FIR Filter Design

- In many cases, FIR design techniques require the desired filter order $N$ to be known.
- The usual approach is to estimate the order using an approximate formula.
  → The filter design can then be carried out.
  
  → Since the order estimation is always approximate, you should always check the designed filter to verify that it meets the spec.
  → If not, increase the order by 1 and try again.
  → Repeat until you get a filter that does meet the spec.
Kaiser Formula:
\[ N \approx \frac{-20 \log_{10} \sqrt{\sigma_p \sigma_s}}{14.6 |\omega_s - \omega_p| / 2\pi} - 13 \] (10.3)

→ Other formulas are given in the book.

→ If you have a normalized spec, set \( \sigma_p = \frac{1}{2\sqrt{1+\epsilon^2}} \)

\[ \sigma_s = \frac{1}{\lambda} \]

Now assume that there is a desired frequency response \( D(e^{j\omega}) \) with impulse response \( h(n) \).

→ This is called \( H_0(e^{j\omega}) \) and \( h_0(n) \) in Mitra's text.

→ If \( d(n) \) is finite length, we're done!

→ But usually it won't be.
Windowed FIR Filter Design

- The strategy here is to multiply the infinite length desired impulse response \( d[n] \) by a finite length window \( w[n] \) to truncate it.

- This gives a finite length impulse response \( h[n] = d[n]w[n] \).

Note: if the desired frequency response \( D(e^{j\omega}) \) is one of the ideal filters, then the impulse response \( d[n] \) will be doubly infinite.

- For this reason, the window functions \( w[n] \) are specified in Mitra's book to be symmetric about \( n=0 \), that is, they are nonzero for \( -M \leq n \leq M \), where the length of \( w[n] \) (and of \( h[n] \)) is \( 2M+1 \) and the filter order is \( N = 2M \).

- Since the designed filter must ultimately be causal and stable, the impulse response obtained from these windows must be shifted right by \( M \) samples to obtain a causal filter.
- In some cases you may have a $\theta(e^{jw})$ that is inherently causal, so that $d[n]=0 \text{ and } n<0$.

- In these cases, you should shift Mitra's windows to the right by $M$ samples before multiplying to get

  $h[n] = w[n\text{mod}N]d[n].$

- For this reason, some authors specify the windows $w[n]$ to be nonzero for $0 \leq n \leq N+1$. This gives formulas that are equivalent to Mitra's, but look different.

Effect of windowing:
- Since $h[n] = w[n\text{mod}N]d[n]$, the obtained frequency response is

\[
H(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(e^{j\theta})D(e^{j(w-\theta)})d\theta
\]

- i.e., the desired spectrum is convolved with the spectrum of the window.
This results in a distortion of the desired spectrum that can be nontrivial.

Over the years, many types of windows have been designed to try and make \( H(e^{j\omega}) \) and \( D(e^{j\omega}) \) as similar as possible in one sense or another.

**Main Types of Windows:**

*Rectangular:*
\[
 w(n) = \begin{cases} 
 1, & -M \leq n \leq M \\
 0, & |n| > M. 
\end{cases} 
\] (10.32)

*Bartlett:*
\[
 w(n) = 1 - \frac{|n|}{M+1} 
\] (10.33)

*Hann:*
\[
 w(n) = \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi n}{M} \right) \right] 
\] (10.34)

*Hamming:*
\[
 w(n) = 0.54 + 0.46 \cos \left( \frac{\pi n}{M} \right) 
\] (10.35)

*Blackman:*
\[
 w(n) = 0.42 + 0.5 \cos \left( \frac{\pi n}{M} \right) + 0.08 \cos \left( \frac{2\pi n}{M} \right) 
\] (10.36)
These windows all have DTFTs that are "sync like": $|W(e^{j\omega})|$

- The relative sidelobe level is the difference between the peak of the mainlobe and the peak of the first sidelobe, expressed in dB.

For good agreement between $H(e^{j\omega})$ and $D(e^{j\omega})$, we want
- Mainlobe width narrow
- Peak sidelobe low.

These are conflicting goals. The different windows trade off between them differently.
- Procedure: start with an ideal lowpass filter with cutoff frequency 
  \[ \omega_c = \omega_p + \frac{\omega_s - \omega_p}{2} \]

- This gives 
  \[ D(e^{j\omega}) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases} \] (10.13) (fundamental period).

- We then have 
  \[ d[n] = \frac{\sin \omega_c n}{\pi n} \] (10.14)

- Estimate the order \( N = 2M \).

- Use Table 10.2, page 540, to select a window that will provide stopband attenuation and transition band width to meet the spec.

- Let \( h[n] = w[n] d[n] \), \(-M \leq n \leq M\)

- Shift \( h[n] \) right by \( M \) samples to obtain a causal filter.
- Check to see if the designed filter meets the spec.

  → If not, increase the order and try again.

**NOTE**: As we said on page 9-12, the rectangular window minimizes the mean squared error (MSE) between $D(e^{j\omega})$ and $H(e^{j\omega})$, which is given by

$$E = \int_{-\pi}^{\pi} \left| D(e^{j\omega}) - H(e^{j\omega}) \right|^2 \, d\omega.$$ 

→ However, this filter usually has significant passband ripple ("Gibbs phenomena") due to the convolution of $D(e^{j\omega})$ with $W(e^{j\omega})$.

→ As the order $N$ is increased, the ripple becomes more localized at the transition band edge, but the peak ripple does not decrease.
Windowed FIR filter designs:

- An "ideal" low pass filter with cutoff frequency $w_c$ has frequency response:

$$H_{LP}(e^{jw})$$

$-\pi \rightarrow -w_c \rightarrow 0 \rightarrow w_c \rightarrow \pi$

- The impulse response is (table):

$$h_{LP}[n] = \frac{\sin w_c n}{\pi n}$$

- This filter is not realizable... it cannot be built...

- because it is **unstable** and

  it is not **causal**.
However, we can make it **stable** if we multiply $h_{lp}[n]$ times a finite-length window to “chop” it off.

- Then we can shift the finite length version to make it causal.

- This is called “windowed FIR filter design”
Common Window Functions for FIR filter design:

**Rectangular:** 
\[ w[n] = 1, \quad -M \leq n \leq M, \]

**Hann:** 
\[ w[n] = \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi n}{M} \right) \right], \quad -M \leq n \leq M, \]

**Hamming:** 
\[ w[n] = 0.54 + 0.46 \cos \left( \frac{\pi n}{M} \right), \quad -M \leq n \leq M, \]

**Blackman:** 
\[ w[n] = 0.42 + 0.5 \cos \left( \frac{\pi n}{M} \right) + 0.08 \cos \left( \frac{2\pi n}{M} \right), \quad -M \leq n \leq M. \]

Main Properties of the Window Functions:

<table>
<thead>
<tr>
<th>Type of Window</th>
<th>Main Lobe Width ( \Delta_{ML} )</th>
<th>Relative Sidelobe Level ( A_{sL} )</th>
<th>Minimum Stopband Attenuation</th>
<th>Transition Bandwidth ( \Delta\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>( 4\pi/(2M + 1) )</td>
<td>13.3 dB</td>
<td>20.9 dB</td>
<td>0.92\pi/M</td>
</tr>
<tr>
<td>Hann</td>
<td>( 8\pi/(2M + 1) )</td>
<td>31.5 dB</td>
<td>43.9 dB</td>
<td>3.11\pi/M</td>
</tr>
<tr>
<td>Hamming</td>
<td>( 8\pi/(2M + 1) )</td>
<td>42.7 dB</td>
<td>54.5 dB</td>
<td>3.32\pi/M</td>
</tr>
<tr>
<td>Blackman</td>
<td>( 12\pi/(2M + 1) )</td>
<td>58.1 dB</td>
<td>75.3 dB</td>
<td>5.56\pi/M</td>
</tr>
</tbody>
</table>
Here's how it works:

1. Look at the "Minimum Stopband Attenuation" column of the table to figure out which windows can meet the spec. Use $\Delta s = -20 \log_{10} \Delta s$.

2. Use the last column of the table to figure out which window can meet the spec with the smallest $M$.
   - The order of the filter is $2M$.

3. Let $\omega_c = \frac{\omega_p + \omega_s}{2}$

4. Let $h_{lp}[n] = \frac{\sin(\omega_c n)}{\pi n}$

5. Compute the "centered" impulse response $w[n] h_{lp}[n]$, $-M \leq n \leq M$

6. Shift right by $M$ to make causal:

$$h[n] = W[n-M] h_{lp}[n-M], \quad 0 \leq n \leq 2M$$
4. 25/20 pts. Use the window design method with an appropriate fixed window from Table 10.2 (p. 540 of the text) to design a causal lowpass FIR digital filter that meets the following specifications:

<table>
<thead>
<tr>
<th>Passband Edge Freq.</th>
<th>( \omega_p = 0.2\pi \text{ rad/sample} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stopband Edge Freq.</td>
<td>( \omega_s = 0.9\pi \text{ rad/sample} )</td>
</tr>
<tr>
<td>Max. Passband Ripple</td>
<td>( \delta_p = 0.005 )</td>
</tr>
<tr>
<td>Max. Stopband Ripple</td>
<td>( \delta_s = 0.005 )</td>
</tr>
</tbody>
</table>

Give the filter impulse response \( h[n] \).

\[ \alpha_s = -20 \log_{10} \delta_s = -20 \log_{10} 0.005 = 46.0206 \text{ dB} \]

Table 10.2: Hamming and Blackman can meet the spec.

\[ \Delta \omega = \omega_s - \omega_p = 0.9\pi - 0.2\pi = 0.7\pi \]

Hamming: \( \Delta \omega = \frac{3.32\pi}{M} \); \( M = \left\lceil \frac{3.32\pi}{0.7\pi} \right\rceil = \left\lceil 4.74 \right\rceil = 5 \)

Blackman: \( \Delta \omega = \frac{5.56\pi}{M} \); \( M = \left\lceil \frac{5.56\pi}{0.7\pi} \right\rceil = \left\lceil 7.94 \right\rceil = 8 \)

\[ \implies \text{Hamming meets the spec with a lower order.} \]

\( M = 5 \), \( \text{order} = N = 2M + 1 = 11 \), \( \text{Length} = 2M + 1 = 11 \).

\( \omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.2\pi + 0.9\pi}{2} = \frac{1.1\pi}{2} = 0.55\pi \).

\[ h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\sin 0.55\pi n}{\pi n} \]

\((10.34)
\[ W[n] = 0.54 + 0.46 \cos \left( \frac{\pi}{5} n \right) \]

\[ W[n] h_{LP}[n] = \left\{ 0.54 + 0.46 \cos \left( \frac{\pi}{5} n \right) \right\} \frac{\sin 0.55\pi n}{\pi n} , \quad -5 \leq n \leq 5. \]

Shift to make causal:

\[ h[n] = \frac{\sin 0.55\pi (n-5)}{\pi (n-5)} \left\{ 0.54 + 0.46 \cos \frac{\pi}{5} (n-5) \right\} , \quad 0 \leq n \leq 10 \]
Kaiser Window:

\[ W(n) = \frac{I_0(\beta \sqrt{1-(n/M)^2})}{I_0(\beta)} \quad -M \leq n \leq M \]  \hspace{1cm} (10.42)

where \[ I_0(x) = 1 + \sum_{r=1}^{\infty} \left( \frac{(x/2)^r}{r!} \right)^2 \]  \hspace{1cm} (10.43)

is a modified Bessel function of zeroth order.

→ it is customary to retain the first 20 terms of the sum.

→ To use the Kaiser window, you need to know \( M \) and \( \beta \).

Let \( \alpha_S = -20 \log_{10} \delta_S \)

Let \( \Delta \omega \) be the desired transition band width, i.e., \( \Delta \omega = |\omega_s - \omega_p| \).

Estimate the order using:

\[ N = \frac{\alpha_S - 8}{2.285 \Delta \omega} \quad (10.45) \]

\[ M = \frac{N-1}{2} \]
<table>
<thead>
<tr>
<th>$\alpha_s$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 50</td>
<td>$0.1102 (\alpha_s - 8.7)$</td>
</tr>
<tr>
<td>$21 \leq \alpha_s \leq 50$</td>
<td>$0.5842 (\alpha_s - 21)^{0.4}$ + $0.07886 (\alpha_s - 21)$</td>
</tr>
<tr>
<td>$\alpha_s &lt; 21$</td>
<td>0</td>
</tr>
</tbody>
</table>

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**FIR Filter Design by Frequency Sampling**

- Set $H[k] = D(e^{j2\pi k/N})$
  
  for $k = 0, 1, \ldots, N-1$.

- Obtain $h[n]$ by inverse DFT:
  
  $$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j2\pi nk/N}, \quad 0 \leq n \leq N-1.$$  

→ If $D(e^{j\omega})$ has a smooth transition band and if there are enough samples on the passband of $D(e^{j\omega})$ so that the samples have a shape that looks like the desired passband,
- Then this approach may work pretty well.

- There will generally be aliasing in $h[n]$:

$$h[n] = \sum_{k=-\infty}^{\infty} d[n+kN], \quad 0 \leq n \leq N-1$$

but this is not a concern if $d[n]$ decays towards zero rapidly for $|n| > N$.

- If the transition band of $D(e^{j\omega})$ is very abrupt, as with an ideal lowpass filter, the aliasing can be reduced by inserting a smoother transition band, i.e.,

$$D(e^{j\omega})$$

or

$$\quad$$

or
Impulse Invariant Design

- This approach is based on a desirable analog prototype filter \( h_a(s) \) with impulse response \( h_a(t) \).

- The strategy is to sample \( h_a(t) \) on a finite interval to get \( h[n] \):

\[
h[n] = h_a(nT_s), \quad 0 \leq n \leq N
\]

\( \rightarrow \) as in windowing, there will be "smearing" of the desired spectrum due to truncation, unless \( h_a(t) \) just happens to have finite support.

\( \rightarrow \) as with frequency sampling, there will generally be aliasing.

\( \rightarrow \) May not be too bad if \( h_a(t) \) decays rapidly toward zero as \( t \) becomes large and there are enough samples so that the shape of \( h[n] \) looks a lot like the shape of \( h_a(t) \).
- It may be shown that
\[ h(e^{jw}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H_a \left( \frac{w}{T_s} - j \frac{2\pi k}{T_s} \right) \]

Parks-McClellan Algorithm:

- It is a numerical technique for designing equiripple linear phase FIR filters. That is, they have:
  - equal ripple in the stop band
  - equal but different ripple in the pass band
  - linear phase

- This is almost always implemented on the computer.

- See the Lab manual & text for details of how to do it using Matlab.