the passband.
  - → This is called "phase compensation" or "phase correction".
- Thus, allpass filters aren't terribly useful by themselves.
  - → But they are frequently used in series with other LTI systems.
  - → For this reason, they are often called "allpass sections".
- Henceforth, assume that the allpass coefficients $d_1, d_2, \ldots, d_M$ are real.
- The filter $A_M(z)$ is then called a "real-coefficient allpass section."

- The numerator and denominator of $A_M(z)$ can be factored to obtain

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)} = \pm \frac{d_M + d_{M-1} z^{-1} + \dots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \dots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

$$= \pm \prod_{k=1}^{M} \left(\frac{-\lambda_k^* + z^{-1}}{1 - \lambda_k z^{-1}} \right).$$

$\Rightarrow$ The poles and zeros of $A_M(z)$ occur in mirror-image pairs.

$\rightarrow$ If $\lambda_k = r_k e^{j\phi_k}$ is a pole (root of denominator), then there is a zero located at

$$-\lambda_k^* + z^{-1} = 0$$

$$-r_k e^{-j\phi_k} + z^{-1} = 0$$

$$z^{-1} = r_k e^{-j\phi_k}$$

$$z = \frac{1}{r_k} e^{j\phi_k} = \frac{1}{r_k e^{-j\phi_k}} = \frac{1}{\lambda_k^*}.$$

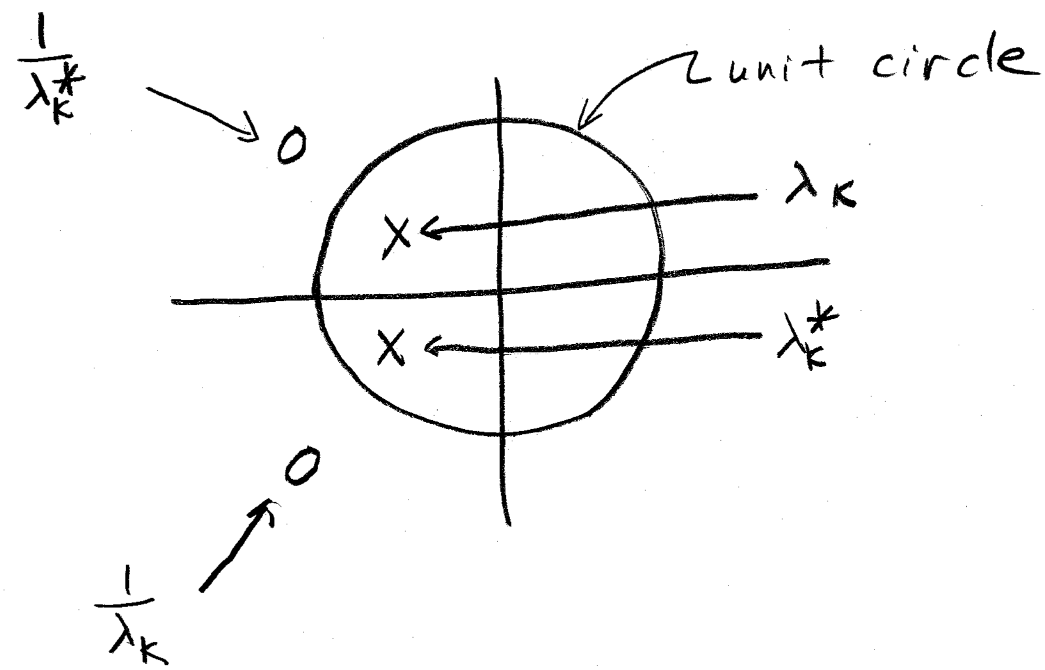
$\rightarrow$ In other words, having a pole at $\lambda_k = r_k e^{j\phi_k}$ implies that there is a zero at $\frac{1}{\lambda_k^*} = \frac{1}{r_k} e^{j\phi_k}$

→ Moreover, since the numerator and denominator of $A_M(z)$ are polynomials in z^{-1} with real coefficients,

→ All complex zeros occur in conjugate pairs

→ All complex poles occur in conjugate pairs.

- So, having a complex zero at $\lambda_k = r_k e^{j\phi_k}$ implies:
  - There is also a zero at $\lambda_k^* = r_k e^{-j\phi_k}$
  - There is a pole at $\frac{1}{\lambda_k^*} = \frac{1}{r_k} e^{j\phi_k}$
  - There is a pole at $\frac{1}{\lambda_k} = \frac{1}{r_k} e^{-j\phi_k}$

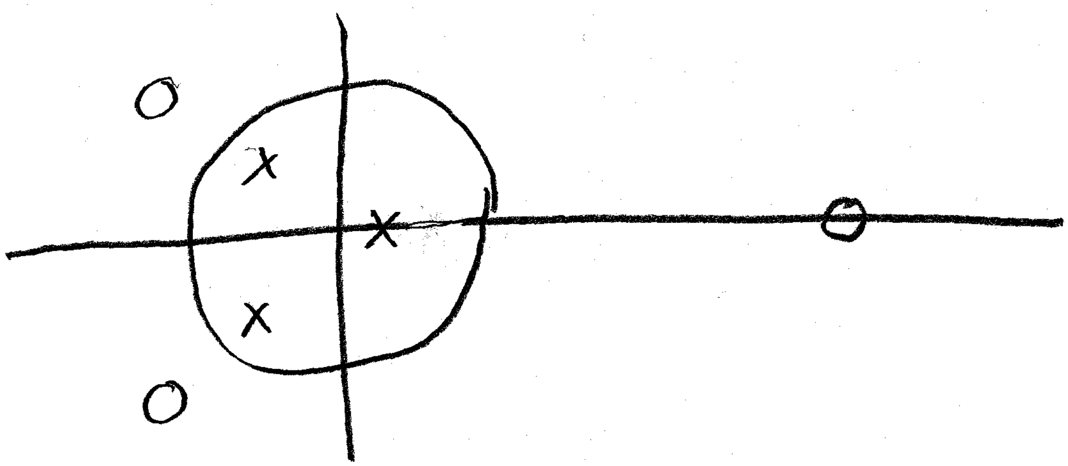


- If the allpass filter $A_m(z)$ is causal and stable, then all the poles must be <u>inside</u> the unit circle.

- Then all the zeros must be <u>outside</u> the unit circle.

- This implies that an allpass filter that is causal and stable <u>CANNOT</u> have a causal, stable inverse.

<u>EX</u>: $A_3(z) = \dfrac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$



- Because $|A_m(e^{j\omega})| = 1 \ \forall \omega$, an allpass section with real coefficients is a lossless structure (see the bottom of page 7.3).

- For a causal, stable allpass section $A_M(z)$,

$|A_M(z)| < 1$ outside the unit circle $(|z|>1)$.

$|A_M(z)| = 1$ on the unit circle $(|z|=1)$.

$|A_M(z)| > 1$ inside the unit circle $(|z|<1)$.

FACTS :

- The phase of a causal, stable real-coefficients allpass filter is a monotonically decreasing function of ω $\{$ for $|\omega| \le \pi$ ... it is 2π-periodic outside this interval $\}$.

$\rightarrow$ The phase derivative is negative.

$\rightarrow$ The group delay is positive.

$\Rightarrow$ The allpass filter adds delay.

- To prove these facts, it will be useful to look at a first-order allpass section.

- On page 7.6, we factored the M-th order allpass section $A_M(z)$ as

$$A_M(z) = \pm \prod_{k=1}^{M} \left(\frac{-\lambda_k^* + z^{-1}}{1 - \lambda_k z^{-1}} \right).$$

- This expresses the $M^{\underline{th}}$ order allpass section as a series connection of M first-order allpass sections.

- The overall magnitude response is the product of the magnitude responses of all the first-order sections.

- To see that they are all equal to 1 $\forall \omega$, consider:

1st order allpass section: $A_1(z) = \dfrac{-\lambda_k^* + z^{-1}}{1 - \lambda_k z^{-1}}$

$$|A_1(e^{j\omega})| = \left| \frac{-\lambda_k^* + e^{-j\omega}}{1 - \lambda_k e^{-j\omega}} \right| = \left| \frac{e^{-j\omega}(1 - \lambda_k^* e^{j\omega})}{1 - \lambda_k e^{-j\omega}} \right|$$

$$= \frac{|e^{-j\omega}| \cdot |1 - \lambda_k^* e^{j\omega}|}{|1 - \lambda_k e^{-j\omega}|} = \frac{|(1 - \lambda_k e^{-j\omega})^*|}{|1 - \lambda_k e^{-j\omega}|}$$

$= 1$, because the numerator and denominator are complex conjugates, so they must have the same magnitude.

$$\Rightarrow |A_1(e^{j\omega})| = 1 \quad \forall \omega$$

$$\Rightarrow |A_M(e^{j\omega})| = \prod_{k=1}^{M} 1 = 1 \quad \forall \omega.$$

- Now we will prove the FACTS on page 7.9.

- $A_M(e^{j\omega})$ is a product of 1st order allpass sections. They are all causal and stable.

- The phase $\arg A_M(e^{j\omega})$ is the sum of the phases of the first-order sections.

- The group delay of $A_M(e^{j\omega})$ is the sum of the group delays of the first-order sections.

→ We will show that any causal, stable 1st-order allpass section has:

  - monotonically decreasing phase
  - negative phase derivative
  - positive group delay

→ Any causal, stable first-order allpass section adds delay.

→ This will establish that the Mth-order causal, stable real-coefficients allpass filter $A_M(z)$ has monotonically decreasing phase, negative phase derivative, positive group delay, and <u>adds</u> <u>delay</u>.

- For the causal, stable first-order allpass section

$$A_1(z) = \frac{-\lambda_k^* + z^{-1}}{1 - \lambda_k z^{-1}}, \quad \text{let } \lambda_k = r_k e^{j\phi_k}.$$

- Because $A_1(z)$ is causal and stable, the pole is inside the unit circle. So $0 < r_k < 1$.

$$\text{phase} = \arg A_1(e^{j\omega}) = \arg\left[\frac{-\lambda_k^* + e^{-j\omega}}{1 - \lambda_k e^{-j\omega}}\right]$$

$$= \arg\left[\frac{e^{-j\omega}(1 - \lambda_k^* e^{j\omega})}{1 - \lambda_k e^{-j\omega}}\right]$$

$$= \arg\left[e^{-j\omega}(1 - \lambda_k^* e^{j\omega})\right] - \arg\left[1 - \lambda_k e^{-j\omega}\right]$$

$$= \arg\left[e^{-j\omega}\right] + \arg\left[1 - \lambda_k^* e^{j\omega}\right] - \arg\left[1 - \lambda_k e^{-j\omega}\right]$$

$$= -\omega + \arg\left[1 - r_k e^{-j\phi_k} e^{j\omega}\right] - \arg\left[1 - r_k e^{j\phi_k} e^{-j\omega}\right]$$

$$= -\omega + \arg\left[1 - r_k e^{j(\omega - \phi_k)}\right] - \arg\left[1 - r_k e^{-j(\omega - \phi_k)}\right]$$

$$= -\omega + \arg\left\{\left[1 - r_k\cos(\omega - \phi_k)\right] - j r_k \sin(\omega - \phi_k)\right\}$$
$$- \arg\left\{\left[1 - r_k\cos(\omega - \phi_k)\right] + j r_k \sin(\omega - \phi_k)\right\}$$

$$= -\omega + \arctan\left[\frac{-r_k \sin(\omega - \phi_k)}{1 - r_k\cos(\omega - \phi_k)}\right] - \arctan\left[\frac{r_k \sin(\omega - \phi_k)}{1 - r_k\cos(\omega - \phi_k)}\right]$$

$$= -\omega - 2\arctan\left[\frac{r_k \sin(\omega - \phi_k)}{1 - r_k\cos(\omega - \phi_k)}\right]$$

$\longrightarrow$

So, the group delay is

$$\tau_g(\omega) = -\frac{d}{d\omega}\left\{-\omega - 2\arctan\left[\frac{r_k \sin(\omega-\phi_k)}{1-r_k\cos(\omega-\phi_k)}\right]\right\}$$

$$= 1 + 2\frac{d}{d\omega}\arctan\left[\frac{r_k \sin(\omega-\phi_k)}{1-r_k\cos(\omega-\phi_k)}\right]$$

$$\longrightarrow \text{use } \frac{d}{d\omega}\arctan u = \frac{1}{1+u^2}\frac{d}{d\omega}u$$

$$= 1+2\frac{1}{1+\frac{r_k^2\sin^2(\omega-\phi_k)}{[1-r_k\cos(\omega-\phi_k)]^2}}\frac{d}{d\omega}\frac{r_k\sin(\omega-\phi_k)}{1-r_k\cos(\omega-\phi_k)}$$

$$= 1+2\frac{1}{1+\frac{r_k^2\sin^2(\omega-\phi_k)}{[1-r_k\cos(\omega-\phi_k)]^2}}\cdot\frac{[1-r_k\cos(\omega-\phi_k)]r_k\cos(\omega-\phi_k)-r_k^2\sin^2(\omega-\phi_k)}{[1-r_k\cos(\omega-\phi_k)]^2}$$

move this to be times the denominator of the second term

$$= 1+2\frac{1}{[1-r_k\cos(\omega-\phi_k)]^2+r_k^2\sin^2(\omega-\phi_k)}$$
$$\times\left\{[1-r_k\cos(\omega-\phi_k)]r_k\cos(\omega-\phi_k)-r_k^2\sin^2(\omega-\phi_k)\right\}$$

$$= 1+2\frac{r_k\cos(\omega-\phi_k)-r_k^2\cos^2(\omega-\phi_k)-r_k^2\sin^2(\omega-\phi_k)}{1-2r_k\cos(\omega-\phi_k)+r_k^2\cos^2(\omega-\phi_k)+r_k^2\sin^2(\omega-\phi_k)}$$

$$= 1+2\frac{r_k\cos(\omega-\phi_k)-r_k^2[\cos^2(\omega-\phi_k)+\sin^2(\omega-\phi_k)]}{1-2r_k\cos(\omega-\phi_k)+r_k^2[\cos^2(\omega-\phi_k)+\sin^2(\omega-\phi_k)]}$$
one
one

$$= 1+2\frac{r_k\cos(\omega-\phi_k)-r_k^2}{1-2r_k\cos(\omega-\phi_k)+r_k^2} \longrightarrow$$

Now combine the two terms over a common denominator:

$$T_g(\omega) = \underbrace{\frac{1-2r_k\cos(\omega-\phi_k)+r_k^2}{1-2r_k\cos(\omega-\phi_k)+r_k^2}}_{1} + \frac{2r_k\cos(\omega-\phi_k)-2r_k^2}{1-2r_k\cos(\omega-\phi_k)+r_k^2}$$

$$= \frac{1-2r_k\cos(\omega-\phi_k)+r_k^2+2r_k\cos(\omega-\phi_k)-2r_k^2}{1-2r_k\cos(\omega-\phi_k)+r_k^2}$$

$$= \frac{1-r_k^2}{1-2r_k\cos(\omega-\phi_k)+r_k^2}$$

$$= \frac{1-r_k^2}{1-2r_k\left[\frac{e^{j(\omega-\phi_k)}+e^{-j(\omega-\phi_k)}}{2}\right]+r_k^2}$$

$$= \frac{1-r_k^2}{1-r_k e^{j(\omega-\phi_k)}-r_k e^{-j(\omega-\phi_k)}+r_k^2}$$

$$= \frac{1-r_k^2}{[1-r_k e^{j(\omega-\phi_k)}][1-r_k e^{-j(\omega-\phi_k)}]}$$

$$= \frac{1-r_k^2}{|1-r_k e^{j(\omega-\phi_k)}|^2}$$

- The numerator is positive because $0 < r_k < 1$, so $0 < r_k^2 < 1$.
- The denominator is positive because it is a squared magnitude.

$$\Rightarrow \underline{\underline{T_g(\omega) > 0}}$$

QED.

Zero-Phase Transfer Function with Real Coefficients

- Suppose that H is an LTI discrete-time system with a real impulse response $h[n]$.
- Then the I/O relation for the system is a difference equation with real coefficients.
  - → The frequeny response $H(e^{j\omega})$ is a ratio of two real-coefficient polynomials in $e^{-j\omega}$.
  - → The transfer function $H(z)$ is a ratio of two real-coefficient polynomials in z^{-1}.
- Moreover, $H(e^{j\omega})$ is conjugate symmetric and

$$\mathrm{Re}\{H(e^{j\omega})\} \xleftrightarrow{DTFT} h_{ev}[n]$$

$$j\mathrm{Im}\{H(e^{j\omega})\} \xleftrightarrow{DTFT} h_{od}[n]$$

where $h[n] = h_{ev}[n] + h_{od}[n]$

$h_{ev}[n] = \frac{1}{2}\{h[n] + h[-n]\}$ is even

$h_{od}[n] = \frac{1}{2}\{h[n] - h[-n]\}$ is odd.

- Often, it is desirable for the filter to change the magnitude spectrum of the input, but not the phase.
- This requires $H(e^{j\omega})$ to have <u>zero phase</u>

- In other words, it requires $H(e^{j\omega})$ to be real. (and non-negative).

- But this means $h_{od}[n]$ must be zero.

→ In other words, $h[n]$ must be even.

⇒ Therefore, the only causal LTI system with a real impulse response and zero phase is the identity filter with $h[n]=\delta[n]$.

- Nontrivial real-valued filters with zero phase must be noncausal (because $h[n]$ must be even).

→ Therefore, these filters are only useful in non-realtime or "offline" applications.

- There is a trick for realizing offline filters with zero phase.

→ Suppose $H(z)$ is a filter with a frequency response $H(e^{j\omega})$ that is approximately one in the passband and zero in the stopband.

- If it is desirable to apply this filter to some signal $x[n]$,

→ Then almost the same result could be obtained by applying the filter <u>twice</u>.

- How the <u>trick</u> works:

① Apply the filter to $x[n]$. Call the output $v[n]$:

$$x[n] \longrightarrow \boxed{H} \longrightarrow v[n]$$

② Let $u[n] = v[-n]$ (time reversal).

③ Apply the filter again to $u[n]$, Call the output $w[n]$:

$$u[n] \longrightarrow \boxed{H} \longrightarrow w[n]$$

④ Let $y[n] = w[-n]$ (time reversal).

- Intuitively, applying the filter the second time to the "backwards" signal is like applying the "backwards" filter... any time delays introduced by the first filtering operation are offset by time advances in the second filtering operation.

- In other words, any phase introduced by the first filtering operation is offset by the second "backwards" filtering operation.

- More formally:

$$x[n] \rightarrow \boxed{H} \rightarrow v[n] \xmapsto{\text{Reverse}} u[n] = v[-n] \rightarrow \boxed{H} \rightarrow w[n]$$

$$\rightarrow \text{Reverse: } y[n] = w[-n].$$

- It is assumed that $x[n]$ and $h[n]$ are real.
  - → Then $v[n]$, $u[n]$, $w[n]$, and $y[n]$ are also real.
  - → So $X(e^{j\omega})$, $H(e^{j\omega})$, $V(e^{j\omega})$, $U(e^{j\omega})$, $W(e^{j\omega})$, and $Y(e^{j\omega})$ are all conjugate symmetric, so that, e.g., $X(e^{-j\omega}) = X^*(e^{j\omega})$.

- We have:

$$V(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$U(e^{j\omega}) = V(e^{-j\omega}) = V^*(e^{j\omega}) = X^*(e^{j\omega})H^*(e^{j\omega}).$$

$$W(e^{j\omega}) = U(e^{j\omega})H(e^{j\omega}) = X^*(e^{j\omega})H^*(e^{j\omega})H(e^{j\omega})$$

→

$$
\begin{aligned}
Y(e^{j\omega}) &= W(e^{-j\omega}) \\
&= W^*(e^{j\omega}) \\
&= X(e^{j\omega})H(e^{j\omega})H^*(e^{j\omega}) \\
&= X(e^{j\omega})\underbrace{|H(e^{j\omega})|^2}_{\text{real and non-negative.}}
\end{aligned}
$$

→ therefore, the overall filter has <u>zero phase</u>.

7.19

Minimum & Maximum Phase Systems

- often, a filter design spec will place requirements on the filter spectral magnitude, but not on the phase.
  - This leaves the designer some flexibility;

- We have argued that
  - ▷ zero phase is desirable in general, but not achievable in most practical scenarios
  - ▷ Linear phase is desirable in many cases.
    - Gives equal delay of the input frequency components.
    - Constant group delay
  - ▷ Generalized zero-phase:

$$e^{j0} = 1 \quad \text{and} \quad e^{j\pi} = -1$$

→ So, for example, a real-valued $H(e^{j\omega})$ that is negative for some ω cannot have a truly zero phase.

→ A filter with a spectral phase that only takes the values 0 and π is said to have a "generalized zero phase".

→ This implies that

→ $H(e^{j\omega}) \in \mathbb{R}$

→ where $H(e^{j\omega}) \geq 0$, $\theta(\omega) = 0$

→ where $H(e^{j\omega}) < 0$, $\theta(\omega) = \pi$

▷ Generalized linear phase:

- The idea is the same as "generalized zero phase".
- The unwrapped phase is piecewise linear, but can have jumps of π to accomodate "sign" flips in $H(e^{j\omega})$.

▷ When linear phase is desired, it's usually sufficient to design for it in the passband only... we usually don't care about the phase in the stopband.

Recall: we have seen that an Allpass section can be used to add phase without affecting the spectral magnitude of a filter.

→ The allpass section has a non-negative group delay for all frequencies $0 \le |\omega| \le \pi$.

→ It adds delay to the system.

Minimum Phase

- Let H be a causal LTI system with I/O relation that's a linear const. coefs. difference equation:

$$\sum_{k=0}^{K} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

→ This implies that the frequency response $H(e^{j\omega})$ is rational in $e^{j\omega}$.

→ This implies that the transfer function $H(z)$ is rational in z^{-1}.

- Let's restrict our attention to the case where the a_k and b_m are real.

→ This implies that

→ The impulse response $h[n]$ is real,

→ Poles are real or occur in conjugate pairs

→ Zeros are real or occur in conjugate pairs.

- Assume further that H is BIBO stable, so that it's both <u>causal</u> and <u>stable</u>.
  - → Implies that the poles of $H(z)$ lie <u>inside</u> the unit circle of the z-plane.
  - → Does not restrict the locations of the zeros.

<u>DEF</u>: $H(z)$ is called "<u>Minimum Phase</u>"

If all of the zeros are <u>inside</u> the unit circle.

→ This implies that a <u>causal</u> & <u>stable</u> inverse exists.

- Why do we call this "minimum phase"?

→ The big idea: Suppose $H_2(z)$ is a causal, stable, rational transfer function that has one or more zeros outside the unit circle.

→ $H_2(z)$ does not have minimum phase.

→ Then $\exists$ a causal, stable minimum phase transfer function $H_1(z)$ such that

$$H_2(z) = H_1(z)\,A(z),$$

→ where $A(z)$ is an allpass section.

→ The allpass section adds phase; it adds delay;

$\Rightarrow$ $H_1(z)$ and $H_2(z)$ have the same magnitude response, but H_2 has excess phase, i.e., excess delay, compared to H_1.

⇒ The minimum group delay,
minimum phase delay,
minimum delay of the input
frequency components,

⇒ Is achieved when $H(z)$ has all
zeros inside the unit circle.

This is called a "minimum phase" system.

→ For the same reason, if all the zeros
are <u>outside</u> the unit circle, then
$H(z)$ is called "Maximum Phase".

→ IF some zeros are inside the circle and
some are outside, the $H(z)$ is
called "Mixed Phase".

- How can we use this idea as a design engineer?
  - → Given a maximum phase or mixed phase $H_2(z)$,
    - → you can factor it into the product of a minimum phase $H_1(z)$ and an allpass function $A(z)$:

$$H_2(z) = H_1(z) A(z),$$

where

- → $H_1(z)$ has the same spectral magnitude as $H_2(z)$
- → But provides the minimum possible group delay out of all $H(z)$ with this same magnitude spectrum.

- How to do it:

  → Given an $H_2(z)$ with maximum or mixed phase,

  - Factor numerator & denominator into products of pole terms and zero terms.

    → For each real-valued zero outside the unit circle, reflect the zero inside the circle & write $H_2(z)$ as a product of the reflected zero and an allpass section.

    → For each conjugate pair of complex zeros, do the same thing for both.

- Here's how:
  - Suppose $H(z)$ has a real-valued zero at $z = \alpha$, where $|\alpha| > 1$.
    - Then $H(z)$ does not have minimum phase because this zero is outside the unit circle.
  - Let

$$H(z) = G(z)(1 - \alpha z^{-1})$$

$$= G(z)(1 - \alpha z^{-1}) \frac{z^{-1} - \alpha}{z^{-1} - \alpha}$$

$$= \underbrace{\left[G(z)(z^{-1} - \alpha)\right]}_{\text{This filter has the same spectral magnitude as } H(z)} \underbrace{\left[\frac{1 - \alpha z^{-1}}{z^{-1} - \alpha}\right]}_{\text{First-order All pass section}}$$

This filter has the same spectral magnitude as $H(z)$, i.e.,

$$\left| G(e^{j\omega})(e^{-j\omega} - \alpha) \right| = |H(e^{j\omega})|,$$

But the zero at $z = \alpha$ has been moved to $\frac{1}{\alpha}$, which is <u>inside</u> the unit circle.

- The same approach works for complex-valued zeros, which occur in conjugate pairs if $h[n]$ is real.

  - Suppose $H(z)$ has a zero at $z=\alpha$, where $\alpha \in \mathbb{C}$ and $|\alpha|>1$.

  - You can reflect this zero inside the unit circle to get a new filter with the same magnitude spectrum as H, but a reduced phase:

$$H(z) = G(z)(1-\alpha z^{-1})$$

$$= G(z)(1-\alpha z^{-1})\,\frac{z^{-1}-\alpha^*}{z^{-1}-\alpha^*}$$

$$= \underbrace{\left[G(z)(z^{-1}-\alpha^*)\right]}_{\text{A filter with the same magnitude as } H\text{, but the zero at } \alpha \text{ moved to } \frac{1}{\alpha^*}.}\;\underbrace{\left[\frac{1-\alpha z^{-1}}{z^{-1}-\alpha^*}\right]}_{\text{First order Allpass Section}}$$

- For any given $H(z)$ that has non-minimum phase (mixed phase or maximum phase),

  - You can repeat this procedure for each zero that is outside the unit circle.

  - This will give you a new filter having:

    1. The same magnitude spectrum as your original $H(z)$,

    2. Minimum phase.

- Why do this?

  → In some applications it may be important to minimize the group delay of the system.

  → In some applications, it may be important to ensure that the system has an inverse that is causal and stable.

7.3 Types of Linear Phase FIR TF's

- Assume $h[n]$ is finite length & Real.

→ CHAPTERS < 7 :

For a finite length signal $h[n]$, we speak of $h[n]$ being "length N" which implies it is defined for $0 \le n < N$...

in other words,

$$h[0], h[1], \ldots, h[\underline{\underline{N-1}}].$$

→ CHAPTERS ≥ 7 :

We now speak of "N" as the order of the filter.

So the rational transfer function will involve polynomials in z^{-1} of degree N :

$$a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

Notice that this polynomial has $\underline{\underline{N+1}}$ terms.

In fact, for an FIR filter,

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

→ If causal and degree N,

$$H(z) = h[0] + h[1]z^{-1} + \cdots + h[N] z^{-N}.$$

So $h[n]$ has length $\underline{\underline{N+1}}$ and is defined for $0 \le n \le N$.

Consider a FIR filter with real impulse response $h[n]$, $0 \le n \le N$.

Transfer Fcn: $H(z) = \sum_{n=0}^{N} h[n] z^{-n}$

Freq Resp: $H(e^{j\omega}) = \sum_{n=0}^{N} h[n] e^{-j\omega n} = H(z)\Big|_{\text{unit circle}}$

- The amplitude response of the filter, also known as the zero-phase response, is a real function $\breve{H}(\omega)$ such that:

  - $|\breve{H}(\omega)| = |H(e^{j\omega})|$
  - $\breve{H}(\omega) \in \mathbb{R} \quad \forall\, \omega \in \mathbb{R}$.

- This implies:

  1. $\breve{H}(\omega)$ has zero phase or generalized zero phase

     zero phase: $\theta(\omega) = 0 \;\; \forall \omega$

     generalized zero phase: $\theta(\omega) = \begin{cases} 0, & \breve{H}(\omega) \geq 0 \\ \pi, & \breve{H}(\omega) < 0 \end{cases}$

  2. $\breve{H}(\omega)$ is 2π-periodic (usually we will only pay attention to the fundamental period from $-\pi \leq \omega < \pi$).

  3. $\breve{H}(\omega)$ is either even or odd in ω.

     - Why? $h[n]$ is real, so $|H(e^{j\omega})|$ is even.

       $\rightarrow$ so $|\breve{H}(\omega)|$ is even.

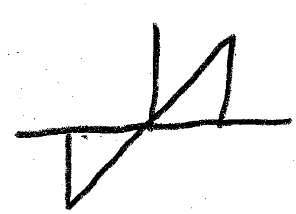
<u>EX</u>:

$|H(e^{j\omega})|$

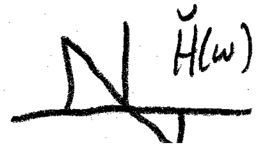
$\Rightarrow$

$\breve{H}(\omega)$

or $\breve{H}(\omega)$



or



$\breve{H}(\omega)$

What's the point?

- Suppose we have an $H(e^{j\omega})$ that's got true zero-phase.

- This would imply that $H(e^{j\omega}) \geq 0 \;\; \forall \omega \in \mathbb{R}$, because

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j \arg H(e^{j\omega})}$$

$$= |H(e^{j\omega})| \quad \text{if } \arg H(e^{j\omega}) = 0.$$

- That's not usually practical. Usually $H(e^{j\omega})$ will be positive for some ω and negative for others.

- So usually we speak of a "generalized zero phase" where

$$\arg H(e^{j\omega}) = \begin{cases} 0, & H(e^{j\omega}) \geq 0 \\ \pi, & H(e^{j\omega}) < 0. \end{cases}$$

→ The "π" accounts for the <u>sign</u> of $H(e^{j\omega})$ which cannot be accounted for in $|H(e^{j\omega})|$.

- Now suppose we have a linear phase FIR filter with $h[n] \in \mathbb{R}$.

- Then $H(e^{j\omega}) = |H(e^{j\omega})| e^{j \arg H(e^{j\omega})} = \breve{H}(\omega) e^{j(c\omega + \beta)}$ (7.42)

for some real constants $c, \beta \in \mathbb{R}$.

- Because $h[n]$ is <u>real</u>, $H(e^{j\omega})$ is conjugate symmetric, so $H(e^{j\omega}) = H^*(e^{-j\omega})$

- Plugging in (7.42),

$$\breve{H}(\omega) e^{j(c\omega+\beta)} = \breve{H}(-\omega) e^{-j(-c\omega+\beta)} \qquad (7.44)$$

- If $\breve{H}(\omega)$ is <u>even</u>, then (7.44) becomes

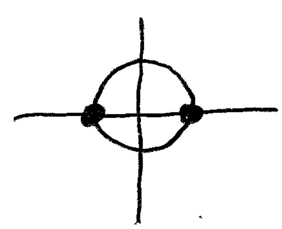
$$\breve{H}(\omega) e^{jc\omega} e^{j\beta} = \breve{H}(\omega) e^{jc\omega} e^{-j\beta}$$

$\Rightarrow e^{j\beta} = e^{-j\beta}$

$\Rightarrow \cos\beta + j\sin\beta = \cos\beta - j\sin\beta$

$\Rightarrow \sin\beta = 0$

$\Rightarrow \beta = 0$ <u>or</u> $\beta = \pi$



- if $\beta = 0$, then (7.42) becomes

$$H(e^{j\omega}) = \breve{H}(\omega) e^{j(c\omega + \underset{\substack{\uparrow \\ 0}}{\beta})} = \breve{H}(\omega) e^{jc\omega} = \underset{\substack{\uparrow \\ \text{(the definition of } H(e^{j\omega}))}}{\sum_{n=0}^{N} h[n] e^{-j\omega n}}$$

- In other words, $\breve{H}(\omega) e^{jc\omega} = \sum_{n=0}^{N} h[n] e^{-j\omega n}$

- Multiply both sides by $e^{-jc\omega}$:

$$\breve{H}(\omega) = \sum_{n=0}^{N} h[n] e^{-j\omega n} e^{-jc\omega} = \sum_{n=0}^{N} h[n] e^{-j\omega(c+n)} \quad (*)$$

- - - - -

- if $\beta = \pi$, then (7.42) instead becomes

$$H(e^{j\omega}) = \breve{H}(\omega) e^{j(c\omega + \pi)} = \breve{H}(\omega) e^{jc\omega} \underbrace{e^{j\pi}}_{-1}$$

$$= -\breve{H}(\omega) e^{jc\omega} = \sum_{n=0}^{N} h[n] e^{-j\omega n}$$

- In other words, $\breve{H}(\omega) e^{jc\omega} = -\sum_{n=0}^{N} h[n] e^{-j\omega n}$

- Multiply both sides by $e^{-jc\omega}$:

$$\breve{H}(\omega) = -\sum_{n=0}^{N} h[n] e^{-j\omega n} e^{-jc\omega} = -\sum_{n=0}^{N} h[n] e^{-j\omega(c+n)} \quad (**)$$

$\Rightarrow$ The Book writes $(*)$ and $(**)$ together as:

$$\breve{H}(\omega) = \pm \sum_{n=0}^{N} h[n] e^{-j\omega(c+n)} \quad (7.45)$$

"+" if $\beta = 0$

"−" if $\beta = \pi$

- From the last page, we have

$$\breve{H}(\omega) = \pm \sum_{n=0}^{N} h[n] e^{-j\omega(c+n)} \qquad (7.45)$$

$\Rightarrow$ Evaluate this at $-\omega$ and write "ℓ" instead of "n":

$$\breve{H}(-\omega) = \pm \sum_{\ell=0}^{N} h[\ell] e^{-j(-\omega)(c+\ell)} = \pm \sum_{\ell=0}^{N} h[\ell] e^{j\omega(c+\ell)} \qquad (7.46)$$

$\Rightarrow$ Now let $\ell = N-n$ and change the sum variable to n in (7.46). This reverses the order in which the terms $h[n]$ / $h[\ell]$ get added:

$$\breve{H}(-\omega) = (7.46) = \pm \sum_{n=0}^{N} h[N-n] e^{j\omega(c+N-n)} \qquad (7.47)$$

$\Rightarrow$ Since we assumed $\breve{H}(\omega)$ is <u>even</u> back on p. 7.35, we must have $\breve{H}(\omega) = \breve{H}(-\omega)$. In other words, (7.45) = (7.47).

- If $\underline{\underline{c = -\frac{N}{2}}}$, then

$$(7.45): \ \breve{H}(\omega) = \pm \sum_{n=0}^{N} h[n] e^{-j\omega(-\frac{N}{2}+n)} = \pm \sum_{n=0}^{N} h[n] e^{j\omega(\frac{N}{2}+n)}$$

$$(7.47): \ \breve{H}(-\omega) = \pm \sum_{n=0}^{N} h[N-n] e^{j\omega(-\frac{N}{2}+N+n)} = \pm \sum_{n=0}^{N} h[N-n] e^{j\omega(\frac{N}{2}+n)}$$

- These <u>must</u> be the same!!

$\Rightarrow$ Therefore, if H is an FIR filter, $h[n]$ is real, and $\breve{H}(\omega)$ is even, the phase will be <u>linear</u> if

$$h[n] = h[N-n], \quad 0 \le n \le N \qquad (7.48)$$

So, an N^{th}-order FIR filter with $h[n]$ real and $\breve{H}(\omega)$ even will have linear phase if $h[n]$ is even symmetric about the point $\frac{N}{2}$. ☆☆☆

$\Rightarrow$ A completely similar argument shows that, if $\breve{H}(\omega)$ is odd, then the filter will have linear phase if $h[n]$ is odd symmetric about the point $\frac{N}{2}$, i.e., if

$$h[n] = -h[N-n] \, , \quad 0 \leq n \leq N \qquad (7.52)$$

☆☆☆

- Summary:

$\left\{\right.$ For a real-valued FIR filter, the phase is linear if $h[n]$ is symmetric or if $h[n]$ is antisymmetric.

[redacted]

This leads to four types of linear phase FIR filters:

$\breve{H}(\omega)$ even $\rightarrow$ $h[n] = h[N-n]$

- N even   Type I
- N odd   Type II

$\breve{H}(\omega)$ odd $\rightarrow$ $h[n] = -h[N-n]$

- N even   Type III
- N odd   Type IV

In all 4 types, the phase is

$c\omega + \beta$ with $c = -\frac{N}{2}$

Type I, II : $\beta = 0$ or $\beta = \pi$

Type III, IV : $\beta = \pi/2$ or $\beta = -\pi/2$

- We have just considered linear phase FIR filters of Type I, II, III, and IV that

  - are <u>causal</u>
  - have an impulse response $h[n]$ of length $N+1$.
  - have a transfer function $H(z)$ that's an N^{th} order polynomial in z^{-1}.

    $\rightarrow$ These are referred to as "N^{th} order filters."

<u>NOTE</u>: Any FIR filter is an <u>all</u> <u>zero</u> filter.

- $$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- if the filter is causal, then $h[n] = 0 \;\forall n < 0$,

  so $$H(z) = h[0] + h[1]z^{-1} + \dots + h[N]z^{-N}.$$

  $\Rightarrow$ There cannot be any poles in the finite z-plane except at $z=0$.

<u>FACT</u>: Any Type I causal linear phase FIR filter can be transformed into a zero-phase filter that is noncausal (this isn't true for the other filter types).

Recall: For true zero phase, $H(e^{j\omega})$ must be real and non-negative.

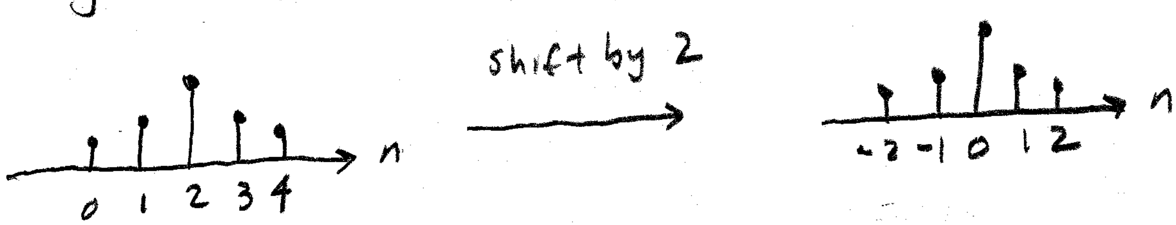
For generalized zero phase, $H(e^{j\omega})$ must be real.

→ In both cases, $H(e^{j\omega})$ real implies that $h[n]$ is even (assuming $h[n] \in \mathbb{R}$).

→ For a Type I filter, N is even and the length of $h[n]$ is odd.

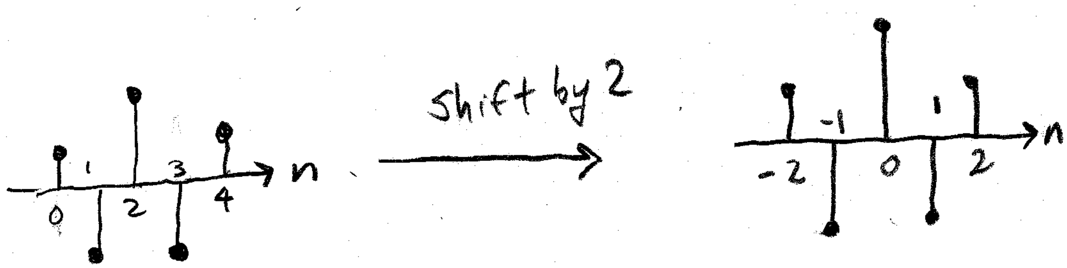
→ So $h[n]$ can be shifted left by $\frac{N}{2}$ samples to make it real and even, resulting in a filter that is not causal but has zero phase or generalized zero phase.

EX:



$N=4$

EX:



$N=4$

NOTE : The book incorrectly states that this can also be done for a Type III filter... it can't.

- For a Type III filter, the shifted impulse response would be real and odd, implying $H(e^{j\omega})$ would be pure imaginary and odd.
- This leads to a phase of

$$\theta(\omega) = \begin{cases} \pi/2, & H(e^{j\omega}) \geq 0 \\ 3\pi/2, & H(e^{j\omega}) < 0 \end{cases}$$

which has zero group delay but does not qualify as zero phase or generalized zero phase.

Locations of the Zeros of the Type I, II, III, and IV Linear Phase FIR Filters:

- For a Type I or Type II filter, $h[n] = h[N-n]$.

  $\rightarrow$ So $H(z) = \sum_{n=0}^{N} h[n] z^{-n} = \sum_{n=0}^{N} h[N-n] z^{-n}$

$$(m = N-n) \qquad = \sum_{m=0}^{N} h[m] z^{m-N}$$

$$= z^{-N} \sum_{m=0}^{N} h[m] z^{m} = z^{-N} H(z^{-1}). \qquad (7.66)$$

- For a Type III or Type IV filter, $h[n] = -h[N-n]$,

  so $H(z) = \sum_{n=0}^{N} h[n] z^{-n} = -\sum_{n=0}^{N} h[N-n] z^{-n}$

$$(m = N-n) \qquad = -\sum_{m=0}^{N} h[m] z^{m-N}$$

$$= -z^{-N} \sum_{m=0}^{N} h[m] z^{m} = -z^{-N} H(z^{-1}). \qquad (7.67)$$

- Polynomials with the symmetry of (7.66),

$$H(z) = z^{-N} H(z^{-1})$$

  are called "mirror image polynomials."

- Those with the symmetry of (7.67), $H(z) = -z^{-N} H(z^{-1})$, are called "anti mirror image polynomials."

- In either case, having a zero at $z = z_0$ implies there is also a zero at $z = \frac{1}{z_0} = z_0^{-1}$.

Mirror Image: $H(z_0) = 0 = z^{-N} H(z_0^{-1})$

$$\Rightarrow H(z_0^{-1}) = 0$$

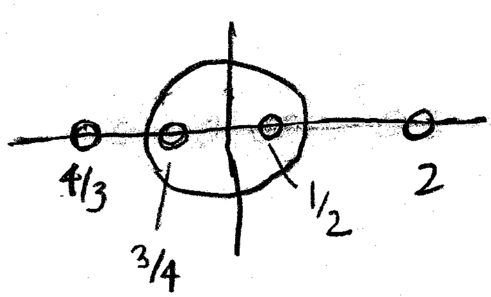
Anti Mirror Image: $H(z_0) = 0 = -z^{-N} H(z_0^{-1})$

$$\Rightarrow H(z_0^{-1}) = 0$$

- So any Type I, II, III, or IV linear phase FIR filter that has a zero at z_0 also has one at $\frac{1}{z_0}$.

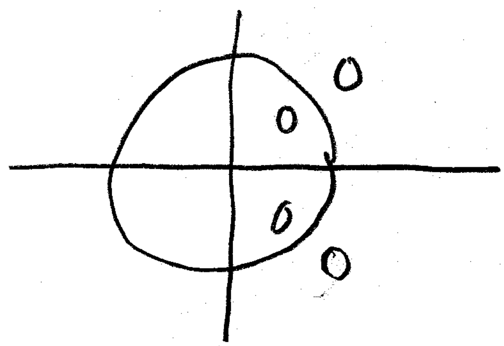
- Moreover, assuming $h[n]$ is real, any complex zeros occur in conjugate pairs.

- So: for any of the four filter types,
  - → Real zeros generally occur in pairs that are reflected through the unit circle.
    - → Having a real zero at $z = \rho$ means there is also one at $z = \frac{1}{\rho}$

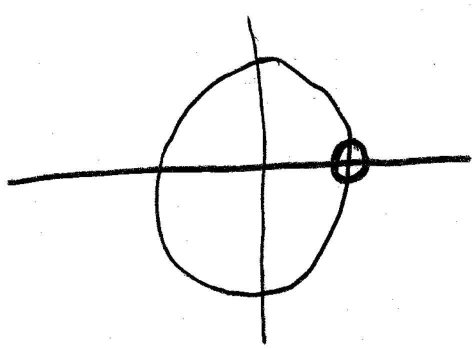


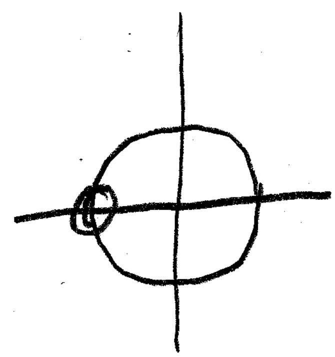
7.44

→ Complex zeros generally occur in groups of four having both conjugate symmetry and mirror image symmetry with respect to the unit circle.

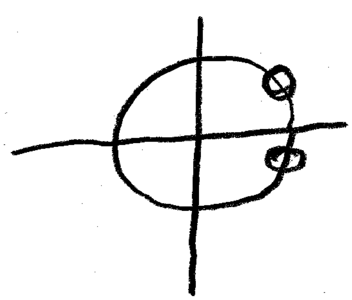


→ Single real zeros can also occur on the unit circle at $z=1$ and $z=-1$, since $\frac{1}{1}=1$ and $\frac{1}{-1}=-1$.



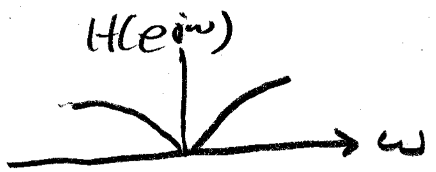


→ Complex zeros can also occur in pairs on the unit circle, since z and z^{-1} both have unit magnitude in this case.



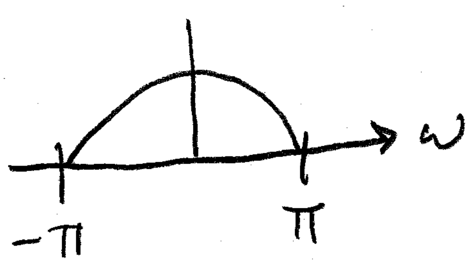
- An important difference between the four filter types has to do with their zeros at $z=1$ and $z=-1$.

- Since $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$ and $1 = e^{j0}$, having a zero in $H(z)$ at $z=1$ means having a zero in $H(e^{j\omega})$ at $\omega = 0$.



$\Rightarrow$ A filter with a zero at $z=1$ can't be a low pass filter or a band stop filter.

- Since $-1 = e^{j\pi} = e^{-j\pi}$, having a zero in $H(z)$ at $z=-1$ means having a zero in $H(e^{j\omega})$ at $\omega = \pm\pi$.



$\Rightarrow$ A filter with a zero at $z=-1$ can't be a high pass filter or a band stop filter.

- A Type I filter must have the symmetry of (7.66) on page 7.43:

$$H(z) = z^{-N} H(z^{-1}).$$

  - For $z=1$, this becomes $H(1) = H(1)$, which does not impose any restrictions.
  - For $z=-1$, we also have N even in a Type I filter, so this becomes

$$H(-1) = -1^{N} H(-1) = H(-1),$$

    which again imposes no restrictions.

$\Rightarrow$ Type I filters are not required to have any zeros at $z=1$ $(\omega=0)$ or $z=-1$ $(\omega=\pm\pi)$.

$\Rightarrow$ So Type I filters can be high pass, low pass, band pass, or band stop.

- For a Type II filter, it is also required that $H(z) = z^{-N} H(z^{-1})$.

- But in this case N is odd.

  - For $z=1$, we have $H(1) = H(1)$, which imposes no restrictions.

  - For $z=-1$, we have $H(-1) = -H(-1)$

    $\Rightarrow$ This means $H(-1) = 0$.

    $\Rightarrow$ A Type II filter must have a zero at $z=-1$ $(\omega = \pm\pi)$.

$\Rightarrow$ So Type II filters cannot be high pass and cannot be band stop.

- A Type III filter must have the symmetry of (7.67) on page 7.43:

$$H(z) = -z^{-N} H(z^{-1}).$$

- For a Type III filter, N is even.

- At $z=1$, this becomes $H(1) = -H(1)$

  $\Rightarrow$ This means $H(1) = 0$.

  $\Rightarrow$ A Type III filter must have a zero at $z=1$ ($\omega=0$).

- At $z=-1$, this becomes $H(-1) = -H(-1)$

  $\Rightarrow$ This means $H(-1) = 0$

  $\Rightarrow$ A Type III filter must also have a zero at $z=-1$ ($\omega = \pm\pi$).

$\Rightarrow$ So Type III filters cannot be high pass, low pass, or band stop. They can only be band pass.

- For a Type IV filter, we must have

$$H(z) = -z^{-N} H(z^{-1})$$

→ N is odd for a Type IV filter.

- At $z=1$, we get $H(1) = -H(-1)$

$\Rightarrow H(1) = 0$

$\Rightarrow$ Any Type IV filter has a zero at $z=1$ $(\omega = 0)$.

- At $z=-1$, we get $H(-1) = H(-1)$, which does not impose any restrictions.

$\Rightarrow$ So Type IV filters cannot be low pass and cannot be band stop.

Summary:

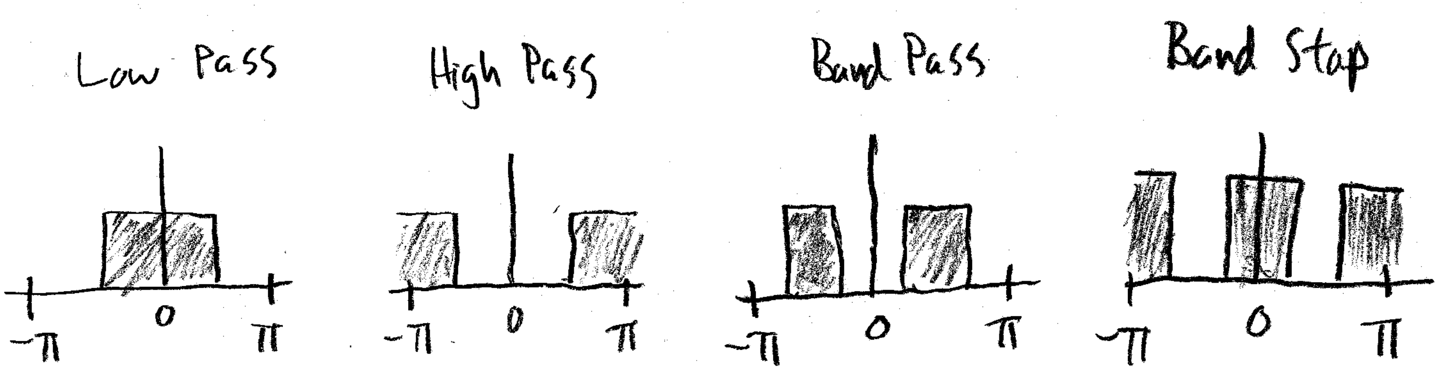
	High Pass	Low Pass	Band Pass	Band Stop
Type I	✓	✓	✓	✓
Type II	X	✓	✓	X
Type III	X	X	✓	X
Type IV	✓	X	✓	X

Type I: Not required to have any zeros at $z=1$ $(\omega=0)$ or $z=-1$ $(\omega=\pm\pi)$

Type II: Must have a zero at $z=-1$ $(\omega=\pm\pi)$

Type III: Must have a zero at $z=1$ $(\omega=0)$ and at $z=-1$ $(\omega=\pm\pi)$

Type IV: Must have a zero at $z=1$ $(\omega=0)$



Simple Digital Filters

provide low: complexity, cost, performance.

Lowpass FIR

- 2-point causal averager:

$$h[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]$$

$$H(z) = \tfrac{1}{2}(1+z^{-1}) = \frac{z+1}{2z}$$

$\rightarrow$ one zero @ $z=-1 \Rightarrow \omega = \pi$

$\rightarrow$ one pole @ $z=0$

$$H(e^{j\omega}) = \tfrac{1}{2}(1+e^{-j\omega})$$

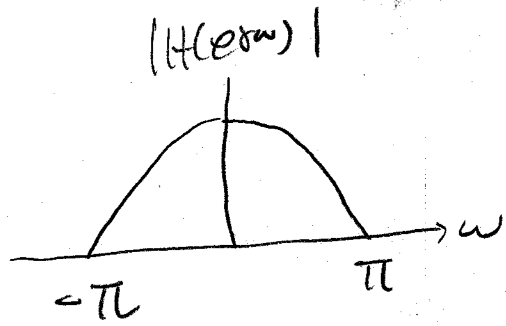
$$= \tfrac{1}{2}e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})$$

$$= \tfrac{1}{2}e^{-j\omega/2} \cdot 2\cos\left(\tfrac{\omega}{2}\right)$$

$$= e^{-j\omega/2}\cos\left(\tfrac{\omega}{2}\right)$$

$$|H(e^{j\omega})| = \cos\left(\tfrac{\omega}{2}\right)$$

$$\theta(\omega) = -\frac{\omega}{2} \qquad \tau_g(\omega) = \frac{1}{2}$$



Gain function: $\mathcal{G}(\omega) = 20\log_{10}|H(e^{j\omega})|$

3dB cutoff frequency ω_c:

$$\mathcal{G}(\omega_c) = -3\text{dB} \Rightarrow |H(e^{j\omega_c})| = \frac{1}{\sqrt{2}}|H(e^{j0})|$$

Solve ω_c:

$$|H(e^{j\omega_c})|^2 = \frac{1}{2} = \cos^2\left(\frac{\omega_c}{2}\right)$$

$$\frac{\omega_c}{2} = \arccos\frac{1}{\sqrt{2}} \approx .7854$$

$$\omega_c \approx 1.5707963$$

$$\underline{\underline{\omega_c = \pi/2}}$$

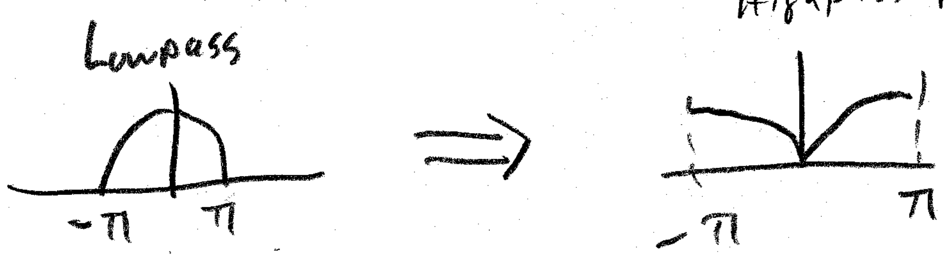
- Filters of this type can be cascaded to lower the 3dB cutoff freq.;
  - Makes the transition sharper
  - Narrows the passband.

7.53

Highpass FIR

- multiply the lowpass impulse response by $(-1)^n = e^{j\pi n}$

→ Shifts $H(e^{j\omega})$ by π, making a highpass filter:



$$h[n] = \tfrac{1}{2}\delta[n] - \tfrac{1}{2}\delta[n-1]$$

$$H(z) = \tfrac{1}{2}(1 - z^{-1}) = \frac{z-1}{2z}$$

- one zero at $z = 1 \Rightarrow \omega = 0$

- one pole at $z = 0$.

$$H(e^{j\omega}) = \tfrac{1}{2}(1 - e^{-j\omega}) = \frac{j}{j} \cdot \frac{1}{2} \cdot e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$= j e^{-j\omega/2}\left[\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}\right]$$

$$= j e^{-j\omega/2} \sin\left(\tfrac{\omega}{2}\right)$$

$$|H(e^{j\omega})| = \left|\sin\tfrac{\omega}{2}\right|$$

NOTE: This has generalized linear phase.

→ For FIR filters, the poles occur only at the origin $z=0$

→ They have the same effect on all points on the unit circle

i.e., all points on the circle are equidistant from the poles.

→ So the shape of $H(e^{j\omega})$ is determined by the locations of the zeros <u>only</u>.

→ But for IIR Filters, the poles and zeros can occur anywhere in the z-plane,

→ So for an IIR filter, the poles and the zeros can be used to shape $|H(e^{j\omega})|$.

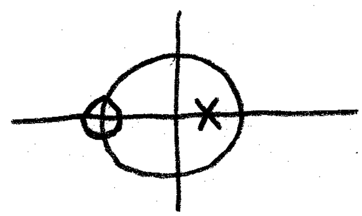
Lowpass IIR

$$H(z) = \left(\frac{1-\alpha}{2}\right) \frac{1+z^{-1}}{1-\alpha z^{-1}}, \quad \alpha \in \mathbb{R}, \quad |\alpha|<1$$

one zero @ $z=-1 \Rightarrow \omega = \pi$

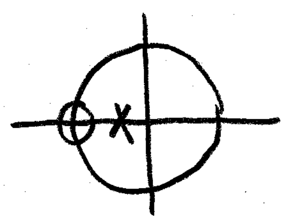
one pole @ $z=\alpha$

$0<\alpha<1$: pole is on the positive side of the real axis:



This tends to have a narrower passband.

$-1<\alpha<0$: pole is on the negative side:



This tends to have a wider passband.

$$|H(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

3dB cutoff freq: $\omega_c = \arccos\left[\frac{2\alpha}{1+\alpha^2}\right]$

Given a design spec for ω_c, $\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$

Highpass IIR

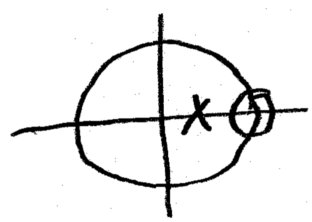
$$H(z) = \left(\frac{1+\alpha}{2}\right) \frac{1-z^{-1}}{1-\alpha z^{-1}} \quad \begin{cases} \alpha \in \mathbb{R} \\ |\alpha| < 1 \end{cases}$$

one zero @ $z = 1 \Rightarrow \omega = 0$

one pole @ $z = \alpha$

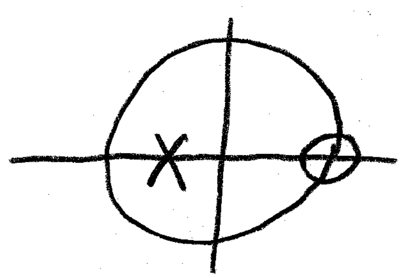
$0 < \alpha < 1$: pole is on positive real axis

$\rightarrow$ wide passband



$-1 < \alpha < 0$: pole is on negative real axis

$\rightarrow$ narrow passband



IIR Bandpass

- A bandpass filter needs to be at least 2nd order.

$\Rightarrow$ Think of it this way:

  - a first order filter can be lowpass or highpass.
  - by cascading a lowpass with a highpass, you get a bandpass characteristic
  - But the order is at least 2.

$$H(z) = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$\alpha, \beta \in \mathbb{R}$
$|\alpha| < 1$
$|\beta| < 1$

Two zeros: $z = 1 \longrightarrow \omega = 0$
$z = -1 \longrightarrow \omega = \pi$

complex conjugate pole pair at $z = re^{\pm j\phi}$

$$r = \sqrt{\alpha}$$
$$\phi = \arccos\left[\frac{\beta(1+\alpha)}{2\sqrt{\alpha}}\right]$$

passband center frequency: $\omega_0 = \arccos(\beta)$

3dB bandwidth: $B_w = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right)$; Quality: $Q = \frac{\omega_0}{B_w}$

Simple IIR Bandstop (notch filter)

$$H(z) = \frac{1+\alpha}{2} \; \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$$\alpha, \beta \in \mathbb{R}, \quad |\alpha| < 1, \quad |\beta| < 1$$

$\omega_0 = \arccos\beta$: "notch frequency"
= stop band center frequency

two zeros at $\omega = \omega_0$.

Notch bandwidth: $B_w = \arccos\left(\frac{2\alpha}{1+\alpha^2}\right)$

Quality; $Q = \frac{\omega_0}{B_w}$

Comb Filters: Read the Book.

Delay Complementary Transfer Fcns:

- The set of filters $H_0(z), H_1(z), \ldots, H_{L-1}(z)$ is <u>delay complementary</u> if

$$H_0(z) + H_1(z) + \cdots + H_{L-1}(z) = \gamma z^{-n_0}$$

where $\gamma \neq 0$ is a constant and $n_0 > 0$ is an integer.

$$\Rightarrow \gamma z^{-n_0} \overset{Z}{\longleftrightarrow} \gamma \delta[n-n_0]$$

in other words, the delay complementary set sums to a pure delayer.

- The frequency response of the sum is

$$\gamma z^{-n_0}\Big|_{z=e^{j\omega}} = \gamma e^{-jn_0\omega}$$

- So the spectral magnitude of the sum is

$$|\gamma e^{-jn_0\omega}| = |\gamma| \text{ --- a constant.}$$

$\longrightarrow$ Means that the stopband of one filter must be covered by passbands of the other filters.

- For example,

  - given a low pass filter, a highpass filter can be designed so that the pair is delay complementary
  - given a bandpass filter, a bandstop filter can be designed so that the pair is delay complementary.

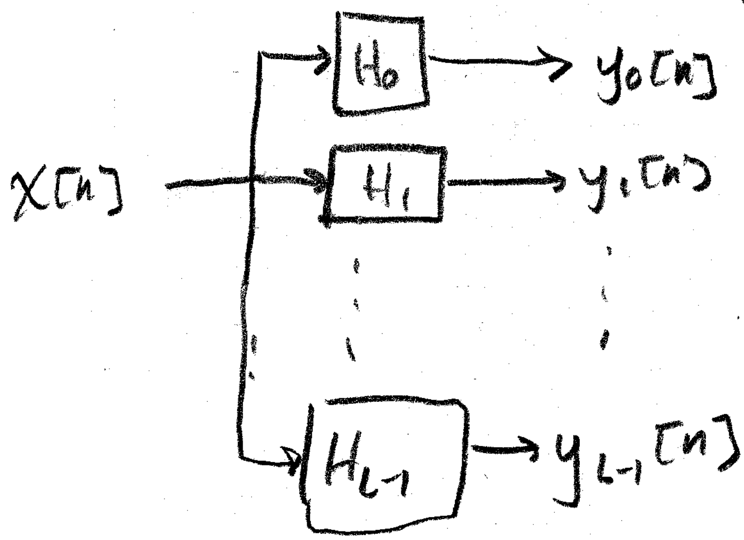
Allpass complementary:

$$H_0(z) + H_1(z) + \cdots + H_{L-1}(z) = \underbrace{A(z)}_{\text{Allpass section.}}$$

- Like the delay complementary set, the magnitude spectrum of the sum is constant.

→ implies that the stop band of one filter is filled in with the passbands of the other filters.

- often delay and allpass complementary sets are used to make parallel filterbanks that split the input signal into "channels" or "bands" that are useful in some way:



- For a delay complementary set, the overall parallel filterbank structure has linear phase; could be used, e.g., for a digital audio crossover network (to separate the bass, midrange, treble into separate channels).

- For an allpass complementary set, the phase of the overall filterbank structure can have arbitrary phase.

Power complementary

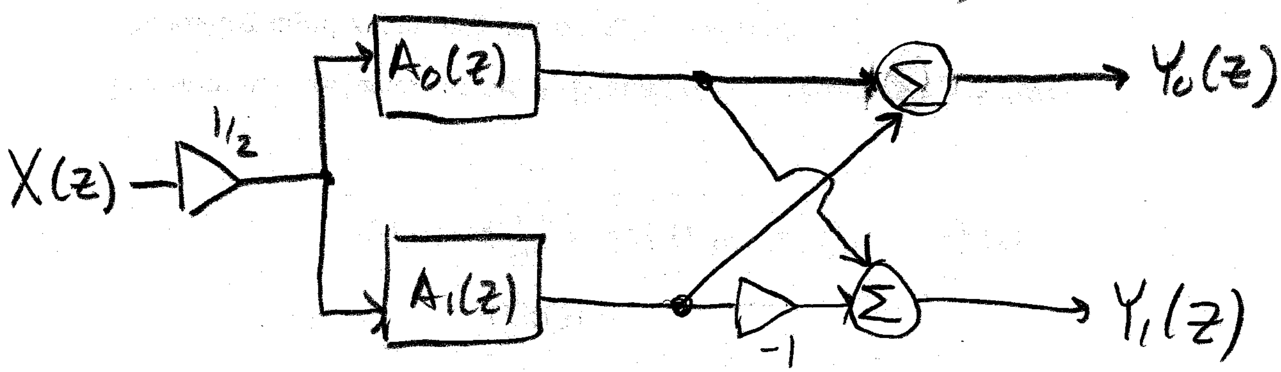
$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 + \cdots + |H_{L-1}(e^{j\omega})|^2 = \text{constant}$$

$\Rightarrow$ A set that is both allpass complementary and power complementary is called "doubly complementary"!

EX: let $A_0(z)$ and $A_1(z)$ be stable allpass sections. A doubly complementary pair can be implemented easily as

$$H_0(z) = \tfrac{1}{2}A_0(z) + \tfrac{1}{2}A_1(z)$$

$$H_1(z) = \tfrac{1}{2}A_0(z) - \tfrac{1}{2}A_1(z)$$



Deconvolution

For an LTI system H with input $x[n]$ and output $y[n]$,

$$x[n] \rightarrow \boxed{H} \rightarrow y[n]$$

→ $y[n] = x[n] * h[n]$

$$Y(z) = X(z)\,H(z)$$

$$X(z) = \frac{Y(z)}{H(z)}$$

→ solving for $x[n]$ given $h[n]$ and $y[n]$ is called "deconvolution."

→ This does <u>not</u> have a general closed form solution in the time domain.

→ However, an <u>iterative</u> time domain solution is possible if all of the following conditions hold:

- → H is causal
- → $h[0] \neq 0$
- → It is known that $x[n] = 0 \quad \forall\, n < 0$.

→ Causality then implies that:

$$y[n] = 0 \quad \forall\, n < 0.$$

- Under those conditions, we have for $n \geq 0$ that

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] \qquad (*)$$

→ sum starts at $k=0$ b/c $x[k]=0$, $k<0$.

→ Sum stops at $k=n$ b/c system is causal... $y[n]$ can't depend on $x[n+1]$.

- For $n=0$, $(*)$ becomes

$$y[0] = x[0]h[0]$$

$$\Rightarrow x[0] = \frac{y[0]}{h[0]}$$

- For $n=1$, we have

$$y[1] = x[0]h[1] + x[1]h[0]$$

$$x[1]h[0] = y[1] - x[0]h[1]$$

$$x[1] = \frac{y[1] - x[0]h[1]}{h[0]}$$

→

For $n=2$, we have

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$$

$$x[2] = \frac{y[2] - x[0]h[2] - x[1]h[1]}{h[0]}$$

And in general:

$$x[n] = \frac{y[n] - \sum_{k=0}^{n-1} x[k]h[n-k]}{h[0]}, \quad n \geq 1$$

System Identification

$$x[n] \longrightarrow \boxed{\begin{matrix} H \\ \text{LTI} \end{matrix}} \longrightarrow y[n]$$

- when the input $x[n]$ and output $y[n]$ are <u>known</u> and the problem is to find $h[n]$,
  it's called "system identification!"

We have:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

- As with deconvolution (and convolution), there is no general time-domain solution unless additional constraints can be imposed.

- Suppose:
  - → H is causal
  - → $x[0] \neq 0$
  - → $x[n] = 0, \quad n < 0$

→ causality then implies $y[n] = 0 \ \forall n < 0$.

$$y[n] = \sum_{k=0}^{n} h[k]\, x[n-k]$$

→ Sum starts at zero b/c system is causal → $h[k] = 0 \ \forall \ k < 0$.

→ Sum stops at $k = n$ b/c $x[k] = 0$, $k < 0$.

<u>n=0</u>

$$y[0] = h[0]\, x[0]$$

$$\Rightarrow \quad h[0] = \frac{y[0]}{x[0]}$$

$\underline{n=1}$

$$y[1] = h[0]x[1] + h[1]x[0]$$

$$h[1]x[0] = y[1] - h[0]x[1]$$

$$\Rightarrow \quad h[1] = \frac{y[1] - h[0]x[1]}{x[0]}$$

$\underline{n=2}$

$$y[2] = h[0]x[2] + h[1]x[1] + h[2]x[0]$$

$$h[2] = \frac{y[2] - h[0]x[2] - h[1]x[1]}{x[0]}$$

And in general . . .

$$h[n] = \frac{y[n] - \sum_{k=0}^{n-1} h[k]x[n-k]}{x[0]}, \quad n \geq 1.$$

- Deconvolution & System Identification can also be approached in terms of power spectra.

- This leads to techniques that can be applied to statistical as well as deterministic signals,
  - → although up to this point the book has only defined the autocorrelation in a deterministic sense.

Recall: input autocorrelation

$$r_{xx}[\ell] = \sum_{k \in \mathbb{Z}} x[k]\, x[k-\ell]$$

"Power spectrum" or "power spectral density" of $x[n]$:

$$S_{xx}(e^{j\omega}) = \text{DTFT}\{r_{xx}[n]\}$$

$$S_{xx}(z) = \mathcal{Z}\{r_{xx}[n]\}. \qquad 7.69$$

- Input-output cross-correlation and cross power spectrum:

$$r_{yx}[\ell] = \sum_{n \in \mathbb{Z}} y[n]\, x[n-\ell]$$

$$S_{yx}(e^{j\omega}) = \text{DTFT}\{r_{yx}[\ell]\}$$

$$S_{yx}(z) = \mathcal{Z}\{r_{yx}[\ell]\}$$

- Output autocorrelation and power spectrum:

$$r_{yy}[\ell], \quad S_{yy}(e^{j\omega}), \quad S_{yy}(z).$$

- It can be shown fairly easily and is derived in the book that, for an LTI system H

$$x[n] \longrightarrow \boxed{\overset{\text{LTI}}{H}} \longrightarrow y[n]$$

the I/O relation in terms of power spectra is

$$\left.\begin{aligned} S_{yx}(z) &= S_{xx}(z)\, H(z) \\ S_{yy}(z) &= S_{xx}(z)\, |H(z)|^2 \end{aligned}\right\} \text{ often called "Wiener-Kintchine Relations"}$$

Along the unit circle, these reduce to

$$S_{yx}(e^{j\omega}) = S_{xx}(e^{j\omega}) H(e^{j\omega}) \qquad (*)$$

$$S_{yy}(e^{j\omega}) = S_{xx}(e^{j\omega}) \left|H(e^{j\omega})\right|^2 \qquad (**)$$

→ If the input power spectrum and the output-input cross power are known, then $(*)$ can be used to solve for $H(e^{j\omega})$.

→ Alternatively, if the input power spectrum is known and the output power spectrum is known or can be measured, then $(**)$ can be used to solve for $|H(e^{j\omega})|$, even if the output-input cross power is not known.

Stability Triangle

- You need to read the book for the details of this.

- The big idea:
  - → we desire a way to check stability of a causal filter without having to factor the denominator to find all the poles.
  - → A causal system with rational $H(z)$ (in z^{-1}) is causal iff all of the poles lie inside the unit circle.
  - → For a 2nd order system, this can be translated into requirements on the denominator polynomial coefficients.
  - → A causal $H(z) = \dfrac{\text{blah}}{1 + d_1 z^{-1} + d_2 z^{-2}}$ is stable if the coefficients d_1 and d_2 lie inside the "Stability Triangle"