DSP STOCHASTICS HANDOUT

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ECE 5213/4213 STOCHASTICS HANDOUT

- This handout summarizes the important results and gives examples.

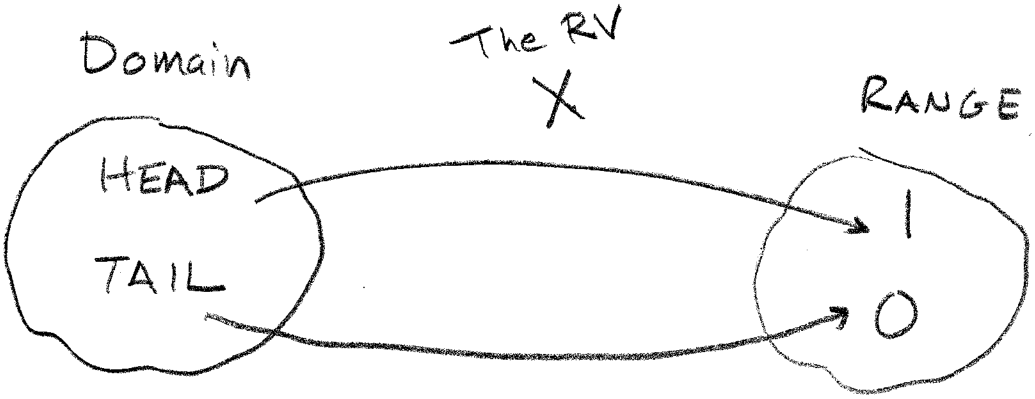
- You may bring this handout to the final exam.

A random variable is a <u>function</u> that maps the outcomes of a chance experiment to numbers.

→ Domain: the set of possible outcomes of the chance experiment.

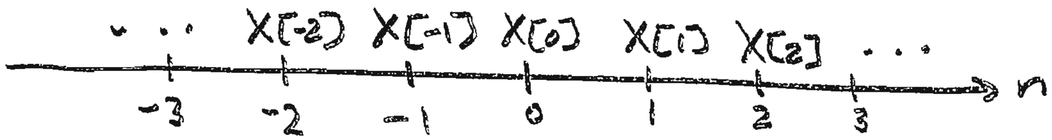
→ Range: a set of numbers (like $\mathbb{R}$, or $\mathbb{Z}$, or $\mathbb{C}$).

<u>EX</u>: Coin Toss:



The random variable (RV) X maps the outcome "HEAD" to the number 1 and the outcome "TAIL" to the number 0.

- Suppose you have a whole bunch of RV's that all have the same domain (the same "underlying experiment space").
  - For each trial of the experiment, every random variable maps the outcome to a number.
  - For each trial, this gives you a bunch of numbers.
- Let's call the RV's ... X[-1], X[0], X[1], X[2], ... and line them up on an "n" axis:



→ This is called a discrete time "random process" or "stochastic process."

- For each trial of the experiment, each RV gets an outcome. Every one of the RV's maps this outcome to a number.

  → That makes a <u>deterministic</u> signal x[n] that is just like the ones from your undergrad signals and systems course

  → This x[n] is called a "sample function" or a "realization" of the stochastic process.

NOTE : Usually, we use the same notation "$x[n]$" to refer to the stochastic process or to a sample function. Which one is meant should be clear from the context.

⇒ To model deterministic signals, we use functions like $x[n] = (\frac{1}{2})^n u[n]$.

⇒ To model statistical signals, we use a stochastic process $x[n]$.

- The autocorrelation function of the stochastic process $x[n]$ is the expected value
  - of the product
    - of two of the RV's from the process.

$$R_x(m,l) = E\Big[x[m]\,x[l]\Big].$$

- If the 2nd order statistics of the RV's from the process don't change with time, then the process is called "Wide Sense Stationary" (WSS).
- This says that all the RV's $x[n]$ have the same mean and the same variance.

☆ For a WSS stochastic process, the autocorrelation function $R_x(m,\ell) = E\left[x[m]x[\ell]\right]$ depends <u>only</u> on the distance <u>between</u> m and ℓ.

- let's call the distance "k", so that $k=\ell-m$.

- For a WSS stochastic process $x[n]$, the autocorrelation is given by $R_x[k] = E\left[x[n]x[n+k]\right]$.

NOTE: this does <u>not</u> depend on "n". It only depends on the distance "k", which is usually called the <u>lag</u>.

NOTE: $R_x[k] = E\left[x[n]\,x[n+k]\right] = E\left[x[n+k]x[n]\right] = R_x[-k]$.

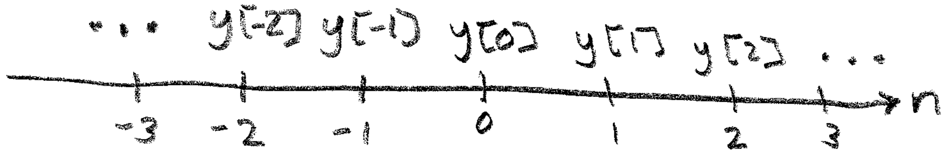
$\Rightarrow$ So the autocorrelation function for a WSS process is <u>always</u> <u>even</u>.

NOTE: For a WSS process, the autocorrelation $R_x[k]$ is like a 1-D discrete-time signal.

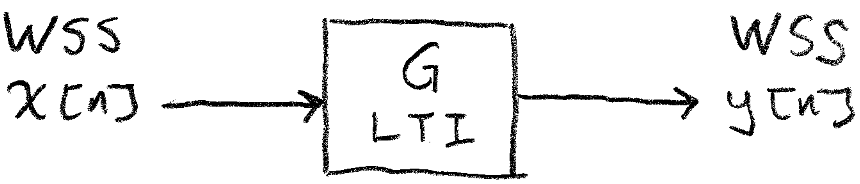
- It has a z-transform $S_x(z) = \mathcal{Z}\{R_x[k]\}$.
  - $\rightarrow$ Since $R_x[k]$ is <u>even</u>, it is two-sided.
    - $\Rightarrow$ The ROC of $S_x(z)$ is <u>always</u> an an <u>annulus</u>.
- It has a DTFT $S_x(e^{j\omega}) = \text{DTFT}\{R_x[k]\}$.

☆ $S_x(z)$ and $S_x(e^{j\omega})$ are <u>both</u> called the "Power Spectrum" of the WSS stochastic process $x[n]$.

- Now suppose you have a second batch of random variables.
- Let's call them ... $y[-1], y[0], y[1], y[2], \ldots$
- If you line them up on an "n" axis, then you will have a second stochastic process $y[n]$:



- Suppose that G is an LTI system with impulse response $g[n]$, frequency response $G(e^{j\omega})$, and transfer function $G(z)$.
- We can use the stochastic process $x[n]$ to model the system input and the stochastic process $y[n]$ to model the system output.
- If $x[n]$ is a WSS stochastic process and if G is LTI, then $y[n]$ is <u>also</u> a WSS stochastic process:



- So the autocorrelation of $y[n]$ is a function of only one variable:

$$R_y[k] = E\big[y[n]y[n+k]\big].$$

NOTE:
$R_y[k]$ is always even.

- The power spectrum of the process $y[n]$ is given by

$$S_y(z) = \mathcal{Z}\{R_y[k]\}$$

or

$$S_y(e^{j\omega}) = \text{DTFT}\{R_y[k]\}.$$

$\Rightarrow$ Moreover, the input $x[n]$ and output $y[n]$ are jointly WSS. This means that the input-output cross-correlation function R_{xy} is a function of only one variable:

$$R_{xy}[k] = E\big[x[n]y[n+k]\big].$$

★ It is given by

$$\begin{aligned} R_{xy}[k] &= R_x[k] * g[k] \\ &= \sum_{m=-\infty}^{\infty} R_x[m]g[m-k] \\ &= \sum_{m=-\infty}^{\infty} R_x[m-k]g[m], \end{aligned}$$

- The input-output cross power is given by

$$S_{xy}(z) = \mathcal{Z}\{R_{xy}[k]\}$$

$$S_{xy}(e^{j\omega}) = \text{DTFT}\{R_{xy}[k]\}.$$

DEF: A stochastic process $x[n]$ is called "independent, identically distributed" or "IID" if all the RV's have the same pdf and they are all mutually independent.

- This implies that all the RV's $x[n]$ have the *same mean* μ_x and the *same variance* σ_x^2.
- This also implies that the process $x[n]$ is WSS.

DEF: A stochastic process $x[n]$ is called "white noise" if the power spectrum $S_x(e^{j\omega})$ is *flat*. In other words, $S_x(e^{j\omega})$ and $S_x(z)$ are *constant*.

- This implies that all of the RV's $x[n]$ have *zero mean*.
- This also implies that the autocorrelation $R_x[k]$ is a constant times $\delta[k]$.

- If $x[n]$ is a stochastic process that is white and IID, then it is WSS and zero mean.
  - The autocorrelation is given by

$$R_x[k] = E\left[x[n]\,x[n+k]\right] = \sigma_x^2\delta[k] = \begin{cases} \sigma_x^2, & k=0 \\ 0, & k\neq 0. \end{cases}$$

$\Rightarrow$ The RVs $x[n]$ and $x[n+k]$ are uncorrelated if $k\neq 0$.

$\rightarrow$ If $k=0$, then

$$R_x[k] = R_x[0] = E\left[x^2[n]\right] = \sigma_x^2,$$

since the mean is zero.

Note: this variance does <u>not</u> depend on "n". Since the process is WSS, all the RVs $x[n]$ have the <u>same</u> <u>variance</u>.

  - The power spectrum is given by

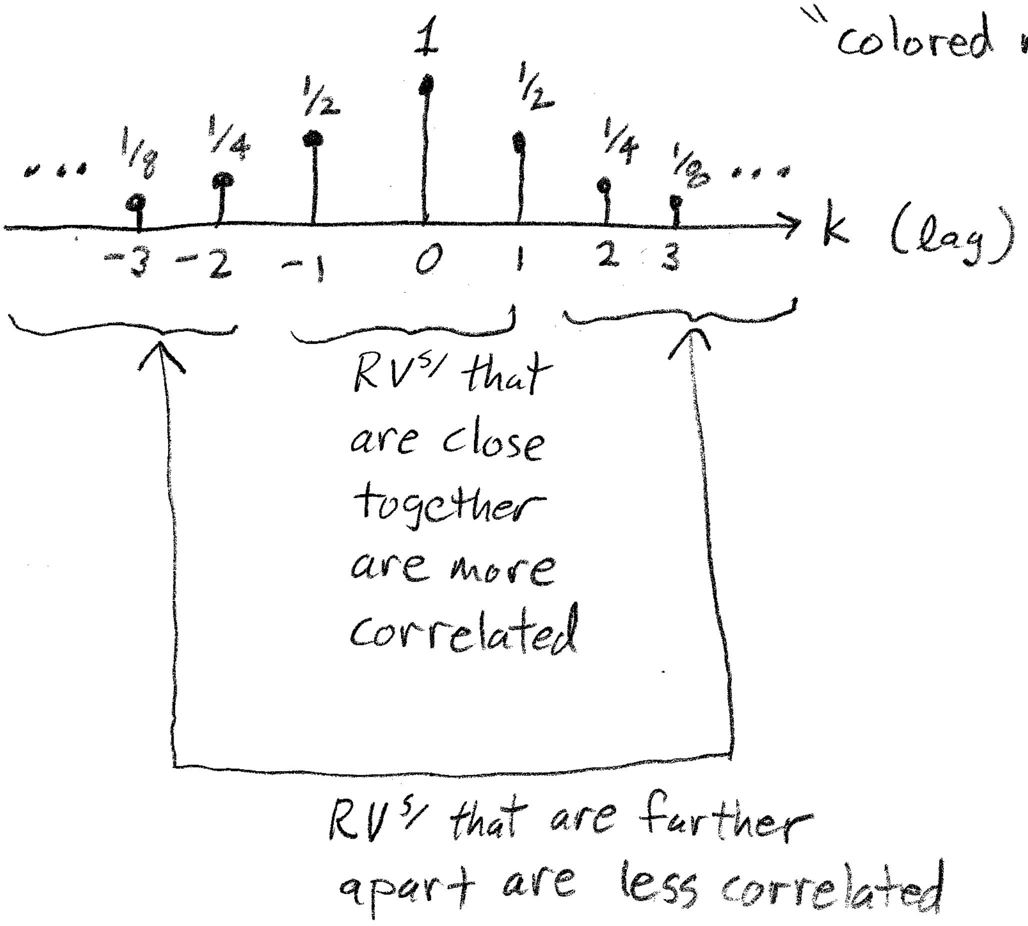
$$S_x(z) = \mathcal{Z}\{R_x[k]\} = \sum_{k=-\infty}^{\infty} R_x[k]\,z^{-k} = \sigma_x^2, \text{ ROC: all } z.$$

$$S_x(e^{j\omega}) = \text{DTFT}\{R_x[k]\} = \sum_{k=-\infty}^{\infty} R_x[k]\,e^{-j\omega k} = \sigma_x^2.$$

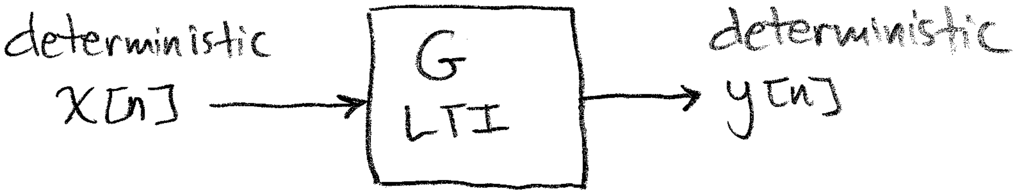
- A stochastic process $x[n]$ that is <u>not</u> white noise is called "colored noise."
  - If $x[n]$ is WSS, the autocorrelation usually "falls off" with increasing lag. In other words, RV's $x[k]$ that are close together are usually correlated with one another, whereas those that are far apart are usually less correlated or uncorrelated.

<u>EX</u>: $R_x[k] = \left(\frac{1}{2}\right)^{|k|}$

"colored noise"



- When an LTI system G has input and output that are deterministic signals $x[n]$ and $y[n]$, then

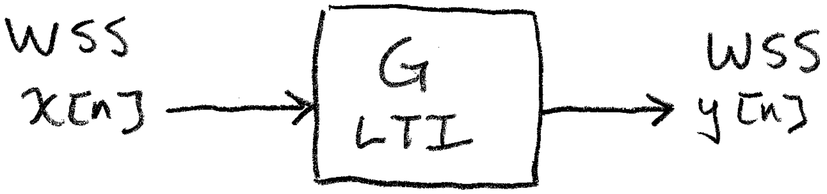


$$y[n] = x[n] * g[n]$$

$$Y(z) = X(z)G(z)$$

$$Y(e^{j\omega}) = X(e^{j\omega})G(e^{j\omega}).$$

- When an LTI system G has input $x[n]$ that is a WSS stochastic process, then the output $y[n]$ is also a WSS stochastic process.



- moreover, $x[n]$ and $y[n]$ are jointly WSS.

☆☆☆ There are $Y = XG$ "like" relations that apply to this situation. They are called "Wiener-Kintchine" relations or "Wiener-Hopf" relations.

⇒ They are summarized on the next two pages.

⇒ Some example problems are given after that.

Definitions

- → $x[n]$ and $y[n]$ are mutually WSS stochastic processes.
- → $g[n]$ is assumed real.

$$g[n] \xleftrightarrow{\text{DTFT}} G(e^{jw}) \qquad g[n] \xleftrightarrow{Z} G(z)$$

Input Autocorrelation: $R_x[k] = E\left[x[n]x[n+k]\right]$

Output Autocorrelation: $R_y[k] = E\left[y[n]y[n+k]\right]$

Input-Output Cross-Correlation:

$$R_{xy}[k] = E\left[x[n]\,y[n+k]\right] = R_x[k] * g[k]$$

Input Power Spectrum:

$$\begin{tikzcd} & S_x(e^{jw}) \\ R_x[k] \arrow[ur, leftrightarrow, "\text{DTFT}"] \arrow[dr, leftrightarrow, "Z"'] & \\ & S_x(z) \end{tikzcd}$$

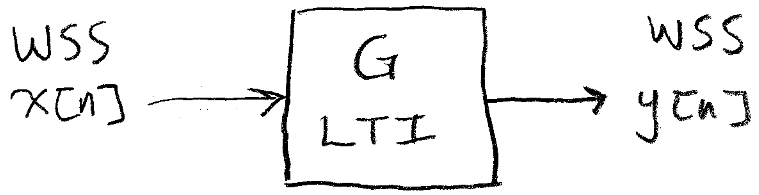
Output Power Spectrum:

$$\begin{tikzcd} & S_y(e^{jw}) \\ R_y[k] \arrow[ur, leftrightarrow, "\text{DTFT}"] \arrow[dr, leftrightarrow, "Z"'] & \\ & S_y(z) \end{tikzcd}$$

Input-Output Cross Power:

$$\begin{tikzcd} & S_{xy}(e^{jw}) \\ R_{xy}[k] \arrow[ur, leftrightarrow, "\text{DTFT}"] \arrow[dr, leftrightarrow, "Z"'] & \\ & S_{xy}(z) \end{tikzcd}$$

What's Important



$$S_y(z) = S_x(z)\,G(z)\,G(z^{-1})$$

$$S_y(e^{j\omega}) = S_x(e^{j\omega})\left|G(e^{j\omega})\right|^2$$

$$= S_x(e^{j\omega})\,G(e^{j\omega})\,G(e^{-j\omega})$$

Assuming g[n] is <u>real</u>.

$$S_{xy}(z) = S_x(z)\,G(z)$$

$$S_{xy}(e^{j\omega}) = S_x(e^{i\omega})\,G(e^{j\omega})$$

Example 1

H is an LTI discrete-time system with impulse response $h[n] = (-\frac{1}{2})^n u[n]$.

The system input is a WSS stochastic process $x[n]$ with autocorrelation $R_X[k] = (\frac{1}{2})^{|k|}$.

The system output is $y[n]$.

→ Find the input-output cross power.

Solution

- We need to break up $R_X[k]$ into a left half and a right half so we can look up the z-transform in a table.

$$R_X[k] = \left(\frac{1}{2}\right)^{|k|} = \left(\frac{1}{2}\right)^k u[k] - \left\{-\left(\frac{1}{2}\right)^{-k} u[-k-1]\right\}$$

$$= \left(\frac{1}{2}\right)^k u[k] - \left\{-(2)^k u[-k-1]\right\}$$

$$\overset{Z}{\swarrow\!\!\!\nearrow} \qquad\qquad\qquad \overset{Z}{\nwarrow\!\!\!\searrow}$$

Table: $\frac{1}{1-\frac{1}{2}z^{-1}}, \ |z| > \frac{1}{2}$ Table: $\frac{1}{1-2z^{-1}}, \ |z| < 2$

$$S_X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{1-2z^{-1}-1+\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$$= \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

⟶

Table: $H(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$

$$S_{xy}(z) = S_x(z)H(z) = \frac{-3/2\, z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1+1/2\, z^{-1})}$$

ROC: $\frac{1}{2} < |z| < 2$

Example 2

H is a discrete-time LTI system with impulse response $h[n] = (-\frac{1}{2})^n u[n]$.

The system input is a WSS stochastic process $x[n]$ with autocorrelation $R_x[k] = (\frac{1}{2})^{|k|}$.

The system output is $y[n]$.

→ Find the output power spectrum.

Solution

$$S_x(z) = \frac{-3/2\, z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

As in the last example.

Table: $H(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$

As in the last example

$\longrightarrow$

Now, the ROC of $H(z)$ is $|z| > \frac{1}{2}$.

$\Rightarrow$ if z is in the ROC of $H(z)$, then $|z| > \frac{1}{2}$.
Then $|z^{-1}| = |\frac{1}{z}| < 2$.

$\Rightarrow$ So the ROC of $H(z^{-1})$ is $|z| < 2$.

$$H(z^{-1}) = \frac{1}{1+\frac{1}{2}z}, \quad |z| < 2$$

$$= \frac{z^{-1}}{z^{-1}+\frac{1}{2}}, \quad |z| < 2$$

$$= \frac{2z^{-1}}{1+2z^{-1}}, \quad |z| < 2.$$

$$H(z)H(z^{-1}) = \frac{2z^{-1}}{(1+\frac{1}{2}z^{-1})(1+2z^{-1})}, \quad \frac{1}{2} < |z| < 2.$$

$$S_y(z) = S_x(z)H(z)H(z^{-1})$$

$$= \frac{-3z^{-2}}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})(1+2z^{-1})(1-2z^{-1})}, \quad \frac{1}{2} < |z| < 2$$

Example 3

Let $x[n]$ be a WSS white noise process with variance $R_x[0] = \sigma_x^2$.

→ Find a discrete-time LTI filter H that will transform $x[n]$ into a colored noise $y[n]$ with power spectrum

$$S_y(z) = \frac{-2z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}, \quad \tfrac{1}{2} < |z| < 2.$$

Solution

Since $x[n]$ is a white noise, the input autocorrelation is

$$R_x[k] = \sigma_x^2\,\delta[k].$$

Table: $S_x(z) = \sigma_x^2$, all z.

From the Wiener-Kintchine relations, we have

$$S_y(z) = S_x(z)\,H(z)\,H(z^{-1}).$$

Plugging in the given $S_y(z)$, we have

$$\frac{-2z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} = \sigma_x^2\,H(z)\,H(z^{-1})$$

$$H(z)H(z^{-1}) = \frac{\frac{-2}{\sigma_x^2}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{\frac{-2}{\sigma_x^2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \quad \tfrac{1}{2}<|z|<2$$

⟶

- We need to work on the last expression on page ⑯ to get it into the form $H(z)H(z^{-1})$.
- We want H to be causal and stable... so we need the poles of $H(z)$ to be inside the unit circle.
  - → Put the term $(1-\frac{1}{2}z^{-1})$ in the denominator of $H(z)$.
  - → Then we need a term $(1-\frac{1}{2}z)$ for the denominator of $H(z^{-1})$.
    - ⇒ We've got $(1-2z^{-1})$ on the last line of page ⑯. If we multiply top and bottom by $-\frac{1}{2}z$, this will turn into

$$-\tfrac{1}{2}z(1-2z^{-1}) = \left(-\tfrac{1}{2}z+1\right) = \left(1-\tfrac{1}{2}z\right) \checkmark$$

which is the term we need.

- So multiply the last line on p. ⑯ by $\frac{-\frac{1}{2}z}{-\frac{1}{2}z}$ $(=1)$:

$$H(z)H(z^{-1}) = \frac{\frac{-2}{\sigma_x^2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \cdot \frac{-\frac{1}{2}z}{-\frac{1}{2}z}, \quad \frac{1}{2} < |z| < 2$$

$$= \frac{1/\sigma_x^2}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} = \underbrace{\frac{1/\sigma_x}{1-\frac{1}{2}z^{-1}}}_{\substack{H(z)\\ |z|>\frac{1}{2}}} \cdot \underbrace{\frac{1/\sigma_x}{1-\frac{1}{2}z}}_{\substack{H(z^{-1})\\ |z|<2}} \longrightarrow$$

So $$H(z) = \frac{1/\sigma_x}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

By table lookup, the impulse response of the required filter is

$$h[n] = \frac{1}{\sigma_x}\left(\frac{1}{2}\right)^n u[n]$$