ECE 4213/5213
Test 1

Thursday, October 18, 2007
12:00 PM - 1:15 PM

Fall 2007

Name: SOLUTION

Dr. Havlicek

Student Num:

Directions: This test is open book. You may also use a calculator and the course notes and formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

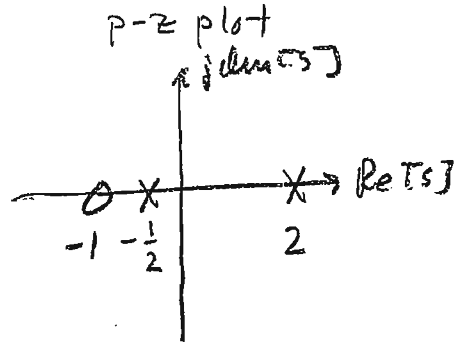
TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________________ Date: ____________________

1. **25/20 pts.** A **STABLE** continuous-time LTI system H has transfer function

$$H(s) = \frac{s+1}{(s-2)(s+\frac{1}{2})}.$$



The system input is given by

$$x(t) = e^{-3t}u(t).$$

Find the system output $y(t)$.

Since the system is stable, the ROC of $H(s)$ must contain the $j\omega$-axis. So the ROC must be $-\frac{1}{2} < Re[s] < 2$. Note that this implies the system cannot be causal.

Table: $X(s) = \frac{1}{s+3}$, $Re[s] > -3$.

$$Y(s) = X(s)H(s) = \frac{s+1}{(s-2)(s+\frac{1}{2})(s+3)} = \frac{A}{s-2} + \frac{B}{s+\frac{1}{2}} + \frac{C}{s+3}$$

ROC: $\{-\frac{1}{2} < Re[s] < 2\} \cap \{Re[s] > -3\} = -\frac{1}{2} < Re[s] < 2.$

PFE:

$$A = \left.\frac{s+1}{(s+\frac{1}{2})(s+3)}\right|_{s=2} = \frac{3}{(\frac{5}{2})(5)} = 3\left(\frac{2}{5}\right)\left(\frac{1}{5}\right) = \frac{6}{25}$$

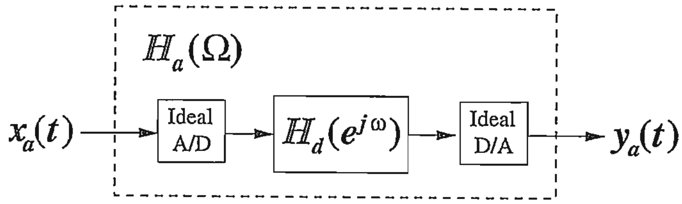
$$B = \left.\frac{s+1}{(s-2)(s+3)}\right|_{s=-\frac{1}{2}} = \frac{1/2}{(-5/2)(5/2)} = \frac{1}{2}\left(\frac{-2}{5}\right)\left(\frac{2}{5}\right) = \frac{-2}{25}$$

$$C = \left.\frac{s+1}{(s-2)(s+\frac{1}{2})}\right|_{s=-3} = \frac{-2}{(-5)(-5/2)} = -2\left(\frac{-1}{5}\right)\left(\frac{-2}{5}\right) = \frac{-4}{25}$$

$$Y(s) = \underbrace{\frac{6/25}{s-2}}_{Re[s]<2} + \underbrace{\frac{-2/25}{s+\frac{1}{2}}}_{Re[s]>-\frac{1}{2}} + \underbrace{\frac{-4/25}{s+3}}_{Re[s]>-3}$$

Table: $y(t) = -\frac{6}{25}e^{2t}u(-t) - \frac{2}{25}e^{-\frac{1}{2}t}u(t) - \frac{4}{25}e^{-3t}u(t)$

2. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



The sampling frequency of the A/D and D/A converters is $F_T = 2$ kHz. $T = \frac{1}{2000}$ sec

The analog input signal is given by $x_a(t) = \sin(1000\pi t)$.

The input/output relation of the digital filter H_d is given by $\Omega_T = 4000\pi$ rad/sec

$$y_d[n] - \frac{1}{3}y_d[n-1] = \frac{2}{3}x_d[n] - 2x_d[n-1].$$

Find the analog output signal $y_a(t)$.

$$Y_d(e^{j\omega})\left[1 - \tfrac{1}{3}e^{-j\omega}\right] = X_d(e^{j\omega})\left[\tfrac{2}{3} - 2e^{-j\omega}\right]$$

$$H_d(e^{j\omega}) = \frac{Y_d(e^{j\omega})}{X_d(e^{j\omega})} = \frac{\frac{2}{3} - 2e^{-j\omega}}{1 - \frac{1}{3}e^{-j\omega}} = \frac{2\left(\frac{1}{3} - e^{-j\omega}\right)}{1 - \frac{1}{3}e^{-j\omega}}$$

<u>METHOD 1: Analog.</u>

$$H_a(\Omega) = H_d(e^{j\Omega T}) = \frac{2\left(\frac{1}{3} - e^{-j\Omega T}\right)}{1 - \frac{1}{3}e^{-j\Omega T}}$$

$$= \frac{2\left(\frac{1}{3} - e^{-j\Omega/2000}\right)}{1 - \frac{1}{3}e^{-j\Omega/2000}}, \quad |\Omega| \leq 2000\pi \ \frac{\text{rad}}{\text{sec}}$$

1 kHz

Table:

$$X_a(\Omega) = \frac{\pi}{j}\left[\delta(\Omega - 1000\pi) - \delta(\Omega + 1000\pi)\right]$$

→

More Workspace for Problem 2...

Because the I/O relation has real coefs, $H_a(\Omega)$ has conj sym.

$$Y_a(\Omega) = X_a(\Omega)H_a(\Omega) = \frac{\pi}{j}H_a(\Omega)\delta(\Omega-1000\pi) - \frac{\pi}{j}H_a(\Omega)\delta(\Omega+1000\pi)$$

$$= \frac{\pi}{j}H_a(1000\pi)\delta(\Omega-1000\pi) - \frac{\pi}{j}H_a(-1000\pi)\delta(-\Omega+1000\pi)$$

$$= \frac{\pi}{j}H_a(1000\pi)\delta(\Omega-1000\pi) - \frac{\pi}{j}H_a^*(1000\pi)\delta(\Omega+1000\pi)$$

$$= \frac{\pi}{j}\frac{2(\frac{1}{3}-e^{-j\pi/2})}{1-\frac{1}{3}e^{-j\pi/2}}\delta(-\Omega-1000\pi) - \frac{\pi}{j}\frac{2(\frac{1}{3}-e^{j\pi/2})}{1-\frac{1}{3}e^{j\pi/2}}\delta(\Omega+1000\pi)$$

$$= -\frac{2\pi}{j}\frac{\frac{1}{3}+j}{1+\frac{1}{3}j}\delta(\Omega-1000\pi) - \frac{2\pi}{j}\frac{\frac{1}{3}-j}{1+\frac{1}{3}j}\delta(-\Omega+1000\pi)$$

Table: $y_a(t) = \frac{1}{j}\frac{\frac{1}{3}+j}{1+\frac{1}{3}j}e^{j1000\pi t} - \frac{1}{j}\frac{\frac{1}{3}-j}{1+\frac{1}{3}j}e^{-j1000\pi t}$

$$\Rightarrow \frac{\frac{1}{3}+j}{1+\frac{1}{3}j} = \frac{\frac{1}{3}+j}{1+\frac{1}{3}j}\cdot\frac{1-\frac{1}{3}j}{1-\frac{1}{3}j} = \frac{\frac{2}{3}+\frac{8}{9}j}{10/9} = \frac{3}{5}+\frac{4}{5}j$$

Let $r = \left|\frac{3}{5}+\frac{4}{5}j\right| = \frac{1}{5}\sqrt{3^2+4^2} = 1$

Let $\theta = \arg\frac{3}{5}+\frac{4}{5}j = \arctan\frac{4}{3}$

$$y_a(t) = \frac{1}{j}re^{j(1000\pi t+\theta)} - \frac{1}{j}e^{-j(1000\pi t+\theta)}$$

$$= 2r\sin(1000\pi t+\theta)$$

$$\boxed{y_a(t) = 2\sin\left(1000\pi t+\arctan\frac{4}{3}\right)}$$

METHOD 2: Digital

$$x_d[n] = x_a(nT) = x_a\left(\frac{n}{2000}\right)$$
$$= \sin\left(\frac{1000\pi}{2000}n\right) = \sin\left(\frac{\pi}{2}n\right)$$

As before, $H_d(e^{j\omega}) = \dfrac{2(\frac{1}{3} - e^{-j\omega})}{1 - \frac{1}{3}e^{-j\omega}}$

Now, $x_d[n] = \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\pi/2 n}$

$$= j\tfrac{1}{2}e^{-j\pi/2 n} - j\tfrac{1}{2}e^{j\pi/2 n}$$

$\rightarrow$ $j\frac{1}{2}e^{-j\pi/2 n}$ is an eigenfunction of any LTI system with associated eigenvalue $H_d(e^{-j\pi/2})$

$\rightarrow$ $-j\frac{1}{2}e^{j\pi/2 n}$ is also an eigenfunction of any LTI system with associated eigenvalue $H_d(e^{j\pi/2})$.

So $y_d[n] = j\frac{1}{2}e^{-j\pi/2 n}H_d(e^{-j\pi/2}) - j\frac{1}{2}e^{j\pi/2 n}H_d(e^{j\pi/2})$

$$|H_d(e^{j\pi/2})| = \left|\frac{2(\frac{1}{3} - e^{-j\pi/2})}{1 - \frac{1}{3}e^{-j\pi/2}}\right| = 2\frac{|\frac{1}{3}+j|}{|1+\frac{1}{3}j|} = 2\frac{\sqrt{\frac{1}{9}+1}}{\sqrt{1+\frac{1}{9}}} = 2$$

$$\arg H_d(e^{j\pi/2}) = \arg\frac{2(\frac{1}{3} - e^{-j\pi/2})}{1 - \frac{1}{3}e^{-j\pi/2}} = \arg\frac{2(\frac{1}{3}+j)}{1+\frac{1}{3}j}$$
$$= \arg\frac{\frac{2}{3}+2j}{1+\frac{1}{3}j}\,\frac{1-\frac{1}{3}j}{1-\frac{1}{3}j} = \arg\frac{\frac{2}{3}+\frac{2}{3}+2j-\frac{2}{9}j}{1+\frac{1}{9}-\frac{1}{3}j+\frac{1}{3}j}$$
$$= \arg\frac{\frac{4}{3}+\frac{16}{9}j}{10/9} = \arg\left[\frac{36}{30}+j\frac{16}{10}\right] \longrightarrow$$

$$\ldots \arg H_d(e^{j\pi/2}) = \arg\left[\tfrac{6}{5} + j\tfrac{8}{5}\right]$$

$$= \arctan \frac{8/5}{6/5} = \arctan \frac{8}{6} = \arctan \frac{4}{3}$$

$\rightarrow$ Since $H_d(e^{j\omega})$ is conjugate symmetric, we have

$$|H_d(e^{-j\pi/2})| = |H_d(e^{j\pi/2})| = 2$$

$$\arg H_d(e^{-j\pi/2}) = -\arg H_d(e^{j\pi/2}) = -\arctan\frac{4}{3}$$

$\rightarrow$ Call $\arctan\frac{4}{3}$ "ϕ".

Then $H_d(e^{j\pi/2}) = 2e^{j\phi}$; $H_d(e^{-j\pi/2}) = 2e^{-j\phi}$

So $y_d[n] = j\frac{1}{2}e^{-j\frac{\pi}{2}n}\cdot 2e^{-j\phi} - j\frac{1}{2}e^{j\frac{\pi}{2}n}\cdot 2e^{j\phi}$

$$= je^{-j(\frac{\pi}{2}n+\phi)} - je^{j(\frac{\pi}{2}n+\phi)}$$

$$= 2\left[\frac{1}{2j}e^{j(\frac{\pi}{2}n+\phi)} - \frac{1}{2j}e^{-j(\frac{\pi}{2}n+\phi)}\right]$$

$$= 2\sin\left[\frac{\pi}{2}n + \phi\right]$$

Apply $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$

with $A = \frac{\pi}{2}n$ and $B = \phi$:

$$y_d[n] = 2\sin\frac{\pi}{2}n\cos\phi + 2\cos\frac{\pi}{2}n\sin\phi$$

$$= 2\cos\phi\sin\frac{\pi}{2}n + 2\sin\phi\cos\frac{\pi}{2}n$$

$\longrightarrow$

So

5C

$$Y_d(e^{j\omega}) = 2\cos\phi \frac{\pi}{j} \sum_{\ell=-\infty}^{\infty} \left[\delta(\omega - \tfrac{\pi}{2} - 2\pi\ell) - \delta(\omega + \tfrac{\pi}{2} - 2\pi\ell) \right]$$
$$+ 2\sin\phi \, \pi \sum_{\ell=-\infty}^{\infty} \left[\delta(\omega - \tfrac{\pi}{2} - 2\pi\ell) + \delta(\omega + \tfrac{\pi}{2} - 2\pi\ell) \right]$$

$Y_a(\Omega)$ = fundamental period of $Y_d(e^{j\Omega T})$

= fundamental period of $Y_d(e^{j\Omega/2000})$

$$= \frac{2\pi}{j} \cos\phi \left[\delta\left(\frac{\Omega}{2000} - \frac{\pi}{2}\right) - \delta\left(\frac{\Omega}{2000} + \frac{\pi}{2}\right) \right]$$
$$+ 2\pi \sin\phi \left[\delta\left(\frac{\Omega}{2000} - \frac{\pi}{2}\right) + \delta\left(\frac{\Omega}{2000} + \frac{\pi}{2}\right) \right]$$

So $y_a(t) = 2\cos\phi \sin(1000\pi t) + 2\sin\phi \cos(1000\pi t)$

Apply the trig identity $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ again to get

$$y_a(t) = 2\sin(1000\pi t + \phi)$$
$$= 2\sin(1000\pi t + \arctan \tfrac{4}{3})$$

3. **25/20 pts.** A causal FIR filter H has impulse response

$$h[n] = \frac{1}{4}\Big(u[n] - u[n-4]\Big)$$
$$= \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3].$$

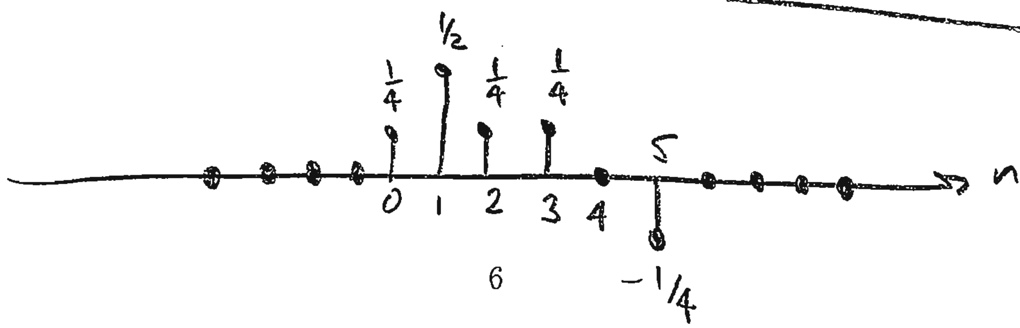
The system input is given by

$$x[n] = u[n] - 2u[n-2] + u[n-3]$$
$$= \delta[n] + \delta[n-1] - \delta[n-2].$$

Use time domain convolution to find the system output $y[n]$.

$$y[n] = h[n] * x[n]$$
$$= h[n] * \big(\delta[n] + \delta[n-1] - \delta[n-2]\big)$$
$$= h[n] + h[n-1] - h[n-2]$$
$$= \tfrac{1}{4}\delta[n] + \tfrac{1}{4}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3]$$
$$+ \tfrac{1}{4}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3] + \tfrac{1}{4}\delta[n-4]$$
$$- \tfrac{1}{4}\delta[n-2] - \tfrac{1}{4}\delta[n-3] - \tfrac{1}{4}\delta[n-4] - \tfrac{1}{4}\delta[n-5]$$

$$\boxed{y[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3] - \tfrac{1}{4}\delta[n-5]}$$



4. **25/20 pts.** A causal FIR filter H has impulse response

$$\begin{aligned} h[n] &= \frac{1}{4}\Big(u[n] - u[n-4]\Big) \\ &= \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3]. \end{aligned}$$

The system input is given by

$$\begin{aligned} x[n] &= u[n] - 2u[n-2] + u[n-3] \\ &= \delta[n] + \delta[n-1] - \delta[n-2]. \end{aligned}$$

(a) **8/6 pts.** Find the transfer function $H(z)$. Be sure to specify the region of convergence.

$$H(z) = \sum_{n \in \mathbb{Z}} h[n] z^{-n}$$

$$= \underline{\underline{\frac{1}{4} z^{0} + \frac{1}{4} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{4} z^{-3}}}$$

ROC: $|z| > 0$ (The whole finite z-plane <u>except</u> the origin.

$$H(z) = \frac{1}{4} + \frac{1}{4} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{4} z^{-3}$$

Problem 4, cont...

(b) **17/14 pts.** Use the z-transform to find the system output $y[n]$.

$$X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n} = z^0 + z^{-1} - z^{-2}$$
$$= 1 + z^{-1} - z^{-2}, \quad |z| > 0.$$

$$Y(z) = X(z)H(z)$$
$$= H(z) + z^{-1}H(z) - z^{-2}H(z)$$
$$= \tfrac{1}{4} + \tfrac{1}{4}z^{-1} + \tfrac{1}{4}z^{-2} + \tfrac{1}{4}z^{-3}$$
$$+ \tfrac{1}{4}z^{-1} + \tfrac{1}{4}z^{-2} + \tfrac{1}{4}z^{-3} + \tfrac{1}{4}z^{-4}$$
$$- \tfrac{1}{4}z^{-2} - \tfrac{1}{4}z^{-3} - \tfrac{1}{4}z^{-4} - \tfrac{1}{4}z^{-5}$$

$$= \tfrac{1}{4} + \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} + \tfrac{1}{4}z^{-3} - \tfrac{1}{4}z^{-5}$$

$$= y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \overset{0}{y[4]}z^{-4} + y[5]z^{-5}$$

$$\underline{\underline{y[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3] - \tfrac{1}{4}\delta[n-5]}}$$

5. **25/20 pts.** A causal FIR filter H has a four-point (finite length) impulse response given by

$$\begin{aligned} h[n] &= [\tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}] \\ &= \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3], \quad 0 \le n \le N-1. \end{aligned}$$

The system input is a four-point (finite length) signal given by

$$\begin{aligned} x[n] &= [1\ 1\ {-1}\ 0] \\ &= \delta[n] + \delta[n-1] - \delta[n-2], \quad 0 \le n \le N-1. \end{aligned}$$

Use the DFT to find the finite length system output $y[n]$.

NOTE: in this problem, you are being asked to use the DFT to implement *linear convolution*, **not** *circular convolution.*

Zero pad both signals to length $4+4-1 = 7$.

$\underline{N=7}$.

$W_7 = e^{-j2\pi/7}$

$h_7[n] = [\tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ 0\ 0\ 0]$

$$H_7[k] = \sum_{n=0}^{6} h_7[n] W_7^{nk}$$

$$= \tfrac{1}{4}W_7^0 + \tfrac{1}{4}W_7^k + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k}$$

$x_7[n] = [1\ 1\ {-1}\ 0\ 0\ 0\ 0]$

$$X_7[k] = \sum_{n=0}^{6} x_7[n] W_7^{nk}$$

$$= W_7^0 + W_7^k - W_7^{2k}$$

$$Y_7[k] = X_7[k]\,H_7[k]$$
$$= \left(W_7^0 + W_7^k - W_7^{2k}\right) H_7[k]$$
$$= \tfrac{1}{4}W_7^0 + \tfrac{1}{4}W_7^k + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k}$$
$$+ \tfrac{1}{4}W_7^k + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k} + \tfrac{1}{4}W_7^{4k}$$
$$- \tfrac{1}{4}W_7^{2k} - \tfrac{1}{4}W_7^{3k} - \tfrac{1}{4}W_7^{4k} - \tfrac{1}{4}W_7^{5k}$$
$$= \tfrac{1}{4}W_7^0 + \tfrac{1}{2}W_7^k + \tfrac{1}{4}W_7^{2k} + \tfrac{1}{4}W_7^{3k} + 0\,W_7^{4k} - \tfrac{1}{4}W_7^{5k}$$
$$\equiv y_7[0]W_7^0 + y_7[1]W_7^k + y_7[2]W_7^{2k} + \cdots + y_7[6]W_7^{6k}$$
$$\Rightarrow \quad y_7[n] = \left[\tfrac{1}{4} \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad 0 \quad -\tfrac{1}{4} \quad 0\right]$$
$$= \tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2] + \tfrac{1}{4}\delta[n-3]$$
$$-\tfrac{1}{4}\delta[n-5], \quad 0 \le n < 7$$