Several important techniques for designing digital IIR filters are based on analog filter prototypes.

To design the digital filter, you begin by transforming the digital specification to an analog spec and doing an analog design.

The problem of designing an analog filter is fundamentally to design an analog function \(|H_a(\omega)|\) that satisfies the passband and stopband requirements.

The general problem of how to design a curve with a desired shape is difficult and has been studied for centuries.

To simplify things, a low pass filter is almost always designed.

If another type is needed (e.g., highpass, bandpass, bandstop), it is obtained by transforming a low pass design.

We will limit our attention to the main results that are useful for designing digital filters.
Analog filter magnitude design spec:
\[ |H_a(\omega)| \]

The ripples are usually specified in dB:
- peak passband ripple: \( \alpha_p = -20 \log_{10} (1 - \delta_p) \) dB
- minimum stopband attenuation: \( \alpha_s = -20 \log_{10} (\delta_s) \) dB

Transition Ratio: \( k = \frac{\Omega_p}{\Omega_s} \)
The spec is also often stated in a normalized form where the peak passband gain is normalized to 1 ... or 0 dB:

\[ |H_a(n)| \]

\[ \frac{1}{\sqrt{1 + 4^2}} \]

\[ \frac{1}{A} \]

Max passband gain: 0 dB  
Max stopband ripple: \( \frac{1}{A} \)  
Min stopband attenuation: \(-20 \log_{10} \left( \frac{1}{A} \right) \) dB  

Discrimination Parameter: \( k_1 = \frac{e}{\sqrt{A^2 - 1}} \)  
- usually \( k_1 \ll 1 \)
Lowpass Butterworth Filter:

- Monotonic passband
- Monotonic stopband (e.g. no ripple).

\[
\left| H_a(\Omega) \right|^2 = \frac{1}{1 + \frac{\Omega^{2N}}{\Omega_c^{2N}}} \quad (4.7)
\]

- N = filter order
- \( \Omega_c \) = 3dB cutoff freq.

- \( H_a(\Omega) \) is fully specified by the two parameters \( N, \Omega_c \).

\[
\text{The first } 2N-1 \text{ derivatives of } \left| H_a(\Omega) \right|^2 \text{ are zero at } \Omega = 0.
\]

Gain: \( G(\Omega) = 10 \log_{10} \left| H_a(\Omega) \right|^2 \text{ dB} \)

@ \( \Omega_c \), \( G(\Omega_c) = 10 \log_{10} \frac{1}{2} \approx -3 \text{ dB} \)

@ DC, \( G(0) = 0 \text{ dB} \)

For large \( \Omega \), \( \left| H_a(\Omega) \right|^2 \approx \frac{1}{(\Omega/\Omega_c)^{2N}} \)

Let \( \Omega_1 \gg \Omega_c \) and \( \Omega_2 = 2\Omega_1 \).

Then \( G(\Omega_2) = G(\Omega_1) - 6N \text{ dB} \)

\text{Butterworth filter rolloff is } 6 \text{ dB per octave for each increase in the filter order.}
The design parameters are:
\[ s_p, s_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{k} \]  

Design:
- First find the order

\[ N = \left\lceil \frac{1}{2} \log_{10} \left[ \frac{(A^2 - 1)/\varepsilon^2}{\log_{10} (s_s/s_p)} \right] \right\rceil = \left\lceil \frac{\log_{10} (1/k)}{\log_{10} (1/k)} \right\rceil \]  

(A.9)

- \( s_c \) can then be solved from

\[ |H_a(s_p)|^2 = \frac{1}{1 + (s_p/s_c)^{2N}} = \frac{1}{1 + \varepsilon^2} \]  

(A.8a)

or

\[ |H_a(s_s)|^2 = \frac{1}{1 + (s_s/s_c)^{2N}} = \frac{1}{A^2} \]  

(A.8b)

- Usually use (A.8b) to meet the stopband spec exactly while exceeding the passband spec.

- Solve for the poles:

\[ P_k = s_c e^{\delta \left[ \pi (N+2k-1)/2N \right]}, \quad 1 \leq k \leq N \]  

(A.11)

- Design \( H_a(s) \):

\[ H_a(s) = \frac{\sum_{k=1}^{N} P_k}{N} (s - P_k) \]  

(A.10)
Butterworth design in Matlab:

- Normalized forms with \( \Omega_c = 1 \), \( \text{order} = N \):

\[
[z, p, g] = \text{buttap}(N);
\]

null poles  gain  order

\[
[N, D] = \text{zp2tf}(z, p, g);
\]

numerator and denominator polynomial coefficients:

\[
H_a(s) = \frac{N(s)}{D(s)} = \frac{g}{\prod_{l=1}^{N} (s - \Omega_l)}
\]

General forms:

\[
[N, W_n] = \text{buttord}(W_p, W_s, R_p, R_s, \Omega, \text{s})\]

3 dB cutoff freq. in rad/sec  max passband loss in dB  min stopband attenuation in dB

\[
[\text{num}, \text{den}] = \text{butter}(N, W_n, \text{s})\]

\( N+1 \) numerator coefs. \( N+1 \) denominator coefs. 3 dB cutoff freq. in rad/sec  order  analog
Lowpass Chebyshev Filters:

- Achieve a narrower transition band than Butterworth for the same order.
- Have ripple in the stopband or in the passband, but not both.

Type I Chebyshev:

\[ |H_a(\omega)|^2 = \frac{1}{1 + \varepsilon^2 T_N^{\alpha}(\frac{\omega}{\omega_p})} \] (A.13)

- \( T_N(\omega) \) is an \( N^{th} \)-order Chebyshev polynomial.
- Design parameters: \( \omega_p, \omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{\lambda} \)
- Design:
  - First find the order:
  \[ N = \left[ \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\omega_s/\omega_p)} \right] = \left[ \frac{\cosh^{-1}(Y_k)}{\cosh^{-1}(Y_k)} \right] \] (A.17)
  - Then find the poles:
  \[ \pi_l = \sigma_l + j \pi_l, \quad 1 \leq l \leq N \] (A.18)
  where
  \[ \sigma_l = -\pi_p \xi \sin\left[ \frac{(2l-1)\pi}{2N} \right], \quad \pi_l = \pi_p \xi \cos\left[ \frac{(2l-1)\pi}{2N} \right] \] (A.19a)
  and
  \[ \xi = \frac{\sqrt{\gamma^2-1}}{2\gamma}, \quad \xi = \frac{\sqrt{\gamma^2+1}}{2\gamma}, \quad \gamma = \left( \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \right)^{1/N} \] (A.19b)
Transfer Function:

\[ H_a(s) = C_0 \prod_{\ell=1}^{N} \frac{-p_\ell}{s-p_\ell} \]

(These formulas are not in the book)

where \( C_0 = \begin{cases} 4, & N \text{ odd} \\ \frac{1}{\sqrt{1+\varepsilon^2}}, & N \text{ even} \end{cases} \)

Type II Chebyshev:

\[ \left| H_a(j\omega) \right| \]

- Monotonic passband
- Equiripple stopband

\[ \left| H_a(j\omega) \right|^2 = \frac{1}{1 + \varepsilon^2 \left[ \frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)} \right]^2} \quad (A.20) \]

Design Parameters: \( \Omega_p, \Omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{\lambda} \)

- The formula for the order is the same as with the Type I Chebyshev:

\[ N = \left[ \frac{\cosh^{-1}(\sqrt{A^2-1}/A)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right] = \left[ \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} \right] \quad (A.17) \]

- Next, solve for the zeros:

\[ z_\ell = \frac{\Omega_s}{\cos \left( \frac{(2\ell-1)\pi}{2N} \right)}, \quad 1 \leq \ell \leq N \quad (A.22) \]
Note: if \( N \) is odd, then the \( \frac{N+1}{2} \) th zero is located at \( s = \infty \) (the denominator of \( (A.22) \) will be zero). Practically, this simply means that the order of the denominator is greater than the numerator.

- Then solve for the poles:

\[ p_k = \sigma_k + j\Omega_k, \quad 1 \leq k \leq N \quad (A.23) \]

where
\[
\sigma_k = \frac{\Omega_0 \alpha_k}{\alpha_k^2 + \beta_k^2}, \quad \Omega_k = -\frac{\Omega_0 \beta_k}{\alpha_k^2 + \beta_k^2} \quad (A.24a)
\]

and
\[
\alpha_k = -\Omega_0 \xi \sin \left[ \frac{(2k-1)\pi}{2N} \right], \quad \beta_k = \Omega_0 \xi \cos \left[ \frac{(2k-1)\pi}{2N} \right] \quad (A.24b)
\]

and
\[
\delta = \frac{\gamma^2 - 1}{2\gamma}, \quad \xi = \frac{\delta^2 + 1}{2\delta}, \quad \gamma = (A + \sqrt{A^2 - 1})^{1/N} \quad (A.24c)
\]

- Finally, the transfer function is given by

\[
H_a(s) = C_0 \frac{\prod_{k=1}^{N} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)} \quad (A.21)
\]

\( \rightarrow \) The constant \( C_0 \) is solved by setting

\[
H_a(0) = 1.
\]
Matlab Routines for Type I and Type II Chebyshev lowpass:

Find zeros/poles/gain for normalized filter with $\Omega_c = 1$:
- cheb1ap, cheb2ap

Find minimum order to meet spec:
- cheb1ord, cheb2ord

Find numerator and denominator polynomial coefficients:
- cheby1, cheby2
Low Pass Elliptic Filters

- For a given order, transition band is narrower than either Butterworth or Chebyshew.
- Have ripple in both stop & pass bands. Both equiripple.

\[ |H_e(e)|^2 = \frac{1}{1 + \varepsilon^2 R_n^2(\omega_p)} \]

\[ R_n^2(\omega): \text{rational in } \omega, \text{ satisfies} \]

\[ R_n(\frac{1}{\omega}) = \frac{1}{R_n(\omega)}. \]

- Described cursorly in books, tables or computer programs generally used.

**Matlab**

- ellipap - find p, z, k
- ellip - find num/den coefs
- ellipord - find min order to meet spec
Butterworth, Chebyshev Type I and II, and Elliptical approximations do not constrain the phase.

→ It will not be linear.

→ If linear phase is needed, can be cascaded with an "all pass section" to try to obtain a nearly or approximately linear phase in the passband.

→ This adds complexity.
Low Pass Bessel Filter

\[ H_a(s) = \frac{c_0}{B_n(s)} \]

\[ B_n(s) = n^{th} \text{ order Bessel polynomial} \]

\[ \rightarrow \text{ poor transition band characteristics} \]

\[ \rightarrow \text{ monotonic magnitude (no ripples)} \]

\[ \rightarrow \text{ approximately linear phase in passband} \]

**Matlab**

besselpap find z, p, k

besself  find num/der coefs

\[ \rightarrow \text{ There is no besselpap, b/c these filters are so crappy they won't be used when the transition band must be specified.} \]
WARNING: for Butterworth, Chebyshev, Elliptical, and Bessel filters of order $N \geq 15$,

**DO NOT** use the "$\text{[num, den]} =$" forms in Matlab.

**→ use only** the "$\text{[z, p, k]} =$" forms.

- Other Filter Types:
  - High pass, band pass, and band stop filters are generally designed by:

    A) Transform the spec to an equivalent lowpass spec.

    B) Design the low pass filter

    C) Do an inverse transform to get the desired filter.

→ This is covered in Appendix B of the book.
APPENDIX B: OTHER FILTER TYPES

- Other filter types (highpass, bandpass, band stop) are not designed directly.
- Instead, the design spec is transformed to an equivalent lowpass spec. The lowpass filter is then designed and a final transformation is applied to obtain the desired filter type.

Notation

\[ H_0(\hat{s}) \]: the desired transfer function (not lowpass).
\[ \rightarrow \text{ independent variable } \hat{s}. \]

\[ H_{lp}(s) \]: equivalent lowpass prototype transfer function.
\[ \rightarrow \text{ independent variable } s. \]

\[ s = \sigma + j \omega \]
\[ \hat{s} = \hat{\sigma} + j \hat{\omega} \]

\[ \omega = \frac{1}{2} s \bigg|_{\text{Re}(s) = 0} = \text{Im} \{s\} \]

\[ \hat{\omega} = \frac{\hat{s}}{2} \bigg|_{\text{Re}(\hat{s}) = 0} = \text{Im} \{\hat{s}\} \]

\[ \rightarrow \text{ The equivalent lowpass filter is specified by a transformation } s = F(\hat{s}). \]
High Pass Filter

\[ s = F(\hat{s}) = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} \]  \hspace{1cm} (B.1)

- on the imaginary axes of the \( s \) and \( \hat{s} \) planes, \( s = j\Omega \) and \( \hat{s} = j\hat{\Omega} \). So (B.1) becomes

\[ j\hat{\Omega} = \frac{\Omega_p \hat{\Omega}_p}{j\hat{\Omega}} \Rightarrow \hat{\Omega} = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}} \]  \hspace{1cm} (B.2)

- The inverse transformation is

\[ \hat{s} = F^{-1}(s) = \frac{\Omega_p \hat{\Omega}_p}{s} \]

Given high pass spec : \( |H_0(\hat{s})| \)

---

Passband:
\[ \hat{\Omega} \in [\hat{\Omega}_p, \infty) \rightarrow \hat{\Omega} \in (-\infty, -\hat{\Omega}_p] \rightarrow \Omega \in [0, \Omega_p] \]  \hspace{1cm} (B.2)

Stopband/Transition band
\[ \hat{\Omega} \in [0, \hat{\Omega}_p] \rightarrow \hat{\Omega} \in [-\hat{\Omega}_p, 0] \rightarrow \Omega \in [\Omega_p, \infty) \]  \hspace{1cm} (B.2)
Equivalent lowpass filter:

- To complete the transformation, we need to choose $\Omega_p$.
- It is common to choose $\Omega_p = 1$.
- Then $\Omega_s = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s} = -\frac{\hat{\Omega}_p}{\hat{\Omega}_s}$.

- $A$ and $\varepsilon$ are the same for both filters.

- So:
  - Given $\hat{\Omega}_p$ and $\hat{\Omega}_s$, you choose $\Omega_p$ (usually $\Omega_p = 1$).
  - You solve $\Omega_s$.
  - You now have an equivalent lowpass design spec.

- Then:
  - You design the lowpass filter $H_{lp}(s)$. 

$\rightarrow$
→ Everywhere that you have $s$ in $H_p(s)$, substitute $s = \frac{\zeta_p \zeta_p^*}{\zeta^*}$.

→ Simplify to obtain $H_p(\zeta^*)$, the desired high pass filter.

→ See example B2 in the book.

→ Main Matlab routine: $lp2hp$.

Note: the Matlab butter, cheby, ellipf functions can be called with type parameters to design highpass, etc., filters directly.
For the transformation to work, the spec must satisfy
\[ \hat{\Omega}_p, \hat{\Omega}_s = \hat{\Omega}_{s_1}, \hat{\Omega}_{s_2}. \] (B.5)

If this isn't the case, then you need to adjust one or more of the frequencies to obtain a new spec that everywhere meets or exceeds the original spec and also satisfies (B.5).

Now define \( \hat{\Omega}_0 \) to be the geometric mean of \( \hat{\Omega}_{p_1} \) and \( \hat{\Omega}_{p_2} \) (and also of \( \hat{\Omega}_{s_1} \) and \( \hat{\Omega}_{s_2} \)):
\[ \hat{\Omega}_0^2 = \hat{\Omega}_{p_1} \hat{\Omega}_{p_2} = \hat{\Omega}_{s_1} \hat{\Omega}_{s_2}. \] (B.5)
In the book, \( \hat{\Omega}_0 \) is called the "center frequency" of the passband...

which is a bit misleading since it's not in the center (like the arithmetic mean would be)... the symmetry is that of the geometric mean.

The transformation to obtain an equivalent low pass design \( H_{lp}(s) \) maps \( \hat{\Omega}_0 \) to DC ... \( \Omega = 0 \).

\[
S = \mathcal{F}(s) = -\Omega_p \frac{\hat{S}^2 + \hat{\Omega}_0^2}{\hat{S}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} \tag{B.3}
\]

You are free to choose \( \Omega_p \).

Typical: choose \( \Omega_p = 1 \).

On the \( j\Omega \)-axis, \( S = j\Omega \), \( \hat{S} = j\hat{\Omega} \) and (B.3) becomes

\[
j\Omega = \Omega_p \frac{(j\hat{\Omega})^2 + \hat{\Omega}_0^2}{j\hat{\Omega}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} = \Omega_p \frac{-\hat{\Omega}^2 + \hat{\Omega}_0^2}{j\hat{\Omega}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}.
\]

Multiply both sides by \( j \): 

\[
\Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}_0^2}{\hat{\Omega}_{p2} - \hat{\Omega}_{p1}} \tag{B.4}
\]
At $\hat{s} = j\hat{\omega}_0$, we have

$$s = F(s) \bigg|_{s = j\hat{\omega}_0} = -\frac{\hat{\omega}_p}{\hat{\omega}_0} \frac{\hat{\omega}_0^2 - \hat{\omega}_p^2}{\hat{\omega}_{p_2} - \hat{\omega}_{p_1}} = 0,$$

→ which shows that the transformation maps the "center frequency" $\hat{\omega}_0$ in $H_0(s)$ to the point $s = 0$ in $H_{lp}(s)$.

Equivalent prototype analog lowpass filter:

$$H_{lp}(\omega)$$

Procedure:

- Given $\hat{\omega}_{p_1}, \hat{\omega}_{p_2}, \hat{\omega}_{s_1}, \hat{\omega}_{s_2}, A, \epsilon$ (the design parameters)

1. check if $\hat{\omega}_{p_1} \hat{\omega}_{p_2} = \hat{\omega}_{s_1} \hat{\omega}_{s_2}$. If not, adjust spec by moving $\hat{\omega}_{s_1}$ up and/or moving $\hat{\omega}_{s_2}$ down.

2. Solve for $\hat{\omega}_0 = \sqrt{\hat{\omega}_{p_1} \hat{\omega}_{p_2}}$.

3. Choose $\hat{\omega}_p$. Usually, we set $\hat{\omega}_p = 1$.

4. Solve for $\hat{\omega}_s$.
   - You can plug in $\hat{\omega}_{s_2}$ for $\hat{\omega}$ in (B.4) to get $\omega_s$,
   - or you can plug in $\hat{\omega}_{s_1}$ for $\hat{\omega}$ in (B.4) to get $-\omega_s$.
   → You get the same answer either way.

⇒ Now you have an equivalent lowpass spec.
- Design $H_p(s)$ using the techniques we've already discussed.

- Use $\hat{s} = F^{-1}(s)$ to convert the designed low pass filter $H_p(s)$ into the desired bandpass filter $H_b(\hat{s})$.

- Discretize to obtain the discrete bandpass filter $H_d(e^{j\omega})$.

Matlab: lpr2bp.

- In this case, $F^{-1}(s)$ is a little bit messy.
- Starting with (B.3), we have

$$s = \Omega_p \frac{\hat{s}^2 + \hat{s}_0^2}{\hat{s}(\hat{s}_p + \hat{s}_p^2)}$$

- Cross multiply:

$$\hat{s} \left[ s \left( \hat{s}_p - \hat{s}_p^2 \right) \right] = \Omega_p \hat{s}^2 + \Omega_p \hat{s}_0^2$$

$$\hat{s} \left[ \frac{s}{\Omega_p} \left( \hat{s}_p - \hat{s}_p^2 \right) \right] = \hat{s}^2 + \hat{s}_0^2$$
\[ \frac{\Lambda^2}{S} - \left[ \frac{S}{\Delta_p} (\hat{\Lambda}_{p_2} - \hat{\Lambda}_{p_1}) \right] \hat{S} + \hat{\Lambda}_0^2 = 0 \]

- Quadratic formula:

\[ \hat{S} = \frac{\frac{S}{\Delta_p} (\hat{\Lambda}_{p_2} - \hat{\Lambda}_{p_1}) \pm \sqrt{\frac{S^2}{\Delta_p^2} (\hat{\Lambda}_{p_2} - \hat{\Lambda}_{p_1})^2 - 4 \hat{\Lambda}_0^2}}{2} \]

- Band Stop filter is similar; see book.