ECE 4213/5213
Test 1

Tuesday, October 22, 2002
5:00 PM - 8:00 PM

Fall 2002 — Name: SOLUTION

Dr. Havlicek — Student Num: ____________

Directions: This test is open book and open notes. You have 180 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work the first four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

1. The input $x[n]$ and output $y[n]$ of a discrete-time system H are related by

$$y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x[k].$$

(a) Is the system H memoryless? Justify your answer.

$y[0] = \sum_{k=-\infty}^{0} \left(\frac{1}{2}\right)^{k} x[k]$, which depends on the past input $x[-1]$.

Therefore, the system is not memoryless

(b) Is the system H causal? Justify your answer.

$y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x[k]$, which depends on inputs $x[k]$ for $k \in (-\infty, n]$. So $y[n]$ depends on the present input and all past inputs, but not on future inputs.

Therefore, the system is causal.

(c) Is the system H linear? Justify your answer.

Let $y_1[n] = H\{x_1[n]\} = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_1[k]$

Let $y_2[n] = H\{x_2[n]\} = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_2[k]$

Let $a, b \in \mathbb{C}$ be constants.

Let $x_3[n] = a x_1[n] + b x_2[n]$.

Then $y_3[n] = H\{x_3[n]\} = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_3[k]$

$= \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} \left(a x_1[k] + b x_2[k]\right)$

$= a \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_1[k] + b \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_2[k]$

$= a y_1[n] + b y_2[n]$ ✓ $\Rightarrow$ The system is linear.

Problem 1, cont...

(d) Is the system H BIBO stable? Justify your answer.

Let $x[n]$ be bounded. Then $\exists B \in \mathbb{R}^+$ s.t. $|x[n]| \leq B \; \forall n \in \mathbb{Z}$.

$$|y[n]| = \left|\sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{n-k} x[k]\right| \leq \left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k} |x[k]|$$

$$\leq \left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} 2^{k} B = B\left(\tfrac{1}{2}\right)^{n} \frac{0 - 2^{n+1}}{1-2} = B\left(\tfrac{1}{2}\right)^{n} 2^{n+1}$$

$$= 2B < \infty.$$

Therefore $y[n]$ is bounded and the system <u>is BIBO stable</u>.

(e) Is the system H invertible? Justify your answer. If it is, then construct the inverse system. If it is not, then find two distinct input signals that produce the same output signal.

$$y[n] = \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{n-k} x[k] = \left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k} x[k]$$

$$= \left(\tfrac{1}{2}\right)^{n} \left\{ \left(\tfrac{1}{2}\right)^{-n} x[n] + \sum_{k=-\infty}^{n-1} \left(\tfrac{1}{2}\right)^{-k} x[k] \right\}$$

$$= x[n] + \tfrac{1}{2} \sum_{k=-\infty}^{n-1} \left(\tfrac{1}{2}\right)^{n-1-k} x[k] = x[n] + \tfrac{1}{2} y[n-1].$$

$$x[n] = y[n] - \tfrac{1}{2} y[n-1]$$

I/O relation for inverse system:

$$\underline{y[n] = x[n] - \tfrac{1}{2} x[n-1]}.$$

2. Consider a frequency selective LTI digital filter H with input $x[n]$ and output $y[n]$. The impulse response $h[n]$ is real-valued and is given by

$$h[n] = \alpha\delta[n] + \beta\delta[n-1] + \alpha\delta[n-2].$$

(a) Find the system frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \sum_{n\in\mathbb{Z}} h[n]e^{-j\omega n} = \alpha + \beta e^{-j\omega} + \alpha e^{-j2\omega}$$

$$= \alpha(1+e^{-j2\omega}) + \beta e^{-j\omega} = \alpha e^{-j\omega}(e^{j\omega}+e^{-j\omega}) + \beta e^{-j\omega}$$

$$= [2\alpha\cos\omega]e^{-j\omega} + \beta e^{-j\omega} = \underline{\underline{[2\alpha\cos\omega + \beta]\, e^{-j\omega}}}$$

(b) The system input is given by

$$x[n] = \cos[0.1n] + \cos[0.4n].$$

Design the coefficients $\alpha, \beta \in \mathbb{R}$ so that the system H meets the following specifications:

- $H(e^{j\omega})$ has linear phase.
- Input term $\cos[0.1n]$ is blocked from reaching the filter output.
- Input term $\cos[0.4n]$ is passed without attenuation.

$\arg H(e^{j\omega}) = -\omega$: phase is linear for all real choices of α and β.

$$|H(e^{j\omega})| = 2\alpha\cos\omega + \beta$$

$$|H(e^{j\omega})|_{\omega=0.1} = 2\alpha\cos(0.1) + \beta = 0 \Rightarrow \beta = -2\alpha\cos(0.1) \quad (*)$$

$$|H(e^{j\omega})|_{\omega=0.4} = 2\alpha\cos(0.4) + \beta = 2\alpha\cos(0.4) - 2\alpha\cos(0.1) = 1$$

→

More Workspace for Problem 2...

$$2\alpha\left[\cos(0.4)-\cos(0.1)\right] = 1$$

$$\underline{\alpha = \left\{2\left[\cos(0.4)-\cos(0.1)\right]\right\}^{-1} \approx -6.76195}$$

plug into (*): $\underline{\beta = -2\alpha\cos(0.1) \approx 13.4563}$

3. Consider a causal digital filter H with transfer function

$$H(z) = \frac{\frac{1}{4}\sin\theta}{z^2 - (\cos\theta)z + \frac{1}{4}},$$

where $\theta \in \mathbb{R}$ is a real constant.

(a) Find the filter impulse response $h[n]$.

Causal: ROC is exterior.

$$H(z) = z^{-1}\,\frac{\frac{1}{4}(\sin\theta)\,z^{-1}}{1-(\cos\theta)z^{-1}+\frac{1}{4}z^{-2}} = \frac{1}{2}z^{-1}\,\frac{\frac{1}{2}(\sin\theta)\,z^{-1}}{1-2(\frac{1}{2})(\cos\theta)z^{-1}+(\frac{1}{2})^2 z^{-2}}$$

TABLE: $h[n] = \frac{1}{2}\left[\left(\frac{1}{2}\right)^m \sin\theta m\right]u[m]\Big|_{m=n-1}$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\sin[\theta(n-1)]\,u[n-1]$$

$$h[n] = \left(\frac{1}{2}\right)^n \sin[\theta(n-1)]\,u[n-1]$$

(b) Is the system H BIBO stable? Justify your answer.

$$\sum_{n\in\mathbb{Z}} |h[n]| = \sum_{n=1}^{\infty} |2^{-n}\sin[\theta(n-1)]| \leq \sum_{n=1}^{\infty} 2^{-n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1/2}{1-1/2} = 1 < \infty$$

Therefore <u>H is BIBO stable</u>

This Solution is Implicitly **REAL**

More Workspace for Problem 3...

(c) Suppose that the input is given by $x[n] = \cos\left(\frac{\pi}{2}n\right)$. Find the system output $y[n]$.

$$x[n] = \frac{1}{2}e^{j\pi/2\, n} + \frac{1}{2}e^{-j\pi/2\, n}$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{\frac{1}{4}\sin\theta}{e^{j2\omega} + (\cos\theta)e^{j\omega} + \frac{1}{4}}$$

$$H(e^{j\pi/2}) = \frac{\frac{1}{4}\sin\theta}{e^{j\pi} - (\cos\theta)e^{j\pi/2} + \frac{1}{4}} = \frac{\frac{1}{4}\sin\theta}{-\frac{3}{4} - j\cos\theta}$$

$$= -\frac{\sin\theta}{3 + j4\cos\theta}$$

$$H(e^{-j\pi/2}) = \frac{\frac{1}{4}\sin\theta}{e^{-j\pi} - (\cos\theta)e^{-j\pi/2} + \frac{1}{4}} = \frac{\frac{1}{4}\sin\theta}{-3/4 + j\cos\theta} = -\frac{\sin\theta}{3 - j4\cos\theta}$$

$$y[n] = \frac{1}{2}H(e^{j\pi/2})e^{j\pi/2\, n} + \frac{1}{2}H(e^{-j\pi/2})e^{-j\pi/2\, n}$$

$$= -\left[\frac{\sin\theta}{6 + j8\cos\theta}e^{j\pi/2\, n} + \frac{\sin\theta}{6 - j8\cos\theta}e^{-j\pi/2\, n}\right]$$

This Solution is Explicitly REAL

More Workspace for Problem 3...

(c) Suppose that the input is given by $x[n] = \cos\left(\frac{\pi}{2}n\right)$. Find the system output $y[n]$.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{\frac{1}{4}\sin\theta}{e^{j2\omega} - (\cos\theta)e^{j\omega} + \frac{1}{4}}$$

$$H(e^{j\pi/2}) = \frac{\frac{1}{4}\sin\theta}{e^{j\pi} - (\cos\theta)e^{j\pi/2} + \frac{1}{4}} = \frac{\frac{1}{4}\sin\theta}{-1 - j\cos\theta + \frac{1}{4}} = \frac{-\frac{1}{4}\sin\theta}{\frac{3}{4} + j\cos\theta}$$

Let $z_1 = \dfrac{1}{\frac{3}{4} + j\cos\theta} = re^{j\varphi}$

Let $z_2 = z_1^{-1} = \frac{3}{4} + j\cos\theta = \frac{1}{r}e^{-j\varphi}$

Then $\frac{1}{r} = |z_2| = \left[\frac{9}{16} + \cos^2\theta\right]^{1/2} \Longrightarrow r = \left[\frac{9}{16} + \cos^2\theta\right]^{-1/2}$

$$-\varphi = \arg z_2 = \arctan\left[\frac{\cos\theta}{3/4}\right] = \arctan\left[\frac{4}{3}\cos\theta\right]$$

$$\Longrightarrow \varphi = -\arctan\left[\frac{4}{3}\cos\theta\right]$$

$$\Longrightarrow z_1 = \left[\frac{9}{16} + \cos^2\theta\right]^{-1/2} \exp\left\{-j\arctan\left[\frac{4}{3}\cos\theta\right]\right\}$$

Let $\beta = \begin{cases} 0, & \sin\theta \leq 0, \\ \pi, & \sin\theta > 0. \end{cases}$

Then $-\frac{1}{4}\sin\theta = \frac{1}{4}|\sin\theta|\, e^{j\beta}$

$$\Longrightarrow H(e^{j\pi/2}) = \left(-\tfrac{1}{4}\sin\theta\right) z_1$$

$$= \frac{1}{4}|\sin\theta| \left[\frac{9}{16} + \cos^2\theta\right]^{-1/2} \exp\left\{j\left(\beta - \arctan\left[\frac{4}{3}\cos\theta\right]\right)\right\}$$

$$\left|H(e^{j\pi/2})\right| = \frac{1}{4}|\sin\theta| \left[\frac{9}{16} + \cos^2\theta\right]^{-1/2}$$

$$\arg H(e^{j\pi/2}) = \beta - \arctan\left[\frac{4}{3}\cos\theta\right]$$

$$y[n] = |H(e^{j\pi/2})| \cos\left\{\frac{\pi}{2}n + \arg H(e^{j\pi/2})\right\}$$

$$\boxed{y[n] = \frac{1}{4}|\sin\theta| \left[\frac{9}{16} + \cos^2\theta\right]^{-1/2} \cos\left\{\frac{\pi}{2}n + \beta - \arctan\left[\frac{4}{3}\cos\theta\right]\right\}}$$

4. Consider an LTI digital filter H with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

Use the convolution property of the Z-transform to find the filter output $y[n]$ when the input is

$$x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n].$$

$$H(z) \overset{\text{Table}}{=\!=} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$X(z) = X(e^{j\omega})\Big|_{e^{j\omega}=z} \overset{\text{Table}}{=\!=} \frac{1}{(1-\frac{1}{4}z^{-1})^2}, \quad |z| > \frac{1}{4} \quad \left(\text{since } x[n] \text{ is right sided}\right)$$

$$Y(z) = X(z)H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})^2}, \quad |z| > \frac{1}{2}$$

Let $$Y(\theta) = \frac{1}{(1-\frac{1}{2}\theta)(1-\frac{1}{4}\theta)^2} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-\frac{1}{4}\theta} + \frac{C}{(1-\frac{1}{4}\theta)^2}$$

$$A = \frac{1}{(1-\frac{1}{4}\theta)^2}\Bigg|_{\theta=2} = \frac{1}{(1-\frac{1}{2})^2} = \frac{1}{1/4} = \underline{4}$$

$$C = \frac{1}{1-\frac{1}{2}\theta}\Bigg|_{\theta=4} = \frac{1}{-1} = \underline{-1}$$

$$\left[\frac{d}{d\theta}(1-\tfrac{1}{2}\theta)^{-1}\right]_{\theta=4} = \left[\frac{d}{d\theta}(1-\tfrac{1}{4}\theta)B\right]_{\theta=4}$$

$$\frac{1/2}{(1-\frac{1}{2}\theta)^2}\Bigg|_{\theta=4} = -\frac{1}{4}B \Rightarrow \frac{1}{2} = -\frac{1}{4}B \Longrightarrow B = \underline{-2}$$

$$Y(z) = Y(\theta)\Big|_{\theta=z^{-1}} = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{4}z^{-1}} - \frac{1}{(1-\frac{1}{4}z^{-1})^2}, \quad |z| > \frac{1}{2}$$

$$\mathcal{Z}^{-1}\left[\frac{4}{1-\frac{1}{2}z^{-1}}\right], \ |z| > \frac{1}{2} = 4\left(\tfrac{1}{2}\right)^n u[n]$$

$$\mathcal{Z}^{-1}\left[\frac{-2}{1-\frac{1}{4}z^{-1}}\right], \ |z| > \frac{1}{4} = -2\left(\tfrac{1}{4}\right)^n u[n]$$

$$\mathcal{Z}^{-1}\left[\frac{-1}{(1-\frac{1}{4}z^{-1})^2}\right], \ |z| > \frac{1}{4} = \mathcal{F}^{-1}\left[\frac{-1}{(1-\frac{1}{4}e^{-j\omega})^2}\right] \overset{\text{Table}}{=\!=} -(n+1)\left(\tfrac{1}{4}\right)^n u[n]$$

$$\underline{y[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n] - (n+1)\left(\tfrac{1}{4}\right)^n u[n]}$$

5. The length-N signal $x[n]$ is defined for $0 \le n \le N-1$. It has a length-N *DFT* $X[k]$ that is defined for $0 \le k \le N-1$.

The length-N^2 signal $y[n]$ is defined for $0 \le n \le N^2-1$. It's values are given by $y[n] = x[\langle n \rangle_N]$; *i.e.*, $y[n]$ is the N-fold periodic repetition of $x[n]$. $Y[k]$, the length-N^2 *DFT* of $y[n]$, is defined for $0 \le k \le N^2-1$.

Express $Y[k]$ in terms of $X[k]$.

$$x[\langle n \rangle_N] = x[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m] e^{j2\pi mn/N}$$

$$Y[k] = \sum_{n=0}^{N^2-1} y[n] e^{-j2\pi nk/N^2} = \sum_{n=0}^{N^2-1} x[\langle n \rangle_N] e^{-j2\pi nk/N^2}$$

$$= \sum_{n=0}^{N^2-1} \left[\frac{1}{N}\sum_{m=0}^{N-1} X[m] e^{j2\pi mn/N}\right] e^{-j2\pi nk/N^2}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X[m] \underbrace{\sum_{n=0}^{N^2-1} e^{j2\pi n(mN-k)/N^2}}_{= \begin{cases} N^2, & k = mN \Longrightarrow m = \frac{k}{N} \\ 0, & \text{otherwise} \end{cases}}$$

$$= \begin{cases} N X[\frac{k}{N}], & k = 0, N, 2N, 3N, \ldots, (N-1)N \\ 0, & \text{otherwise} \end{cases}$$