ECE 4213/5213
Test 1

Monday, October 30, 2006
3:00 PM - 4:15 PM

Fall 2006

Dr. Havlicek

Name: SOLUTION

Student Num: ______

Directions: This test is open book and closed notes. You may also use a calculator and the formula sheet from the course web site. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______ Date: ______

1. A discrete-time LTI system H has impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

and input

$$x[n] = 3^n u[-n].$$

Use the z-transform to find the system output $y[n]$.

ROC: $|\frac{1}{3}z| < 1$

$\frac{1}{3}|z| < 1$

$|z| < 3$

Table: $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{0} 3^n z^{-n} \overset{m=-n}{=} \sum_{m=0}^{\infty} 3^{-m} z^m \overset{n=m}{=} \sum_{n=0}^{\infty} \left(\tfrac{1}{3}z\right)^n$$

$$= \frac{1}{1-\frac{1}{3}z} \cdot \frac{3z^{-1}}{3z^{-1}} = \frac{3z^{-1}}{3z^{-1}-1} = \frac{-3z^{-1}}{1-3z^{-1}}, \quad |z| < 3$$

$$Y(z) = X(z)H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{-3z^{-1}}{1-3z^{-1}} = \frac{-3z^{-1}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}$$

$\frac{1}{2} < |z| < 3.$

PFE: $$\frac{-3\theta}{(1-\frac{1}{2}\theta)(1-3\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-3\theta}$$

$$A = \left.\frac{-3\theta}{1-3\theta}\right|_{\theta=2} = \frac{-6}{1-6} = \frac{-6}{-5} = \frac{6}{5}$$

$$B = \left.\frac{-3\theta}{1-\frac{1}{2}\theta}\right|_{\theta=\frac{1}{3}} = \frac{-1}{1-\frac{1}{6}} = \frac{-1}{5/6} = -\frac{6}{5}$$

$$Y(z) = \underbrace{\frac{6}{5}\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{6}{5}\frac{1}{1-3z^{-1}}}_{|z|<3}$$

Table: $\boxed{y[n] = \frac{6}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{6}{5}(3)^n u[-n-1]}$

2. A discrete-time system H has input

$$x[n] = \frac{\sin\left(\frac{\pi}{4}n\right)}{n\pi}$$

and output

$$y[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{n\pi}$$

Is H an LTI system? *(justify your answer)*

Table: $X(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases}$ (Fundamental Period)

$=$

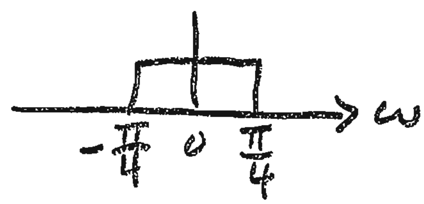
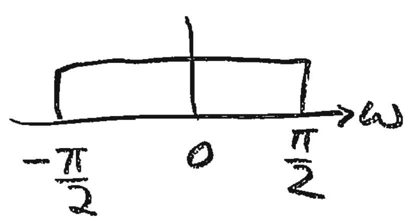


Table: $Y(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| < \pi \end{cases}$ (Fundamental Period)

$=$

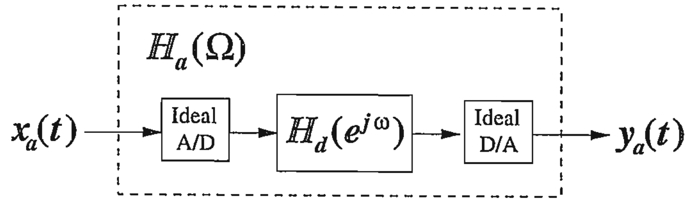


For an LTI system with frequency response $H(e^{j\omega})$, we must have that $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$.

In this case, $Y(e^{j\omega})$ is nonzero for $\frac{\pi}{4} < \omega < \frac{\pi}{2}$, but $X(e^{j\omega})$ is zero in this range. So there is <u>no possible</u> $H(e^{j\omega})$ that will give $Y(e^{j\omega}) = 1 = 0 \cdot H(e^{j\omega})$ for $\frac{\pi}{4} < \omega < \frac{\pi}{2}$. Therefore, H is NOT LTI.

For an LTI system, the output cannot contain frequencies that are not in the input.

3. The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



The sampling frequency of the A/D and D/A converters is $F_T = 2kHz$.

The analog input signal is given by $x_a(t) = \sin(1000\pi t)$.

$F_T = 2\ kHz = 2000\ Hz$

The input/output relation of the digital filter H_d is given by

$T = \frac{1}{F_T} = \frac{1}{2000}$ sec

$$y_d[n] - \frac{1}{3}y_d[n-1] = \frac{2}{3}x_d[n] - 2x_d[n-1].$$

Find the analog output signal $y_a(t)$.

DTFT: $Y_d(e^{j\omega})\left[1-\frac{1}{3}e^{-j\omega}\right] = X_d(e^{j\omega})\left[\frac{2}{3} - 2e^{-j\omega}\right]$

$$H_d(e^{j\omega}) = \frac{Y_d(e^{j\omega})}{X_d(e^{j\omega})} = \frac{2\left(\frac{1}{3} - e^{-j\omega}\right)}{1-\frac{1}{3}e^{-j\omega}}$$

The digital input signal is

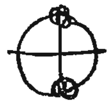
$$x_d[n] = x_a(nT) = x_a\left(\frac{n}{2000}\right) = \sin\left(\frac{1000\pi n}{2000}\right) = \sin\left(\frac{\pi}{2}n\right).$$

$\Rightarrow$ NOTE that $x_d[n]$ is an eigen-function of the digital system. The associated eigenvalue is $H_d(e^{j\pi/2})$. Therefore, the digital output signal is

$$y_d[n] = H_d(e^{j\omega})\, x_d[n]\Big|_{\omega=\pi/2}$$

$$= \left|H_d(e^{j\pi/2})\right| \sin\left(\frac{\pi}{2}n + \arg H(e^{j\pi/2})\right) \quad (*)$$

$\longrightarrow$

 $e^{-j\pi/2} = -j$

More Workspace for Problem 3...

$$|H_d(e^{j\pi/2})| = \left| \frac{2\left(\frac{1}{3} - e^{-j\pi/2}\right)}{1 - \frac{1}{3}e^{-j\pi/2}} \right| = 2\,\frac{|\frac{1}{3}+j|}{|1-\frac{1}{3}j|} = 2\,\frac{\sqrt{(\frac{1}{3})^2+1^2}}{\sqrt{(\frac{1}{3})^2+1^2}}$$

$$= 2$$

$$\arg H_d(e^{j\pi/2}) = \arg \frac{\frac{1}{3} - e^{-j\pi/2}}{1-\frac{1}{3}e^{-j\pi/2}} = \arg \frac{\frac{1}{3}+j}{1+\frac{1}{3}j}$$

$$= \arg \frac{(\frac{1}{3}+j)(1-\frac{1}{3}j)}{(1+\frac{1}{3}j)(1-\frac{1}{3}j)} = \arg \frac{\frac{1}{3} - \frac{1}{9}j + j + \frac{1}{3}}{1 + 1/9}$$

$$= \arg \frac{\frac{2}{3} + 8/9\,j}{10/9} = \arg \frac{9}{10}\left(\frac{2}{3} + \frac{8}{9}j\right) = \arg\left(\frac{3}{5} + j\,\frac{4}{5}\right)$$

$$= \arctan \frac{4/5}{3/5} = \arctan \frac{4}{3} \approx 0.9273 \text{ rad}$$

$$\approx 0.2952\pi \text{ rad}$$

plugging into (*) on p. 4,

$$y_d[n] = 2\sin\left(\frac{\pi}{2}n + \underline{0.2952\,\pi}\right)$$

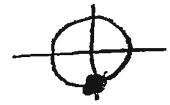
$$y_a(t) = y_d\left[\frac{t}{T}\right] = 2\sin\left[2000\left(\frac{\pi}{2}t + 0.2952\pi\right)\right]$$

$$= 2\sin\left(1000\pi t + 5904\,\pi\right)$$

$$\boxed{y_a(t) = 2\sin\left[1000\pi\,(t + 0.5904)\right]}$$

4. Consider a finite-length discrete-time signal $x[n]$ with length 4. The signal is given by

$$\begin{aligned} x[n] &= [1\ 0\ 2\ 1] \\ &= \delta[n] + 2\delta[n-2] + \delta[n-3]. \end{aligned}$$



(a) Find the four-point DFT $X[k]$ of $x[n]$. $W_4 = e^{-j2\pi/4} = e^{-j\pi/2} = -j$

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{nk} = 1W_4^0 + 2W_4^{2k} + W_4^{3k}$$

$$= 1 + 2(-j)^{2k} + (-j)^{3k} = 1 + 2[(-j)^2]^k + [(-j)^3]^k$$

$$= 1 + 2(-1)^k + j^k$$

$$\boxed{\begin{aligned} X[k] &= 1 + 2(-1)^k + j^k \\ &= [4\ \ (-1+j)\ \ 2\ \ (-1-j)] \end{aligned}}$$

Another way:

$$X[k] = x[0]e^{-j\pi/2 \cdot 0 \cdot k} + x[1]e^{-j\pi/2 \cdot 1 \cdot k} + x[2]e^{-j\pi/2 \cdot 2 \cdot k} + x[3]e^{-j\pi/2 \cdot 3 \cdot k}$$

$$= 1 \cdot 1 + 0 \cdot e^{-j\pi/2 k} + 2e^{-j\pi k} + 1 \cdot e^{-j3\pi/2 k}$$

$$= 1 + 2e^{-j\pi k} + e^{-j\frac{3\pi}{2}k} = \text{ANSWER ABOVE}$$

Problem 4, cont...

(b) Let $y[n]$ be the four-point circular convolution of $x[n]$ with itself. Use the DFT to find $Y[k]$ and $y[n]$.

$$Y[k] = X^2[k] = \left(1 + 2W_4^{2k} + W_4^{3k}\right)\left(1 + 2W_4^{2k} + W_4^{3k}\right)$$

$$\begin{aligned} = 1 &+ 2W_4^{2k} + W_4^{3k} \\ &+ 2W_4^{2k} + 4W_4^{4k} + 2W_4^{5k} \\ &+ W_4^{3k} + 2W_4^{5k} + W_4^{6k} \end{aligned}$$

$$= 1 + 4W_4^{2k} + 2W_4^{3k} + 4W_4^{4k} + 4W_4^{5k} + W_4^{6k}$$

$$= 1 + 4[(-j)^2]^k + 2[(-j)^3]^k + 4[(-j)^4]^k + 4[(-j)^5]^k + [(-j)^6]^k$$

$$= 1 + 4(-1)^k + 2j^k + 4(1)^k + 4(-j)^k + (-1)^k$$

$$= 5 + 4(-j)^k + 5(-1)^k + 2j^k \quad (*)$$

$$\boxed{Y[k] = 5 + 4(-j)^k + 5(-1)^k + 2j^k}$$

$$= \underset{\substack{\uparrow \\ y[0]}}{5W_4^{0k}} + \underset{\substack{\uparrow \\ y[1]}}{4W_4^{1k}} + \underset{\substack{\uparrow \\ y[2]}}{5W_4^{2k}} + \underset{\substack{\uparrow \\ y[3]}}{2W_4^{3k}} = \sum_{n=0}^{3} y[n] W_4^{nk}$$

$$\boxed{\begin{aligned} y[n] &= 5\delta[n] + 4\delta[n-1] + 5\delta[n-2] + 2\delta[n-3] \\ &= [5\ 4\ 5\ 2] \end{aligned}}$$

5. A *causal* discrete-time LTI system H has transfer function

$$H(z) = \frac{\frac{1}{3} + j\frac{1}{3}}{1 - \left(\frac{1}{2} + j\frac{1}{2}\right) z^{-1}} + \frac{\frac{1}{3} - j\frac{1}{3}}{1 - \left(\frac{1}{2} - j\frac{1}{2}\right) z^{-1}}.$$

(a) Is the system H BIBO stable? *(justify your answer)*

Let $a = \frac{1}{3} + j\frac{1}{3}$ and $b = \frac{1}{2} + j\frac{1}{2}$.

If you were to combine $H(z)$ into a single ratio of polynomials in z^{-1}, the denominator would be $(1 - bz^{-1})(1 - b^* z^{-1})$. So the poles are at b and b^*.

$$|b| = |b^*| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{1}{2}} = 1/\sqrt{2} < 1.$$

Since the system is causal, $h[n]$ is right sided and the ROC of $H(z)$ is exterior; i.e., the ROC is $|z| > 1/\sqrt{2}$.

Since the ROC of $H(z)$ includes the unit circle,

$$\boxed{H \text{ is BIBO STABLE}}$$

Problem 5, cont...

Let $a = \frac{1}{3} + j\frac{1}{3}$; $b = \frac{1}{2} + j\frac{1}{2}$

Let $r = |b| = |b^*| = 1/\sqrt{2}$

(b) Find the system impulse response $h[n]$.

$\omega_0 = \arg b = \pi/4$

$\Rightarrow \arg b^* = -\omega_0 = -\pi/4$

$$H(z) = \frac{a}{1-bz^{-1}} + \frac{a^*}{1-b^*z^{-1}}, \quad |z|>|b|$$

Table: $h[n] = ab^n u[n] + a^*(b^*)^n u[n]$

$$= \left(ar^n e^{j\omega_0 n} + a^* r^n e^{-j\omega_0 n}\right)u[n]$$

$$= \left(\tfrac{1}{3}r^n e^{j\omega_0 n} + j\tfrac{1}{3}r^n e^{j\omega_0 n} + \tfrac{1}{3}r^n e^{-j\omega_0 n} - j\tfrac{1}{3}r^n e^{-j\omega_0 n}\right)u[n]$$

$$= \tfrac{1}{3}r^n\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)u[n] + j\tfrac{1}{3}r^n\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)u[n]$$

$$= \tfrac{2}{3}r^n\cos(\omega_0 n)u[n] + j\tfrac{1}{3}r^n\left[2j\sin(\omega_0 n)\right]u[n]$$

$$= \tfrac{2}{3}r^n\cos(\omega_0 n)u[n] - \tfrac{2}{3}r^n\sin(\omega_0 n)u[n]$$

$$\boxed{h[n] = \tfrac{2}{3}(\sqrt{2})^{-n}\cos\left(\tfrac{\pi}{4}n\right)u[n] - \tfrac{2}{3}(\sqrt{2})^{-n}\sin\left(\tfrac{\pi}{4}n\right)u[n]}$$

ALTERNATE SOLUTION

Problem 5, cont...

(b) Find the system impulse response $h[n]$.

Let $a = \frac{1}{3} + j\frac{1}{3}$ $\quad b = \frac{1}{2} + j\frac{1}{2}$

$r = |b| = |b^*| = 1/\sqrt{2}$

$\omega_0 = \arg b = \pi/4$

$\Rightarrow \arg b^* = -\omega_0 = -\pi/4$

$$H(z) = \frac{a}{1-bz^{-1}} + \frac{a^*}{1-b^*z^{-1}}$$

$$= \frac{a(1-b^*z^{-1}) + a^*(1-bz^{-1})}{(1-bz^{-1})(1-b^*z^{-1})} = \frac{a - ab^*z^{-1} + a^* - a^*bz^{-1}}{1-bz^{-1}-b^*z^{-1}+bb^*z^{-2}}$$

$$= \frac{\frac{2}{3} - (ab^* + a^*b)z^{-1}}{1 - 2\mathrm{Re}[b]z^{-1} + r^2z^{-2}} = \frac{\frac{2}{3} - \left[(\frac{1}{3}+j\frac{1}{3})re^{-j\omega_0} + (\frac{1}{3}-j\frac{1}{3})re^{j\omega_0}\right]z^{-1}}{1-2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$$

$$= \frac{\frac{2}{3} - \left[\frac{1}{3}re^{-j\omega_0} + j\frac{1}{3}re^{-j\omega_0} + \frac{1}{3}re^{j\omega_0} - j\frac{1}{3}re^{j\omega_0}\right]z^{-1}}{1-2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$$

$$= \frac{\frac{2}{3} - \left[\frac{1}{3}r(e^{j\omega_0}+e^{-j\omega_0}) - j\frac{1}{3}r(e^{j\omega_0}-e^{-j\omega_0})\right]z^{-1}}{1-2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$$

$$= \frac{\frac{2}{3} - \frac{2}{3}r\cos(\omega_0)z^{-1} - \frac{2}{3}r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$$

$$= \frac{2}{3}\,\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}} - \frac{2}{3}\,\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$$

Table: $h[n] = \frac{2}{3}r^n\cos(\omega_0 n)u[n] - \frac{2}{3}r^n\sin(\omega_0 n)u[n]$

$$\boxed{h[n] = \frac{2}{3}(\sqrt{2})^{-n}\cos\left(\frac{\pi}{4}n\right)u[n] - \frac{2}{3}(\sqrt{2})^{-n}\sin\left(\frac{\pi}{4}n\right)u[n]}$$