ECE 4213/5213

MODULE 9

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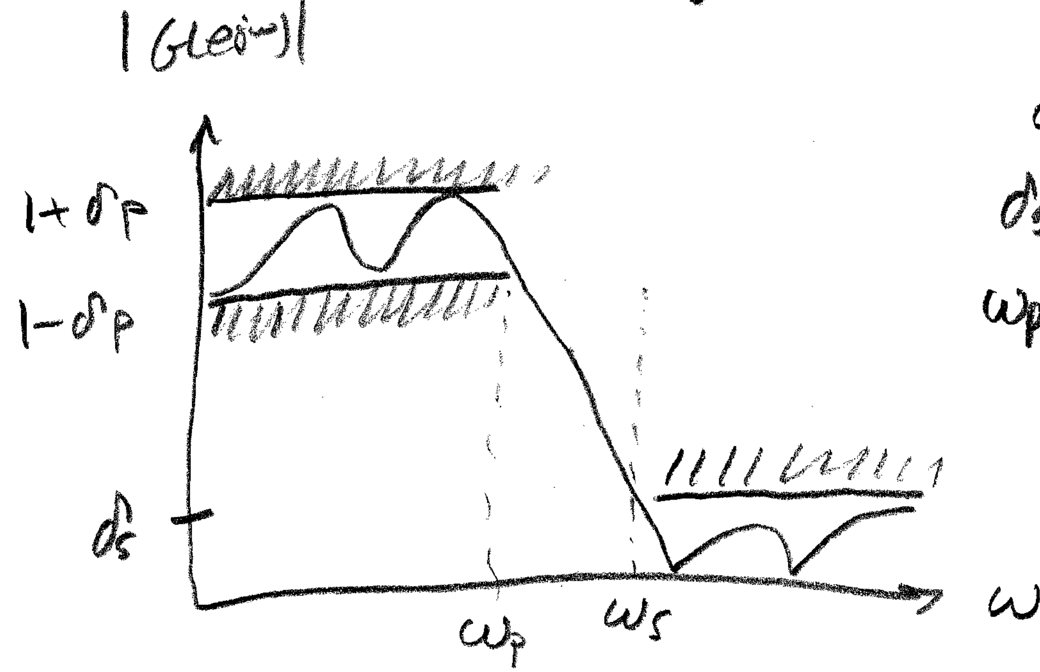
IIR Filter Design

→ Design a practical, realizable filter that, to within tolerances, approximates a desired frequency response $G(e^{j\omega})$.

→ $G(e^{j\omega})$ is 2π-periodic

→ if $G(z)$ has real coefficients (usually the case), so that $g[n]$ is real, then $|G(e^{j\omega})|$ is even

→ Desired magnitude response usually specified only for $0 \le \omega \le \pi$.



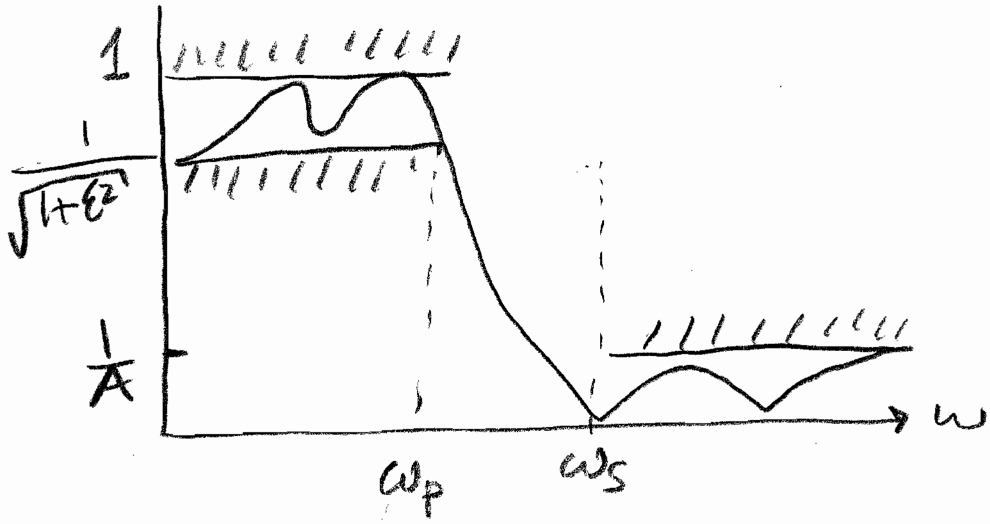
δ_p : passband ripple

δ_s : stopband ripple

ω_p : passband edge frequency

ω_s : stopband edge frequency

Normalized Spec:



The critical frequencies may be specified in (analog) Hz or radians along w/ a sampling frequency ... in cases where the the digital filter will be used to implement an analog filter.

Ω_T : sampling frequency in radians/sec

$F_T = \frac{\Omega_T}{2\pi}$: sampling frequency in Hz

$T = \frac{1}{F_T} = \frac{2\pi}{\Omega_T}$: sampling interval in sec.

Ω_p, F_p : passband edge freq. in rad/sec, Hz

Ω_s, F_s : stopband edge freq. in rad/sec, Hz

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T \quad (9.7)$$

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→ and similar for ω_s.

- Why use an FIR design?
  - → always stable, even after quantization
  - → can be designed for linear phase by enforcing symmetry on the impulse response.

- Why use IIR?
  - → Can achieve spec w/ a lower order... often <u>much</u> lower.
  - → But the advantage may be lost if linear or approximately linear phase is required: the designed filter will have to be cascaded with a delay equalizer.

NOTE: You will <u>almost always</u> design a <u>lowpass</u> filter. If another type is needed, you will first transform the spec to a lowpass spec. After design, you can transform back.

BILINEAR TRANSFORM

→ Map $[-\pi, \pi]$ on the unit circle of the z-plane to $(-\infty, \infty)$ on the $j\Omega$ axis of the s-plane.

→ Through this mapping, the digital critical frequencies are mapped to analog critical frequencies.

→ Use the techiques of Appendix A to design the analog lowpass filter (the digital specs on passband ripple, stopband ripple, $\frac{1}{\sqrt{1+\varepsilon^2}}$, $\frac{1}{A}$ can be taken directly over to the analog spec.)

Digital Spec

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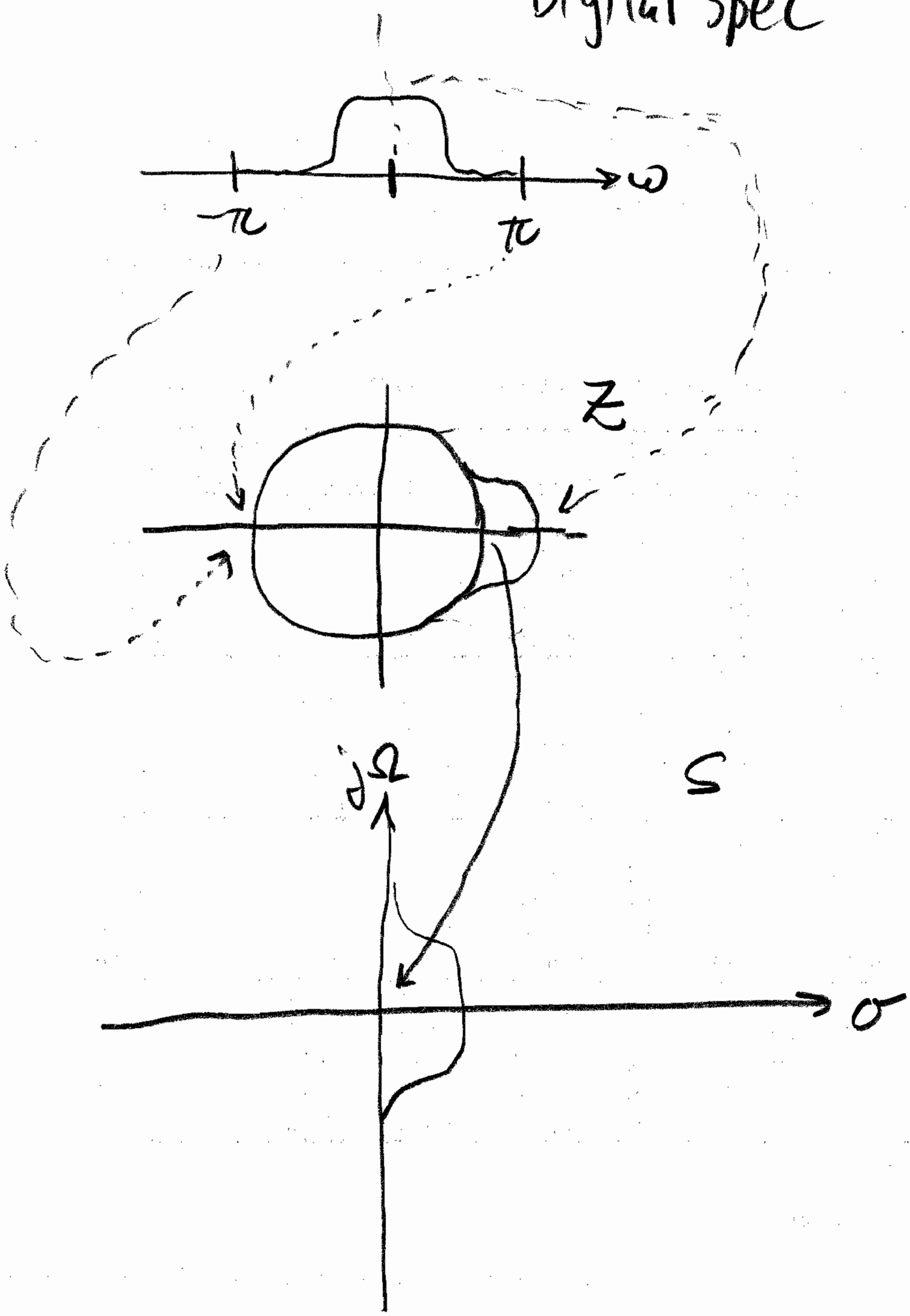
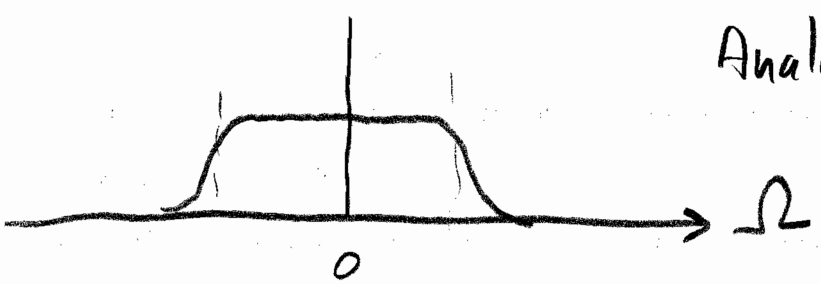


$-\pi$ goes to $-\infty$

$\leftarrow$

Analog Spec



$\longrightarrow$ π goes to ∞

The bilinear transformation between the s-plane and z-plane:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad (9.14)$$

→ T is included for historical reasons. It changes the "rate" of the warping that sends $\pi \to \infty$ and $-\pi \to -\infty$.

- Choosing different values for T maps a given digital spec into different analog specs, but that has no practical significance today.

⇒ Choose T=2. It makes the math cleaner.

→ On the unit circle of the z-plane, which maps to the $j\Omega$-axis of the s-plane, (9.14) becomes

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right).$$

with T=2, $\Omega = \tan\frac{\omega}{2}$.

Procedure:

① "prewarp" the digital frequencies ω_p and ω_s:

$$\Omega_p = \tan\frac{\omega_p}{2} \; ; \; \Omega_s = \tan\frac{\omega_s}{2} .$$

② Carry the ripple specs δ_p, δ_s or $\frac{1}{\sqrt{1+\varepsilon^2}}$, A directly over from the digital spec to the analog spec.

③ Design the analog filter $H_a(s)$.

④ Change "s" to $\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$.

This gives you the digital transfer function $G(z)$.

Here is an example of using the bilinear transform to design a digital Butterworth filter:

Use the bilinear transform to design a lowpass digital Butterworth filter that meets the following specifications:

Passband Edge Freq.	$\omega_p = \pi/8$ rad/sample
Stopband Edge Freq.	$\omega_s = 2\pi/5$ rad/sample
Max. Passband Ripple	$1/\sqrt{1+\epsilon^2} = 0.9$
Min. Stopband Atten.	$1/A = 0.2$

- Prewarp the digital critical frequencies ω_p and ω_s to get analog critical frequencies Ω_p and Ω_s:

$$\Omega_p = \tan\left(\frac{\omega_p}{2}\right) = \tan(\pi/16) = 0.198912$$

$$\Omega_s = \tan\left(\frac{\omega_s}{2}\right) = \tan(\pi/5) = 0.726543$$

- Use the given information to find A and ε:

$$\frac{1}{A} = 0.2 = \frac{2}{10}$$

$$2A = 10$$

$$A = 5$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{9}{10}$$

$$\sqrt{1+\varepsilon^2} = \frac{10}{9}$$

$$1 + \varepsilon^2 = \frac{100}{81}$$

$$\varepsilon^2 = \frac{100}{81} - 1 = \frac{100-81}{81}$$

$$= \frac{19}{81}$$

$$\varepsilon = \sqrt{\frac{19}{81}} = 0.48432$$

- Now we have the analog filter spec:

$\Omega_p = 0.198912 \quad \Omega_s = 0.726543 \quad A = 5 \quad \varepsilon = 0.48432$

- Design the analog filter:

  - Find the order:

$$N = \left\lceil \frac{\frac{1}{2}\log_{10}\left[(A^2-1)/\varepsilon^2\right]}{\log_{10}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\frac{1}{2}\log_{10}\left[24 \cdot \frac{81}{19}\right]}{\log_{10} \frac{0.726543}{0.198912}} \right\rceil$$

$$= \left\lceil \frac{\frac{1}{2}\log_{10} 102.316}{\log_{10} 3.65258} \right\rceil = \left\lceil \frac{\frac{1}{2}(2.00994)}{0.562599} \right\rceil$$

$$= \lceil 1.78630 \rceil = \underline{\underline{2}}$$

  - Find Ω_c:

$$\frac{1}{1 + (\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2}$$

$$A^2 = 1 + (\Omega_s/\Omega_c)^{2N}$$

$$A^2 - 1 = \frac{\Omega_s^{2N}}{\Omega_c^{2N}}$$

$\longrightarrow$

$$\Omega_c^{2N} = \frac{\Omega_s^{2N}}{A^2 - 1}$$

$$\Omega_c = \frac{\Omega_s}{[A^2-1]^{\frac{1}{2N}}} = \Omega_s [A^2-1]^{-\frac{1}{2N}}$$

plugging in the numbers:

$$\Omega_c = 0.726543(24)^{-\frac{1}{4}}$$
$$= 0.726543(0.451801)$$
$$= 0.328253.$$

- Find the poles: $p_\ell = \Omega_c e^{j[\pi(N+2\ell-1)/(2N)]}$

$\ell = 1, 2$

$$p_1 = \Omega_c e^{j\pi(2+2-1)/4} = \Omega_c e^{j3\pi/4} = 0.328253 e^{j3\pi/4}$$

$$p_2 = \Omega_c e^{j\pi(2+4-1)/4} = \Omega_c e^{j5\pi/4} = 0.328253 e^{j5\pi/4}$$

$\longrightarrow$

- plug in the designed values of N, Ω_c, p_1, and p_2 to get the analog transfer function $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^N}{\prod_{l=1}^{N}(s-p_l)} = \frac{\Omega_c^2}{(s-p_1)(s-p_2)}$$

$$= \frac{\Omega_c^2}{(s-\Omega_c e^{j3\pi/4})(s-\Omega_c e^{j5\pi/4})}$$

$$= \frac{\Omega_c^2}{s^2 - s\Omega_c e^{j5\pi/4} - s\Omega_c e^{j3\pi/4} + \Omega_c^2 e^{j8\pi/4}}$$

NOTE: $e^{j5\pi/4} = e^{j(5\pi/4 - 2\pi)} = e^{-j3\pi/4}$

$$H_a(s) = \frac{\Omega_c^2}{s^2 - s\Omega_c(e^{j3\pi/4} + e^{-j3\pi/4}) + \Omega_c^2 \underbrace{e^{j2\pi}}_{\text{one}}}$$

$$= \frac{\Omega_c^2}{s^2 - s\Omega_c 2\cos\frac{3\pi}{4} + \Omega_c^2} = \frac{\Omega_c^2}{s^2 + \Omega_c\sqrt{2}\,s + \Omega_c^2}$$

$\longrightarrow$

plugging in the numbers:

$$H_a(s) = \frac{0.107750}{s^2 + 0.464219s + 0.107750}$$

- Finally, invert the bilinear transform by letting $s = \frac{1-z^{-1}}{1+z^{-1}}$. This gives us the transfer function $H(z)$ of the digital filter:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.107750}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.464219\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.107750}$$

$\rightarrow$ Multiply by $\frac{(1+z^{-1})^2}{(1+z^{-1})^2} = 1$ to clear the denominators out of the denominator...

$$H(z) = \frac{0.107750\,(1+z^{-1})^2}{(1-z^{-1})^2 + 0.464219\,(1-z^{-1})(1+z^{-1}) + 0.107750\,(1+z^{-1})^2}$$

$$= \frac{0.107750\,(1+2z^{-1}+z^{-2})}{1-2z^{-1}+z^{-2} + 0.464219\,(1-z^{-2}) + 0.107750\,(1+2z^{-1}+z^{-2})}$$

$$= \frac{0.10775 + 0.2155\,z^{-1} + 0.10775\,z^{-2}}{(1+0.464219+0.107750) + (-2+0.2155)\,z^{-1} + (1-0.464219+0.10775)\,z^{-2}}$$

$$= \frac{0.10775 + 0.2155\,z^{-1} + 0.10775\,z^{-2}}{1.57197 - 1.7845\,z^{-1} + 0.64353\,z^{-2}}$$

divide everybody on top and bottom by this to get a leading "1" downstairs... in other words, multiply $H(z)$ by

$$1 = \frac{\frac{1}{1.57197}}{\frac{1}{1.57197}}$$

$$H(z) = \frac{0.0685445 + 0.137089\,z^{-1} + 0.0685445\,z^{-2}}{1 - 1.1352\,z^{-1} + 0.409379\,z^{-2}} \longrightarrow$$

ROC: $|z| > |\text{largest pole}|$

(To find the poles, you would have to factor the denominator. They will be complex in general).

→ To get $H(e^{j\omega})$, plug in $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{0.0685445 + 0.137089 e^{-j\omega} + 0.0685445 e^{-j2\omega}}{1 - 1.1352 e^{-j\omega} + 0.409379 e^{-j2\omega}}$$

Transformations for other Filter Types:

- If the desired digital filter is highpass, bandpass, or bandstop, then you apply frequency transformations to obtain an equivalent lowpass analog design spec.

1. prewarp the digital critical frequencies to obtain an analog spec. This spec is for the same type of filter as the desired filter... i.e., highpass, bandpass, bandstop.

2. Use the analog frequency transformations of Appendix B to convert this spec to an equivalent analog <u>lowpass</u> spec.

3 Design the analog lowpass filter. 9-9

⇒ Now there are two paths you can take to get the digital filter

PATH A

4A Use the bilinear transform (9.14) to convert the analog lowpass filter to a digital lowpass filter.

5A Use a digital-to-digital frequency transformation from Table 7.1 to convert the digital lowpass filter to the desired type.

PATH B

4B Apply an analog-to-analog transformation that is the inverse of the one used in step 2 to convert the analog lowpass filter to an analog filter of the desired type

5B Use the bilinear transform (9.14) to convert the analog filter from 4B into a digital filter of the desired type.

Numerical Optimization techiques for IIR Filters

- Suppose that we have a desired frequency response $D(e^{j\omega})$.

- The goal is to design

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}}$$

So that the error

$$\left| H(e^{j\omega}) - D(e^{j\omega}) \right|$$

is minimized in some sense.

- Note that this strategy can be used to design FIR filters by taking $a_1 = a_2 = \cdots = a_p = 0$.

- Usually, we care about the errors at certain frequencies more than other.
  - → For example, we might care a lot in the passband(s), less in the stopband(s), and not at all in the transition band(s).

- This can be taken into account by introducing a "weighting function" into the error:

$$W(e^{j\omega})\left| H(e^{j\omega}) - D(e^{j\omega}) \right|$$

  - at frequences where we care a lot about the error (e.g. passband), set $W(e^{j\omega}) = 1$.

  - at frequencies where we care less set $0 \le W(e^{j\omega}) < 1$.

- The objective is to solve for the coefficients a_k and b_k that minimize the weighted error in some sense.

- This usually has to be done numerically.

e.g., integer programming
dynamic programming
least squares
etc.

- The figure of merit that is usually optimized is of the form

$$\varepsilon = \int_{-\pi}^{\pi} \left| W(e^{j\omega}) \left[H(e^{j\omega}) - D(e^{j\omega}) \right] \right|^{p} d\omega$$

where $p \in \mathbb{Z}$.

$p = 2$: "least squares" optimization

- If $W(e^{j\omega}) = 1$ and $p = 2$, the least squares solution is always an FIR filter.
  → It is the one obtained by truncating $d[n]$, the impulse response of the desired filter $D(e^{j\omega})$, to a finite length.