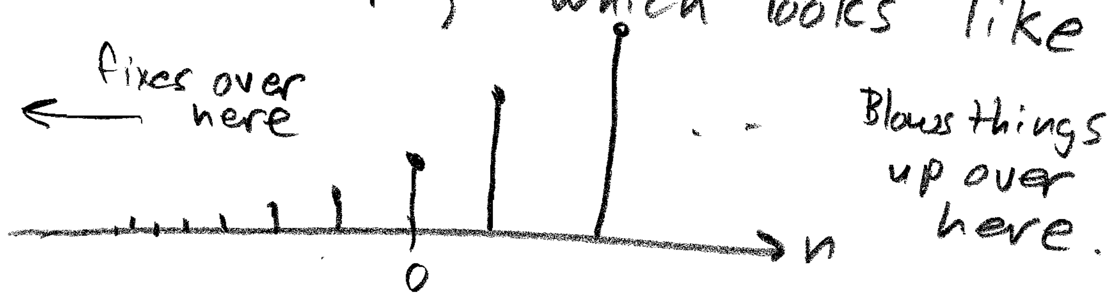
For the z-transform, the "fixer-upper" is r^{-n}, where $r \in \mathbb{R}$ and $r \geq 0$.

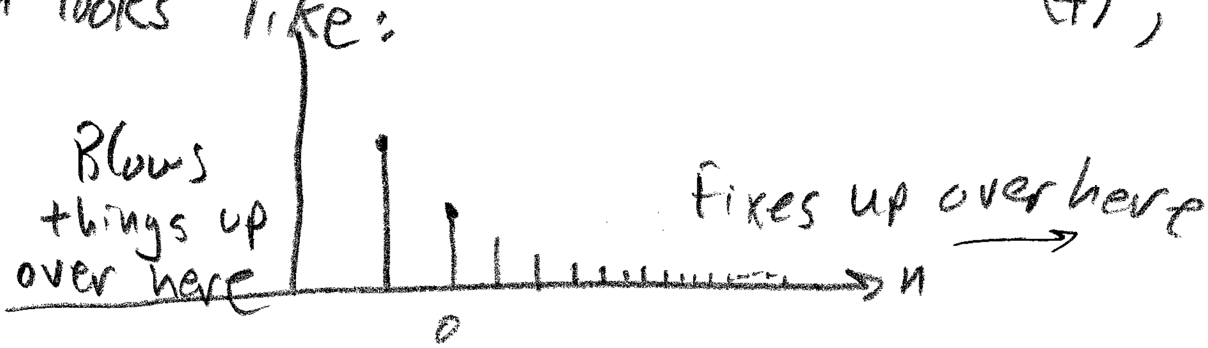
ZA1

→ if $0 < r < 1$, then r^{-n} fixes up on the left. For example, if $r = \frac{1}{4}$, then $r^{-n} = 4^n$, which looks like



→ if $r = 1$, then $r^{-n} = 1$ and the fixer-upper does nothing.

→ if $r > 1$, then r^{-n} fixes up on the right. For example, if $r = 4$, then $r^{-n} = \left(\frac{1}{4}\right)^n$, which looks like:



Now, the z-transform is just the DTFT of the fixed up guy $x[n]\, r^{-n}$. zA2

$$X(z) = \mathcal{F}\{x[n]\, r^{-n}\}$$

$$= \sum_{n \in \mathbb{Z}} x[n]\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n \in \mathbb{Z}} x[n] \, (\underbrace{re^{j\omega}}_{\substack{z \text{ in polar form} \\ r = \text{magnitude} \\ \omega = \text{angle}}})^{-n}$$

$$= \sum_{n \in \mathbb{Z}} x[n]\, z^{-n}.$$

→ For each choice of r, you either <u>do</u> or <u>do not</u> get a convergent Fourier transform.

→ when you do, the sum converges at all the $\omega \in [-\pi, \pi)$ for that r.

→ So you get a circle of radius r as part of the region of convergence (ROC)

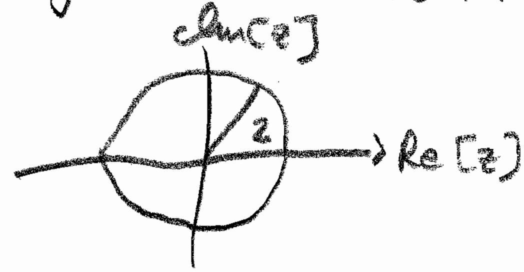
EX: Suppose that $X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n}$

$$= \sum_{n \in \mathbb{Z}} x[n] r^{-n} e^{-j\omega n}$$

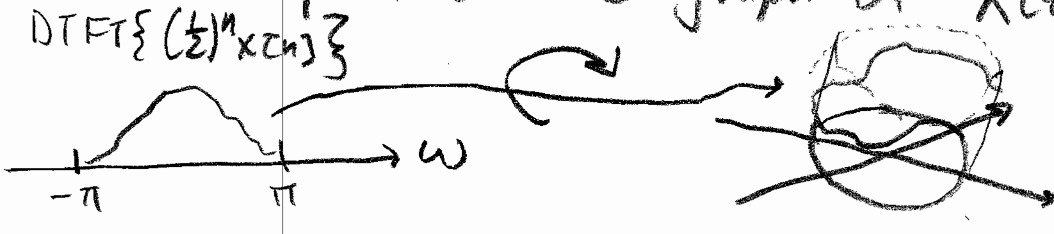
converges for $r = 2$.

→ This means that the fixed up guy $2^{-n} x[n] = \left(\frac{1}{2}\right)^n x[n]$ has a convergent Fourier transform.

→ This means that $X(z)$ converges for $r = |z| = 2$. In other words, it converges for all the ω's at $r = 2$. So the circle of z's with magnitude 2 is in the ROC of $X(z)$:



→ The graph of the DTFT of the fixed up guy $\left(\frac{1}{2}\right)^n x[n]$ is wrapped around this circle as part of the graph of $X(z)$:



So you should think of the z-transform as a collection of Fourier transforms of fixed up guys $r^{-n}x[n]$ for all the different choices of $r \geq 0$.

- For each choice of r, you get the Fourier transform (DTFT) of $r^{-n}x[n]$ around the circle of radius r in the z-plane
  - → If this converges, then all the z's with $|z|=r$ are part of the ROC of $X(z)$.
  - → if it doesn't converge, then the z's with $|z|=r$ are not part of the ROC of $X(z)$.

→

Now here's what I said backwards in class on Thu. 9/13, because I momentarily forgot about the "–" in r^{-n}: ZA5

→ Suppose $x[n]$ is a right-sided guy.

→ then there is some $n_0 \in \mathbb{Z}$ such that $x[n] = 0 \quad \forall n \leq n_0$.

→ In other words, $x[n]$ is all zero if you go far enough out to the left (i.e., further left than n_0).

→ So <u>if</u> $x[n]$ is a bad guy, his bad behaviour must be on the <u>right</u> <u>side</u>.

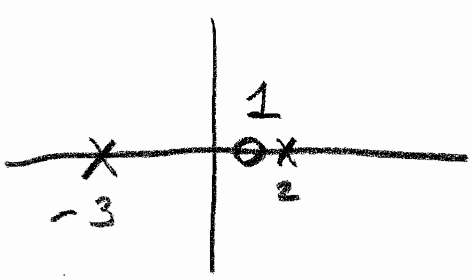
→ Fixer-uppers r^{-n} with $r > 1$ will be able to fix this, because they fix up on the <u>right</u>.

ZA6

→ Fixer-uppers with $0 < r < 1$ will <u>not</u> fix up this $x[n]$. In fact, if he has bad behaviour, these fixer-uppers will make things even worse, because these fixer-uppers are growing <u>fast</u> out on the right side.

In other words: for a right-sided $x[n]$, the ROC includes r's from fixer-uppers that fix on the right... so the ROC is the <u>exterior</u> of the circle that passes through the largest pole of $X(z)$.

<u>EX:</u> given that $x[n]$ is right sided and $X(z) = \dfrac{z-1}{(z-2)(z+3)}$.



The ROC must be $|z| > 3$.

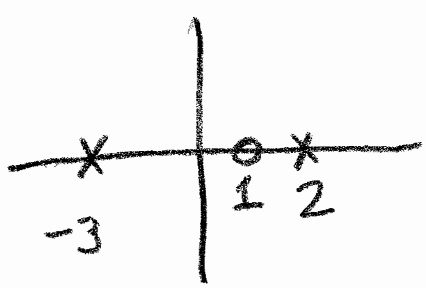
→ Note: $X(e^{j\omega})$ doesn't exist... unit circle not in ROC.

Similarly, if $x[n]$ is left-sided, than any bad behaviour is on the left. So the ROC includes r's from fixer-uppers that fix on the left. These are the <u>small</u> r's.

So the ROC of $X(z)$ must be the <u>interior</u> of a circle that passes through the smallest pole.

<u>EX</u>: given that $x[n]$ is left sided and

$$X(z) = \frac{z-1}{(z-2)(z+3)}$$

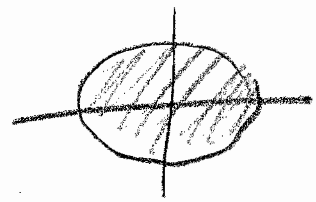


The ROC must be $|z| < 2$.

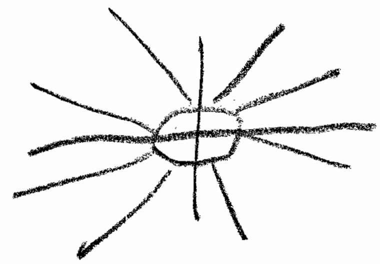
NOTE: in this case, $X(e^{j\omega})$ does exist, because the ROC of $X(z)$ contains the unit circle, and that's exactly where $r=1$ so that $X(z)$ equals $X(e^{j\omega})$.

Any two-sided $x[n]$ can be broken into the sum of a left-sided guy and a right-sided guy.

The left-sided part has a z-transform with an <u>interior</u> ROC:



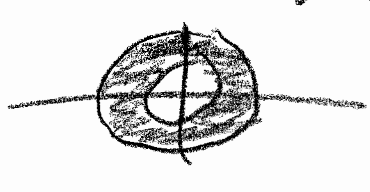
The right-sided part has a z-transform with an <u>exterior</u> ROC:



The overall z-transform $X(z) = \mathcal{Z}\{x[n]\}$ is the sum of the transforms of the two parts.

It has a ROC that includes only those z for which both parts converge.

In other words, the ROC of $X(z)$ is the intersection of the two ROCs, which is an annulus or "ring".



How does all this apply to the transfer function of a discrete-time LTI system?

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

impulse response: $h[n]$

frequency response: $H(e^{j\omega})$

transfer function: $H(z)$.

① we already know that the system is BIBO stable iff $h[n] \in \ell^1(\mathbb{Z})$, i.e. iff $\sum_{n\in\mathbb{Z}} |h[n]| < \infty$.

Note: this is a sufficient condition to ensure that $X(e^{j\omega})$ converges.

② if H is causal, then $h[n]$ must be zero for the negative n's. In other words $h[n]$ is either right-sided or finite length. In other words, the ROC of $H(z)$ is either the whole z-plane or it is <u>exterior</u>.

① and ② imply this: A causal discrete-time LTI system is stable if all the poles of $H(z)$ are <u>inside the unit circle</u>. ZA10

→ If $H(z)$ is rational in z, this becomes "if and only if."

More generally, any LTI discrete-time system (causal or not) is stable if the ROC of $H(z)$ includes the unit circle.

Also, if the system is causal, then the ROC of $H(z)$ must be exterior.

→ But this is <u>not</u> "if and only if," because $h[n]$ could be right-sided but start <u>before</u> $n=0$. This would give you an exterior ROC, but a non-causal system.