ECE 4213/5213

MODULE AB

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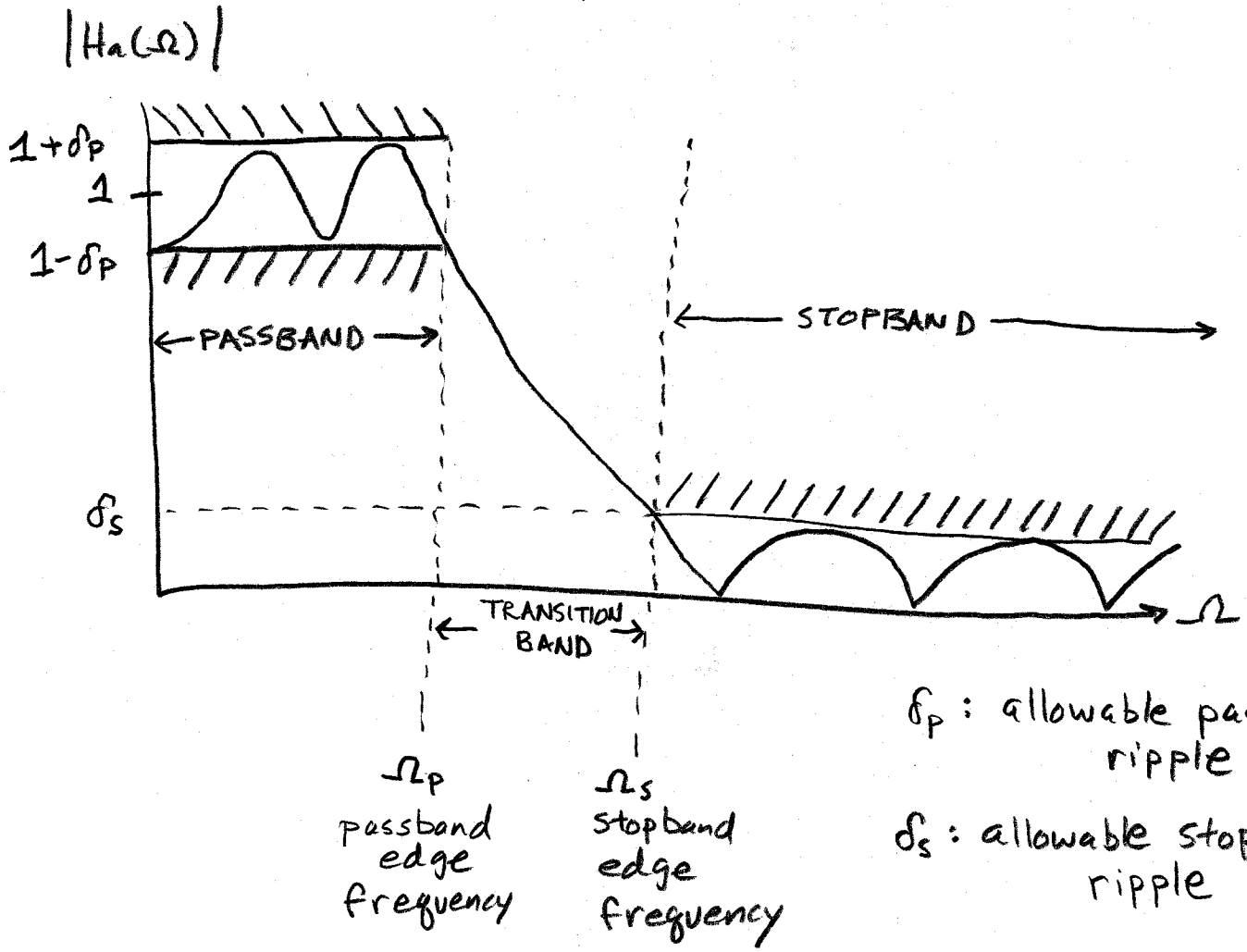
APPENDIX A: ANALOG FILTER DESIGN

- Several important techniques for designing digital IIR filters are based on analog filter prototypes.

- To design the digital filter, you begin by transforming the digital specification to an analog spec and doing an analog design.

- The problem of designing an analog filter is fundamentally to design an analog function $|H_a(\Omega)|$ that satisfies the passband and stopband requirements.
- The general problem of how to design a curve with a desired shape is difficult and has been studied for centuries.
  - To simplify things, a low pass filter is almost always designed.
  - If another type is needed (e.g., highpass, bandpass, bandstop), it is obtained by transforming a low pass design.
- We will limit our attention to the main results that are useful for designing digital filters.

Analog filter magnitude design spec:



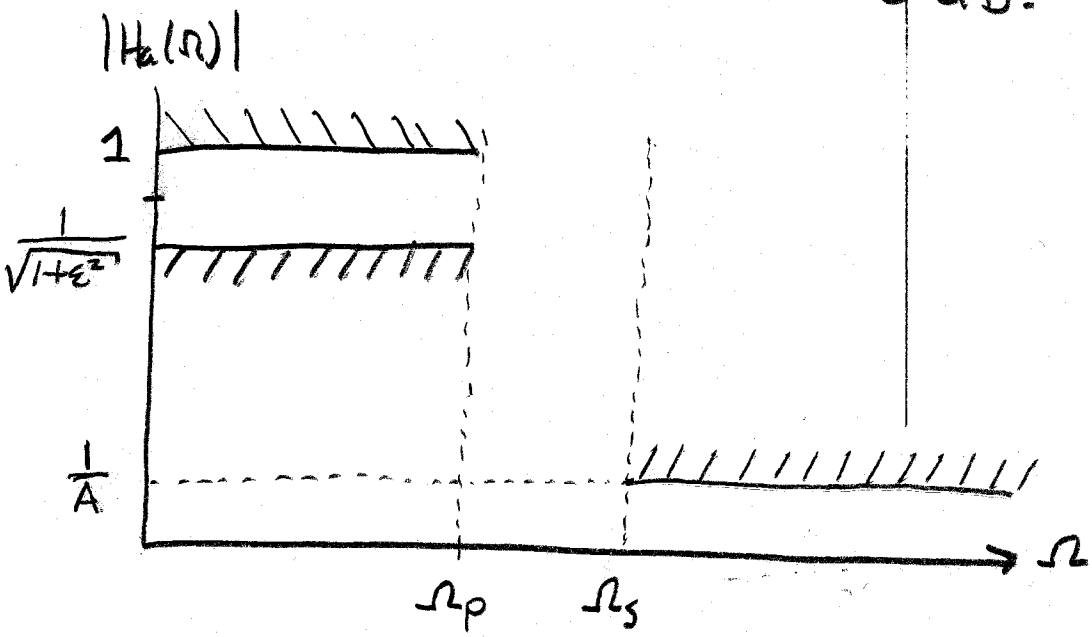
δ_p : allowable passband ripple

δ_s : allowable stopband ripple

- The ripples are usually specified in dB:
  - peak passband ripple: $\alpha_p = -20\log_{10}(1-\delta_p)$ dB
  - minimum stopband attenuation: $\alpha_s = -20\log_{10}(\delta_s)$ dB

Transition Ratio: $k = \dfrac{\Omega_p}{\Omega_s}$

- The spec is also often stated in a normalized form where the peak passband gain is normalized to 1 ... or 0 dB:



Max passband gain: 0 dB

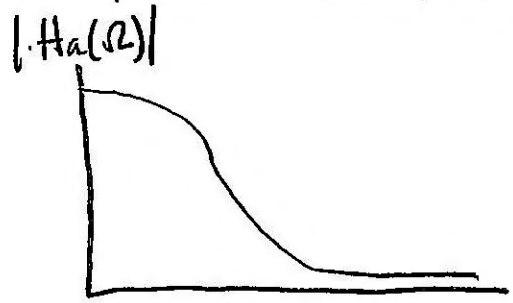
Max stopband ripple: $\frac{1}{A}$

min stopband attenuation: $-20\log_{10}\left(\frac{1}{A}\right)$ dB

Discrimination Parameter: $k_1 = \frac{\varepsilon}{\sqrt{A^2-1}}$

- usually $k_1 \ll 1$.

Lowpass Butterworth Filter:



- Monotonic passband
- Monotonic stopband (e.g. no ripple).

$$|H_a(\Omega)|^2 = \frac{1}{1 + \frac{\Omega^{2N}}{\Omega_c^{2N}}} \quad (A.7)$$

N = filter order
Ω_c = 3dB cutoff freq.

→ $H_a(\Omega)$ is fully specified by the two parameters N, Ω_c.

- The Butterworth filter is "maximally flat" at DC:
→ The first $2N-1$ derivatives of $|H_a(\Omega)|^2$ are zero at $\Omega = 0$.

Gain: $G(\Omega) = 10\log_{10}|H_a(\Omega)|^2$ dB

@ Ω_c, $G(\Omega_c) = 10\log_{10}\frac{1}{2} \approx -3$dB

@ DC, $G(0) = 0$ dB

For large Ω, $|H_a(\Omega)|^2 \approx \dfrac{1}{(\Omega/\Omega_c)^{2N}}$

Let $\Omega_1 \gg \Omega_c$ and $\Omega_2 = 2\Omega_1$.

Then $G(\Omega_2) = G(\Omega_1) - 6N$ dB

→ Butterworth filter rolloff is 6 dB per octave / 20dB per decade for each increase in the filter order.

The design parameters are:

$$\Omega_p,\ \Omega_s,\ \frac{1}{\sqrt{1+\varepsilon^2}},\ \frac{1}{A}.$$

Design:

- First find the order

$$N = \left\lceil \frac{1}{2}\,\frac{\log_{10}\left[(A^2-1)/\varepsilon^2\right]}{\log_{10}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} \right\rceil \qquad (A.9)$$

- Ω_c can then be solved from

$$|H_a(\Omega_p)|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2} \qquad (A.8a)$$

or

$$|H_a(\Omega_s)|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2} \qquad (A.8b)$$

→ usually use (A.8b) to meet the stopband spec exactly while exceeding the passband spec.

- Solve for the poles:

$$p_\ell = \Omega_c e^{j[\pi(N+2\ell-1)/2N]},\quad 1 \le \ell \le N \qquad (A.11)$$

- Design $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^N}{\prod_{\ell=1}^{N}(s-p_\ell)} \qquad (A.10)$$

Butterworth design in Matlab:

- Normalized forms with $\Omega_c = 1$, order $= N$:

```
[z,p,g] = buttap(N);
```

null, poles, gain, order

```
[N,D] = zp2tf(z,p,g);
```

numerator and denominator polynomial coefficients:

$$H_a(s) = \frac{N(s)}{D(s)} = \frac{g}{\prod_{\ell=1}^{N}(s-p_\ell)}$$

General forms:

```
[N,Wn] = buttord(Wp, Ws, Rp, Rs, 's');
```

passband edge freq. in rad/sec.

stopband edge freq. in rad/sec.

min order

3 dB cutoff freq. in rad/sec

max passband loss in dB

min stopband attenuation in dB

'analog'

```
[num, den] = butter(N, Wn, 's');
```

N+1 numerator coefs.

N+1 denominator coefs.

order

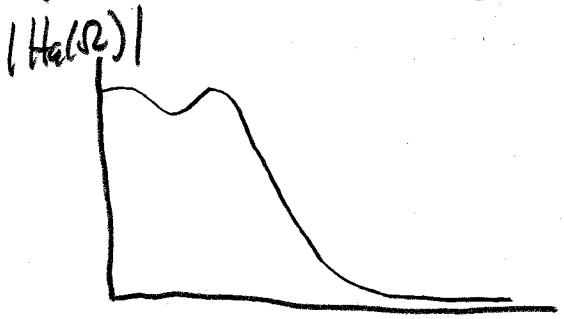
3dB cutoff freq in rad/sec

analog

Lowpass Chebyshev Filters:

- Achieve a narrower transition band than Butterworth for the same order.
- Have ripple in the stopband or in the passband, but not both.

Type I Chebyshev:



- Equiripple passband
- Monotonic stopband

$$|H_a(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)} \quad (A.13)$$

- $T_N(\Omega)$ is an Nth-order Chebyshev polynomial.

- Design parameters: $\Omega_p, \Omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{A}$
- Design:
  - First find the order:

$$N = \left\lceil \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} \right\rceil \quad (A.17)$$

  - Then find the poles:

$$p_\ell = \sigma_\ell + j\Omega_\ell, \quad 1 \le \ell \le N \quad (A.18)$$

where

$$\sigma_\ell = -\Omega_p \xi \sin\left[\frac{(2\ell-1)\pi}{2N}\right], \quad \Omega_\ell = \Omega_p \zeta \cos\left[\frac{(2\ell-1)\pi}{2N}\right] \quad (A.19a)$$

and

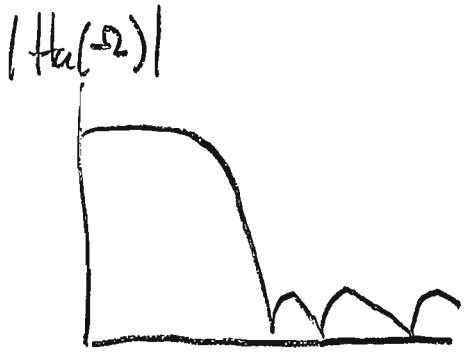
$$\xi = \frac{\gamma^2-1}{2\gamma}, \quad \zeta = \frac{\gamma^2+1}{2\gamma}, \quad \gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{1/N} \quad (A.19b)$$

Transfer Function:

$$H_a(s) = C_0 \prod_{\ell=1}^{N} \frac{-p_\ell}{s - p_\ell},$$

$$\text{where } C_0 = \begin{cases} 1, & N \text{ odd} \\ \frac{1}{\sqrt{1+\varepsilon^2}}, & N \text{ even} \end{cases}.$$

Type II Chebyshev:



- Monotonic passband
- Equiripple stopband

$$|H_a(\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left[\frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}\right]^2} \qquad (A.20)$$

Design Parameters: $\Omega_p, \Omega_s, \frac{1}{\sqrt{1+\varepsilon^2}}, \frac{1}{A}$

- The formula for the order is the same as with the Type I Chebyshev:

$$N = \left\lceil \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} \right\rceil \qquad (A.17).$$

- Next, solve for the zeros:

$$z_\ell = \frac{j\Omega_s}{\cos\left[\frac{(2\ell-1)\pi}{2N}\right]}, \quad 1 \le \ell \le N \qquad (A.22)$$

Note: if N is odd, then the $\frac{N+1}{2}$ th zero is located at $s=\infty$ (the denominator of (A.22) will be zero).

→ practically, this simply means that the order of the denominator is greater than the numerator.

- Then solve for the poles:

$$p_\ell = \sigma_\ell + j\Omega_\ell, \quad 1 \le \ell \le N \qquad (A.23)$$

where

$$\sigma_\ell = \frac{\Omega_s \alpha_\ell}{\alpha_\ell^2 + \beta_\ell^2}, \quad \Omega_\ell = -\frac{\Omega_s \beta_\ell}{\alpha_\ell^2 + \beta_\ell^2} \qquad (A.24\,a)$$

and

$$\alpha_\ell = -\Omega_p \zeta \sin\left[\frac{(2\ell-1)\pi}{2N}\right], \quad \beta_\ell = \Omega_p \xi \cos\left[\frac{(2\ell-1)\pi}{2N}\right] \qquad (A.24b)$$

and

$$\zeta = \frac{\gamma^2-1}{2\gamma}, \quad \xi = \frac{\gamma^2+1}{2\gamma}, \quad \gamma = \left(A + \sqrt{A^2-1}\right)^{1/N} \qquad (A.24c)$$

- Finally, the transfer function is given by

$$H_a(s) = C_0 \frac{\prod_{\ell=1}^{N} (s - z_\ell)}{\prod_{\ell=1}^{N} (s - p_\ell)} \qquad (A.21)$$

→ The constant C_0 is solved by setting

$$H_a(0) = 1.$$

Matlab Routines for Type I and Type II Chebyshev lowpass:

Find zeros/poles/gain for normalized filter with $\Omega_c = 1$: cheb1ap, cheb2ap

Find minimum order to meet spec: cheb1ord, cheb2ord

Find numerator and denominator polynomial coefficients: cheby1, cheby2

Low Pass Elliptic Filters

- For a given order, transition band is narrower than either Butterworth or Chebyshev.

- have ripple in both stop & pass bands. Both equiripple.

$$|H_a(\Omega)|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\Omega/\Omega_p)}$$

$R_N^2(\Omega)$: rational in Ω, satisfies

$$R_N\left(\frac{1}{\Omega}\right) = \frac{1}{R_N(\Omega)}.$$

- described cursorily in book; tables or computer programs generally used.

Matlab

ellipap - find p, z, k
ellip - find num/den coefs
ellipord - find min order to meet spec

Butterworth, Chebyshev Type I and II, A-12
and Elliptical approximations do
not constrain the phase.

→ It will not be linear.

→ If linear phase is needed, can be
cascaded with an "all pass section".
to try to obtain a nearly or
approximately linear phase in the
passband.

→ This adds complexity.

<u>WARNING</u>: for Butterworth, Chebyshev, Elliptical, and Bessel filters of order $N \geq 15$,

<u>DO</u> <u>NOT</u> use the "[num, den] =" forms in Matlab.

→ use <u>only</u> the "[z, p, k] =" forms.

- Other Filter Types:

- High pass, band pass, and band stop filters are generally designed by:

  A) Transform the spec to an equivalent lowpass spec.

  B) Design the low pass filter

  C) Do an inverse transform to get the desired filter.

APPENDIX B: OTHER FILTER TYPES

- other filter types (highpass, bandpass, band stop) are not designed directly.
- Instead, the design spec is transformed to an equivalent lowpass spec. The lowpass filter is then designed and a final transform is applied to obtain the desired filter type.

Notation

$H_D(\hat{s})$: the desired transfer function (not lowpass).
→ independent variable $\hat{s}$.

$H_{LP}(s)$: equivalent lowpass prototype transfer function.
→ independent variable s.

$$s = \sigma + j\Omega \qquad \hat{s} = \hat{\sigma} + j\hat{\Omega}$$

$$\Omega = \frac{1}{j} s \Big|_{Re[s]=0} = Im[s]$$

$$\hat{\Omega} = \frac{\hat{s}}{j} \Big|_{Re[\hat{s}]=0} = Im[\hat{s}]$$

→ The equivalent lowpass filter is specified by a transformation $s = F(\hat{s})$.

High Pass Filter

$$s = F(\hat{s}) = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} \qquad (B.1)$$

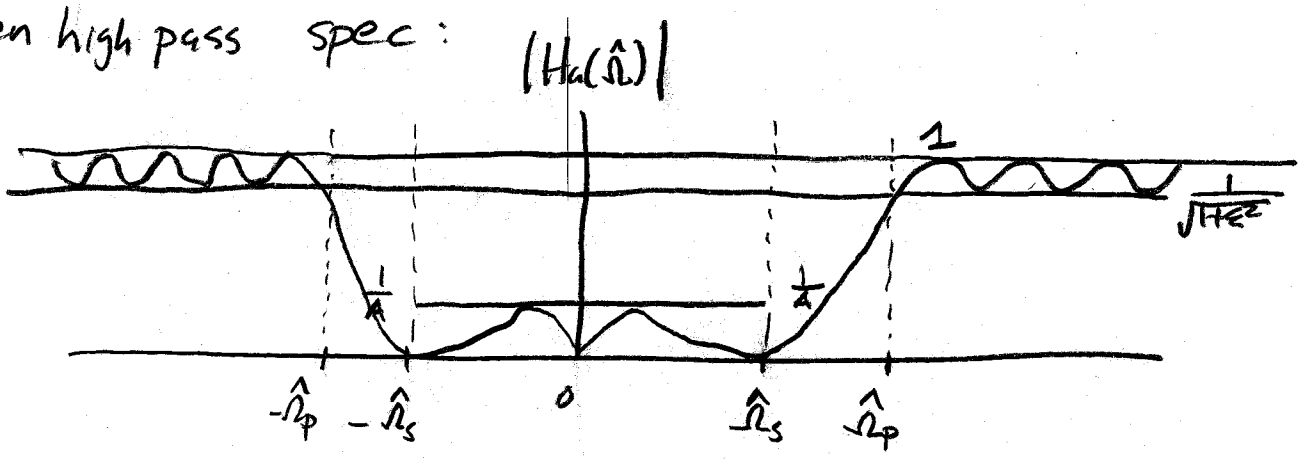
- on the imaginary axes of the s and $\hat{s}$ planes, $s = j\Omega$ and $\hat{s} = j\hat{\Omega}$. So (B.1) becomes

$$j\Omega = \frac{\Omega_p \hat{\Omega}_p}{j\hat{\Omega}} \Longrightarrow \Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}} \qquad (B.2)$$

- The inverse transformation is

$$\hat{s} = F^{-1}(s) = \frac{\Omega_p \hat{\Omega}_p}{s}.$$

Given high pass spec:



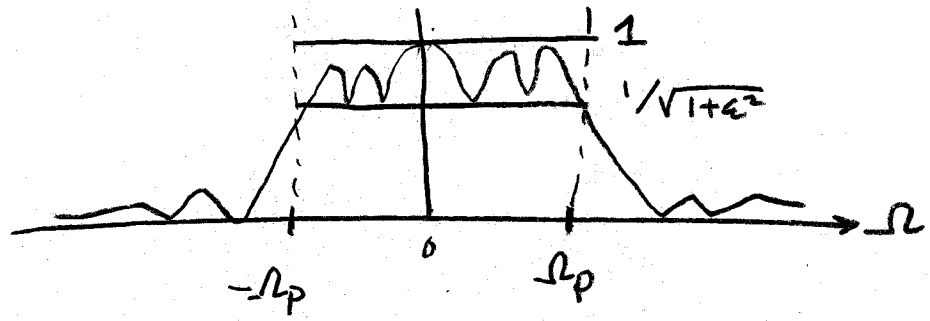
Passband:

$$\hat{\Omega} \in [\hat{\Omega}_p, \infty) \longmapsto \hat{\Omega} \in (-\infty, -\hat{\Omega}_p] \xmapsto{(B.2)} \Omega \in [0, \Omega_p]$$

Stopband/Transition band

$$\hat{\Omega} \in [0, \hat{\Omega}_p] \longmapsto \hat{\Omega} \in [-\hat{\Omega}_p, 0] \xmapsto{(B.2)} \Omega \in [\Omega_p, \infty)$$

Equivalent lowpass filter:



- To complete the transformation, we need to choose Ω_p.
- It is common to choose $\Omega_p = 1$.
- Then $\Omega_s = -\dfrac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s} = -\dfrac{\hat{\Omega}_p}{\hat{\Omega}_s}$.
- A and ε are the same for both filters.
- So:
  - Given $\hat{\Omega}_p$ and $\hat{\Omega}_s$, you choose Ω_p (usually $\Omega_p = 1$).
  - You solve Ω_s.
  - You now have an equivalent low pass design spec.
- Then:
  - You design the lowpass filter $H_{LP}(s)$.

$\longrightarrow$

→ Everywhere that you have s in $H_{LP}(s)$, substitute

$$s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}} .$$

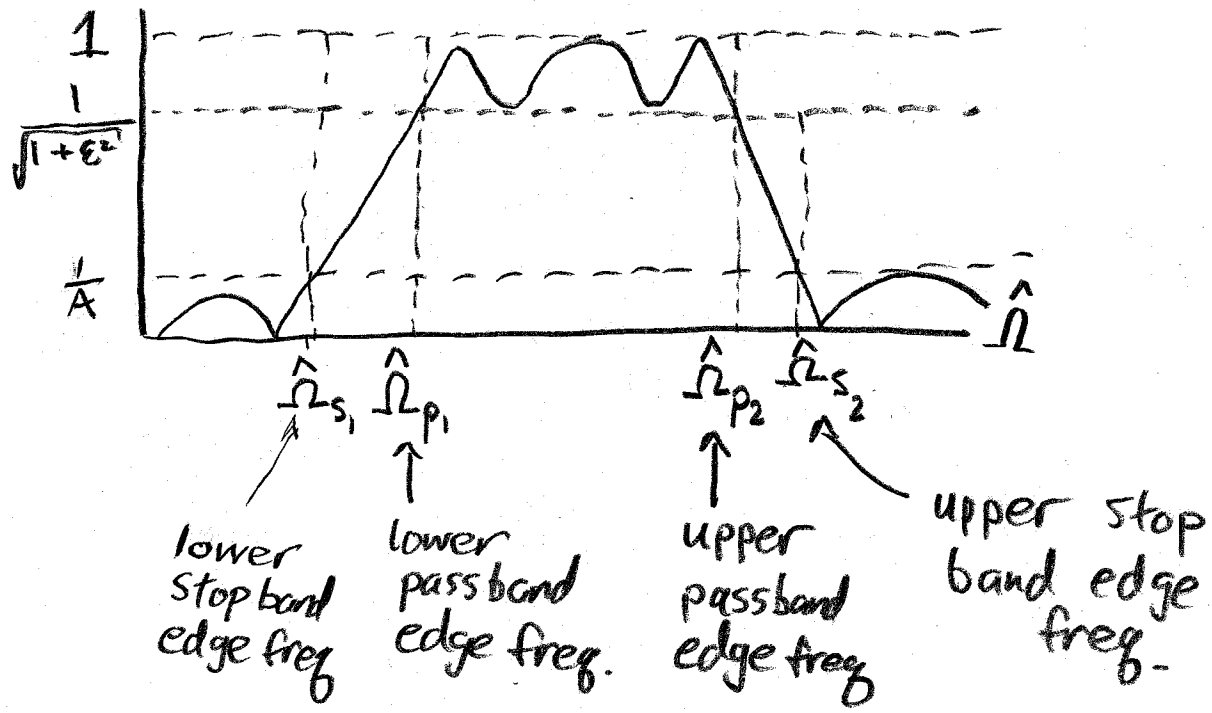
→ Simplify to obtain $H_D(\hat{s})$, the desired high pass filter

→ Main Matlab routine: $l_p 2 h_p$.

<u>Note:</u> the Matlab butter, cheby, ellipf functions can be called with type parameters to design highpass, etc., filters directly.

BANDPASS FILTER

DESIRED FILTER: $H_{BP}(\hat{\Omega})$



- For the transformation to work, the spec must satisfy

$$\hat{\Omega}_{p_1}\hat{\Omega}_{p_2} = \hat{\Omega}_{s_1}\hat{\Omega}_{s_2}. \quad (B.5)$$

$\Rightarrow$ If this isn't the case, then you need to adjust one or more of the frequencies to obtain a new spec that everywhere meets or exceeds the original spec and also satisfies (B.5)

- Now define $\hat{\Omega}_0$ to be the geometric mean of $\hat{\Omega}_{p_1}$ and $\hat{\Omega}_{p_2}$ (and also of $\hat{\Omega}_{s_1}$ and $\hat{\Omega}_{s_2}$):

$$\hat{\Omega}_0^2 = \hat{\Omega}_{p_1}\hat{\Omega}_{p_2} = \hat{\Omega}_{s_1}\hat{\Omega}_{s_2} \quad (B.5)$$

- Often, $\hat{\Omega}_0$ is called the "center frequency" of the passband...
  - which is a bit misleading since it's not in the center (like the arithmetic mean would be)... the symmetry is that of the geometric mean.
- The transformation to obtain an equivalent low pass design $H_{LP}(s)$ maps $\hat{\Omega}_0$ to DC ... $\Omega = 0$.

$$s = F(\hat{s}) = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_0^2}{\hat{s}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})} \qquad (B.3)$$

  - You are free to choose Ω_p.
    Typical: choose $\Omega_p = 1$.
- on the $j\Omega$-axis, $s = j\Omega$, $\hat{s} = j\hat{\Omega}$ and (B.3) becomes

$$j\Omega = \Omega_p \frac{(j\hat{\Omega})^2 + \hat{\Omega}_0^2}{j\hat{\Omega}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})} = \Omega_p \frac{-\hat{\Omega}^2 + \hat{\Omega}_0^2}{j\hat{\Omega}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}.$$

Multiply both sides by $\frac{1}{j}$:

$$\Omega = \frac{-\Omega_p}{\hat{\Omega}} \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}} \qquad (B.4)$$

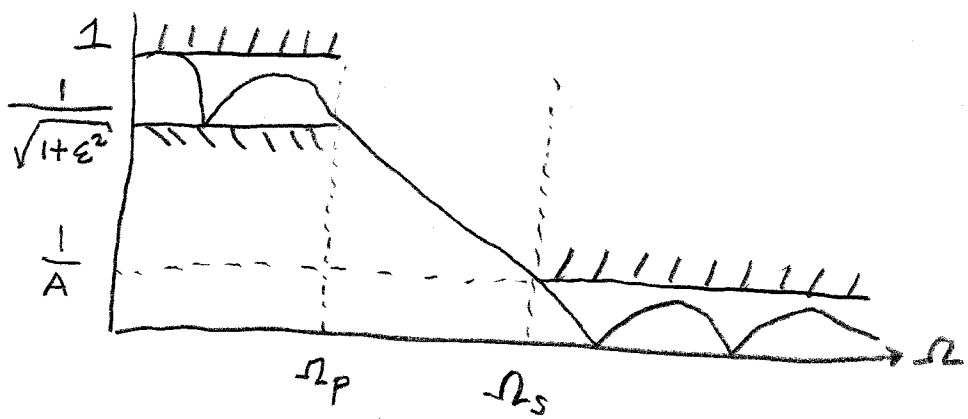
At $\hat{s} = j\hat{\Omega}_0$, we have

$$s = F(\hat{s})\Big|_{\hat{s}=j\hat{\Omega}_0} = \frac{-\Omega_p}{\hat{\Omega}_0}\,\frac{\hat{\Omega}_0^2 - \hat{\Omega}_0^2}{\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}} = 0,$$

→ which shows that the transformation maps the "center frequency" $\hat{\Omega}_0$ in $H_D(\hat{s})$ to the point $\Omega = 0$ in $H_{LP}(s)$.

Equivalent prototype analog lowpass filter:

$|H_{LP}(\Omega)|$



<u>Procedure:</u>

- Given $\hat{\Omega}_{p_1}, \hat{\Omega}_{p_2}, \hat{\Omega}_{s_1}, \hat{\Omega}_{s_2}, A, \varepsilon$ (the design parameters)

① Check if $\hat{\Omega}_{p_1}\hat{\Omega}_{p_2} \overset{?}{=} \hat{\Omega}_{s_1}\hat{\Omega}_{s_2}$. If not, adjust spec by moving $\hat{\Omega}_{s_1}$ up and/or moving $\hat{\Omega}_{s_2}$ down.

② Solve for $\hat{\Omega}_0 = \sqrt{\hat{\Omega}_{p_1}\hat{\Omega}_{p_2}}$.

③ Choose Ω_p. Usually, we set $\Omega_p = 1$.

④ Solve for Ω_s.

- You can plug in $\hat{\Omega}_{s_2}$ for $\hat{\Omega}$ in (B.4) to get Ω_s,
- or you can plug in $\hat{\Omega}_{s_1}$ for $\hat{\Omega}$ in (B.4) to get $-\Omega_s$.

→ You get the same answer either way.

⇒ Now you have an equivalent low pass spec.

- Design $H_{LP}(s)$ using the techniques we've already discussed.
  - Use $\hat{s} = F^{-1}(s)$ to convert the designed low pass filter $H_{LP}(s)$ into the desired bandpass filter $H_D(\hat{s})$.
- Discretize to obtain the discrete bandpass filter $H_d(e^{j\omega})$.

Matlab: lp2bp.

- In this case, $F^{-1}(s)$ is a little bit messy.
- Starting with (B.3), we have

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_0^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}$$

- Cross multiply:

$$\hat{s}\left[s(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})\right] = \Omega_p \hat{s}^2 + \Omega_p \hat{\Omega}_0^2$$

$$\hat{s}\left[\frac{s}{\Omega_p}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})\right] = \hat{s}^2 + \hat{\Omega}_0^2$$

$\longrightarrow$

$$\hat{s}^2 - \left[\frac{s}{\Omega_p}\left(\hat{\Omega}_{p2} - \hat{\Omega}_{p1}\right)\right]\hat{s} + \hat{\Omega}_0^2 = 0$$

- Quadratic formula:

$$\hat{s} = \frac{\frac{s}{\Omega_p}\left(\hat{\Omega}_{p2} - \hat{\Omega}_{p1}\right) \pm \sqrt{\frac{s^2}{\Omega_p^2}\left(\hat{\Omega}_{p2} - \hat{\Omega}_{p1}\right)^2 - 4\hat{\Omega}_0^2}}{2}$$