ECE 4213/5213
Test 1

Thursday, October 30, 2008
12:00 PM - 1:15 PM

Fall 2008

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, Circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

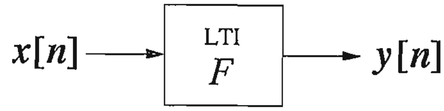
1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________ Date: ____________

1. **25/20 pts.** The **causal** discrete-time LTI system F shown below has input $x[n]$ and output $y[n]$ related by the difference equation $y[n] - 4y[n-1] = x[n]$.



(a) **8/6 pts.** Find the transfer function $F(z)$. Be sure to specify the ROC.

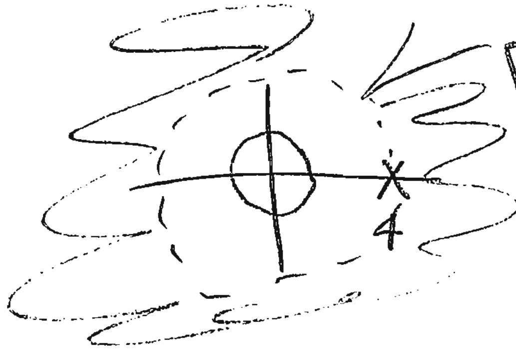
$$Y(z)\left[1-4z^{-1}\right] = X(z)$$

$$F(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-4z^{-1}}$$

one pole @ $z=4$

Since the system is causal, the ROC is exterior...

$\boxed{\text{ROC: } |z|>4}$



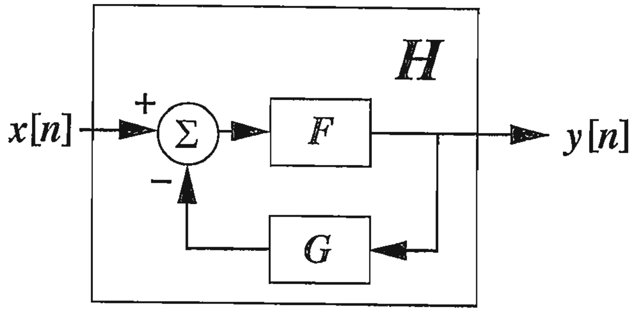
(b) **5/4 pts.** The system F is unstable; briefly explain how we know this.

A causal LTI system with a rational transfer function is BIBO stable iff all the poles are inside the unit circle.

In this case, we have a pole at $z=4$, which is <u>outside</u> the unit circle... so F is unstable.

Problem 1, cont...

(c) 12/10 pts. As shown in the figure below, we now add negative feedback to the system F from part (a) to form a new causal discrete-time LTI system H.



The purpose of the feedback is to stabilize the system F by moving the pole. The impulse response of LTI system G is given by $g[n] = K\delta[n]$, where $K \in \mathbb{R}$ is a real constant.

For what values of the constant K is the overall system H BIBO stable?

Table: $G(z) = K$, ROC: all z.

$$H(z) = \frac{F(z)}{1+F(z)G(z)} = \frac{\frac{1}{1-4z^{-1}}}{1+\frac{K}{1-4z^{-1}}} \cdot \frac{1-4z^{-1}}{1-4z^{-1}} = \frac{1}{1+K-4z^{-1}}$$

$$= \left(\frac{1}{1+K}\right) \frac{1}{1-\frac{4}{K+1}z^{-1}}$$

There is one pole at $z = \frac{4}{K+1}$

The system H will be BIBO stable provided that this pole is inside the unit circle, i.e., provided that $\left|\frac{4}{K+1}\right| = \frac{4}{|K+1|} < 1$.

$\Rightarrow K+1 < -4$ or $K+1 > 4$

$$\boxed{K < -5 \text{ or } K > 3}$$

2. **25/20 pts.** A causal and stable LTI digital filter H has input $x[n]$ and output $y[n]$ related by the difference equation $y[n] - \frac{1}{2}y[n-1] + \frac{1}{16}y[n-2] = x[n] + \frac{1}{2}x[n-1]$.

The input is given by $x[n] = \left(\frac{1}{4}\right)^n u[n]$.

Find the output signal $y[n]$.

Hint: any table of DTFT pairs can be converted into a table of z-transform pairs by making the change of variable $z = e^{j\omega}$. The resulting z-transform pairs are valid for the region of convergence that includes the unit circle.

$$Y(z)\left[1 - \tfrac{1}{2}z^{-1} + \tfrac{1}{16}z^{-2}\right] = X(z)\left[1 + \tfrac{1}{2}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{16}z^{-2}} = \frac{1+\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1+\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})^2},$$

ROC: $|z| > \frac{1}{4}$
(exterior since causal)

$$X(z) = \left(\tfrac{1}{4}\right)^n u[n] \overset{\text{Table}}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \tfrac{1}{4}$$

$$Y(z) = X(z)H(z) = \frac{1+\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})^3}, \quad |z| > \tfrac{1}{4}$$

METHOD I: no PFE

$$Y(z) = \underbrace{\frac{1}{(1-\frac{1}{4}z^{-1})^3}}_{|z|>\frac{1}{4}} + \frac{1}{2}z^{-1}\underbrace{\frac{1}{(1-\frac{1}{4}z^{-1})^3}}_{|z|>\frac{1}{4}} \qquad (*)$$

Now, the last entry in the DTFT table on the formula sheet is:

$$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n], \quad |a|<1 \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{(1-ae^{-j\omega})^r}.$$

With $z = e^{j\omega}$, this becomes

$$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n], \quad |a|<1 \overset{Z}{\longleftrightarrow} \frac{1}{(1-az^{-1})^r}, \quad |z|>|a|$$

must include unit circle.

So, we have $\frac{1}{(1-\frac{1}{4}z^{-1})^3}$, $|z|>\frac{1}{4}$ $\overset{Z}{\longleftrightarrow}$ $\frac{(n+2)!}{n!\,2!}\left(\frac{1}{4}\right)^n u[n]$

$$= \frac{(n+2)(n+1)n!}{n!\,2}\left(\frac{1}{4}\right)^n u[n]$$

$$= \frac{1}{2}(n+2)(n+1)\left(\frac{1}{4}\right)^n u[n] \quad ☆$$

Time Shift Property:

$$\frac{z^{-1}}{(1-\frac{1}{4}z^{-1})^3}, |z|>\frac{1}{4} \overset{Z}{\longleftrightarrow} \frac{1}{2}(n+1)n\left(\frac{1}{4}\right)^{n-1} u[n-1]$$

$$= (n^2+n)\left(\tfrac{1}{2}\right)4\left(\tfrac{1}{4}\right)^n u[n-1]$$

$$= (2n^2+2n)\left(\tfrac{1}{4}\right)^n u[n-1]$$

So $\frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})^3}$, $|z|>\frac{1}{4}$ $\overset{Z}{\longleftrightarrow}$ $(n^2+n)\left(\frac{1}{4}\right)^n u[n-1]$ ☆

From (*) on page 4,

$$\boxed{y[n] = \tfrac{1}{2}(n+2)(n+1)\left(\tfrac{1}{4}\right)^n u[n] + (n^2+n)\left(\tfrac{1}{4}\right)^n u[n-1]}$$

METHOD II: PFE

$$Y(z) = \frac{1+\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})^3} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{(1-\frac{1}{4}z^{-1})^2} + \frac{C}{(1-\frac{1}{4}z^{-1})^3}$$

Multiply both sides by $(1-\frac{1}{4}z^{-1})^3$

For C: evaluate at $z^{-1}=4$

For B: differentiate then evaluate at $z^{-1}=4$

For A: differentiate twice then evaluate at $z^{-1}=4$.

$$C = 1+\tfrac{1}{2}\theta \Big|_{\theta=4} = 1+2 = 3.$$

5A

$$B: \frac{d}{d\theta}\left\{ 1+\tfrac{1}{2}\theta = A(1-\tfrac{1}{4}\theta)^2 + B(1-\tfrac{1}{4}\theta)+C \right\}\Big|_{\theta=4}$$

$$\left\{\tfrac{1}{2} = 2A(1-\tfrac{1}{4}\theta)(-\tfrac{1}{4}) - \tfrac{1}{4}B\right\}\Big|_{\theta=4}$$

$$\tfrac{1}{2} = -\tfrac{1}{4}B \Rightarrow B = -2$$

$$A: \frac{d^2}{d\theta^2}\left\{ 1+\tfrac{1}{2}\theta = A(1-\tfrac{1}{4}\theta)^2 + B(1-\tfrac{1}{4}\theta)+C \right\}\Big|_{\theta=4}$$

$$\frac{d}{d\theta}\left\{ \tfrac{1}{2} = 2A(1-\tfrac{1}{4}\theta)(-\tfrac{1}{4}) - \tfrac{1}{4}B\right\}\Big|_{\theta=4}$$

$$\left\{0 = \tfrac{1}{16}2A\right\}_{\theta=4} \Rightarrow A = 0$$

$$Y(z) = \frac{3}{(1-\frac{1}{4}z^{-1})^3} - \frac{2}{(1-\frac{1}{4}z^{-1})^2}$$

Check: $$Y(z) = \frac{3-2(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{4}z^{-1})^3} = \frac{3-2+\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})^3} = \frac{1+\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})^3} \checkmark$$

Again using the DTFT Table final entry:

$$\frac{-2}{(1-\frac{1}{4}z^{-1})^3},\ |z|>\tfrac{1}{4} \longleftrightarrow -2(n+1)(\tfrac{1}{4})^n u[n]$$

$$\frac{3}{(1-\frac{1}{4}z^{-1})^3},\ |z|>\tfrac{1}{4} \longleftrightarrow \frac{3(n+3-1)!}{n!\,2!}(\tfrac{1}{4})^n u[n]$$

$$= \tfrac{3}{2}(n+1)(n+2)(\tfrac{1}{4})^n u[n]$$

$\longrightarrow$

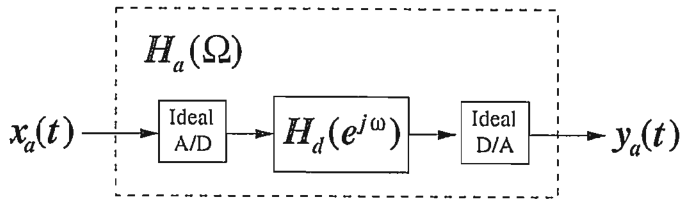
Putting it all together, 5B

$$\boxed{y[n] = \frac{3}{2}(n+2)(n+1)\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n]}$$

$\Rightarrow$ you can use matlab to verify that this is the same as the first answer $\forall n \in \mathbb{Z}$.

3. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter. The sampling frequency of the A/D and D/A converters is $F_T = 44$ kHz. H_d is causal and has input/output relation

$$y[n] - \frac{2}{3}y[n-1] + \frac{1}{9}y[n-2] = x[n].$$



Assume that all input signals $x_a(t)$ are band limited with $|X_a(\Omega)| = 0 \ \forall\ |\Omega| > \pi/T$ so that aliasing does not occur.

(a) **9/8 pts.** Find the frequency response $H_d(e^{j\omega})$ of the digital filter.

$$Y(e^{j\omega})\left[1 - \frac{2}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega}\right] = X(e^{j\omega})$$

$$H_d(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{2}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega}}$$

$$= \frac{1}{\left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{-j\omega}\right)}$$

$$= \frac{1}{\left(1 - \frac{1}{3}e^{-j\omega}\right)^2}$$

Problem 3, cont...

(b) **8/6 pts.** Find the impulse response $h_d[n]$ of the digital filter.

DTFT Table, penultimate entry:

$$h_d[n] = (n+1)\left(\tfrac{1}{3}\right)^n u[n]$$

(c) **8/6 pts.** Find the frequency response $H_a(\Omega)$ of the overall analog system with input $x_a(t)$ and output $y_a(t)$.

$$\Omega_s = \frac{2\pi}{T} = 44\text{kHz}\ \frac{2\pi\ \text{rad}}{\text{Hz}} = 88{,}000\pi\ \frac{\text{rad}}{\text{sec}}\ ;\quad \frac{\Omega_s}{2} = 44{,}000\pi\ \frac{\text{rad}}{\text{sec}}$$

$$T = \frac{2\pi}{\Omega_s} = \frac{2\pi}{88000\pi}\ \text{sec} = \frac{1}{44{,}000}\ \text{sec}$$

$$H_a(\Omega) = \begin{cases} H_d(e^{j\Omega T}) & , |\Omega| < \frac{\Omega_s}{2} \\ 0 & , |\Omega| > \Omega_s/2 \end{cases}$$

$$= \begin{cases} \dfrac{1}{\left(1-\frac{1}{3}e^{-j\Omega/44{,}000}\right)^2} & , |\Omega| < 44{,}000\pi \\ 0 & , |\Omega| > 44{,}000\pi \end{cases}$$

4. **25/20 pts.** Let $w[n]$ and $x[n]$ be complex-valued discrete-time signals with domain $\mathbb{Z}$ and range $\mathbb{C}$. Let $y[n]$ be the correlation of $w[n]$ and $x[n]$, which is given by

$$y[\ell] = \sum_{n=-\infty}^{\infty} w[n]x[n-\ell].$$

Assume that the signals $w[n]$, $x[n]$, and $y[n]$ are all in the spaces $\ell^1(\mathbb{Z})$ and $\ell^2(\mathbb{Z})$ so that they are all both absolutely summable and square summable.

Find an expression for the DTFT $Y(e^{j\omega})$ in terms of the DTFT's $W(e^{j\omega})$ and $X(e^{j\omega})$.

(A) Direct Solution:

$$Y(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} y[\ell]e^{-j\omega\ell} = \sum_{\ell=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} w[n]x[n-\ell] \right\} e^{-j\omega\ell}$$

$$= \sum_{n=-\infty}^{\infty} w[n] \left\{ \sum_{\ell=-\infty}^{\infty} x[n-\ell]e^{-j\omega\ell} \right\}$$

Let $m = n - \ell$

$\ell = n - m$

$\ell \to \infty$; $m \to -\infty$

$\ell \to -\infty$; $m \to \infty$

$$= \sum_{n=-\infty}^{\infty} w[n] \left\{ \sum_{m=+\infty}^{-\infty} x[m]e^{-j\omega(n-m)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} w[n] \left\{ \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega n} e^{j\omega m} \right\}$$

$$= \sum_{n=-\infty}^{\infty} w[n] e^{-j\omega n} \left\{ \sum_{m=-\infty}^{\infty} x[m]e^{-j\theta m} \Big|_{\theta=-\omega} \right\}$$

$$= W(e^{j\omega}) X(e^{j\theta}) \Big|_{\theta=-\omega}$$

$$= \underline{\underline{W(e^{j\omega}) X(e^{-j\omega})}}$$

(B) Solution using DTFT properties:

$$y[\ell] = \sum_{n=-\infty}^{\infty} w[n]\,x[n-\ell]$$

$$= \sum_{n=-\infty}^{\infty} w[n]\,x[-(\ell-n)]$$

$$= w[\ell] * x[-\ell] \qquad \{\text{Book } (2.124)\}$$

So, by the convolution property of the DTFT,

$$Y(e^{j\omega}) = \text{DTFT}\{w[\ell]\}\,\text{DTFT}\{x[-\ell]\}$$

Time reversal property:

$$\underline{\underline{Y(e^{j\omega}) = W(e^{j\omega})\,X(e^{-j\omega})}}$$

5. **25/20 pts.** Let $x[n]$ and $v[n]$ be a pair of 4-point discrete-time signals with

$$\begin{aligned} x[n] &= [0\ \ 1\ \ 2\ \ 3] \\ &= \delta[n-1] + 2\delta[n-2] + 3\delta[n-3],\ 0 \le n \le 3. \end{aligned}$$

The 4-point circular convolution of $x[n]$ with $v[n]$ is given by

$$\begin{aligned} y[n] &= x[n] \; ④ \; v[n] \\ &= [-4\ \ 0\ \ 4\ \ 0] \\ &= -4\delta[n] + 4\delta[n-2],\ 0 \le n \le 3. \end{aligned}$$

$$W_4 = e^{-j2\pi/4} = e^{-j\pi/2}$$
$$= -j$$

Find the **linear convolution**

$$c[n] = x[n] * v[n]$$

of the signals $x[n]$ and $v[n]$.

Hint: use the DFT to find $v[n]$ from the given information. Then perform linear convolution in the time domain to find $c[n]$.

$$X[k] = \text{DFT}_4\{x[n]\} = \sum_{n=0}^{3} x[n] W_4^{nk} = \sum_{n=0}^{3} x[n] (-j)^{nk}$$

$$= (-j)^k + 2(-j)^{2k} + 3(-j)^{3k}$$

$k=0$: $X[0] = 1+2+3 = 6$

$k=1$: $X[1] = -j + 2(-1) + 3(j) = -2+2j$

$k=2$: $X[2] = -1 + 2(1) + 3(-1) = -2$

$k=3$: $X[3] = j + 2(-1) + 3(-j) = -2-2j$

$$X[k] = [6\ \ (-2+2j)\ \ -2\ \ (-2-2j)]$$

$$Y[k] = DFT_4\{y[n]\} = \sum_{n=0}^{3} y[n] W_4^{nk} = \sum_{n=0}^{3} y[n](-j)^{nk} = -4(-j)^0 + 4(-j)^{2k}$$
$$= -4 + 4\left[(-1)^{2k}(j)^{2k}\right]$$
$$= -4 + 4(1)^k(-1)^k$$
$$= -4 + 4(-1)^k$$

$k=0: Y[0] = -4+4 = 0$

$k=1: Y[1] = -4-4 = -8$

$k=2: Y[2] = -4+4 = 0$

$k=3: Y[3] = -4-4 = -8$

$$Y[k] = [0 \ -8 \ 0 \ -8]$$

Now, $Y[k] = X[k]V[k]$, so $V[k] = \dfrac{Y[k]}{X[k]}$

$$V[k] = \left[\frac{0}{6} \ \left(\frac{-8}{-2+2j}\cdot\frac{-2-2j}{-2-2j}\right) \ \frac{0}{-2} \ \left(\frac{-8}{-2-2j}\cdot\frac{-2+2j}{-2+2j}\right)\right]$$

$$= \left[0 \ \left(\frac{16+16j}{8}\right) \ 0 \ \left(\frac{16-16j}{8}\right)\right] = [0 \ (2+2j) \ 0 \ (2-2j)]$$

$$v[n] = DFT_4^{-1}\{V[k]\} = \frac{1}{4}\sum_{k=0}^{3} V[k] W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} V[k] j^{nk}$$

$$= \frac{1}{4}(2+2j)j^n + \frac{1}{4}(2-2j)j^{3n} = \left(\tfrac{1}{2}+\tfrac{1}{2}j\right)j^n + \left(\tfrac{1}{2}-\tfrac{1}{2}j\right)(-j)^n$$

$n=0: v[0] = \frac{1}{2}+\frac{1}{2}j + \frac{1}{2} - \frac{1}{2}j = 1$

$n=1: v[1] = \left(\frac{1}{2}+\frac{1}{2}j\right)j + \left(\frac{1}{2}-\frac{1}{2}j\right)(-j) = -\frac{1}{2}+\frac{1}{2}j - \frac{1}{2} - \frac{1}{2}j = -1$

$n=2: v[2] = \left(\frac{1}{2}+\frac{1}{2}j\right)(-1) + \left(\frac{1}{2}-\frac{1}{2}j\right)(-1) = -\frac{1}{2}-\frac{1}{2}j - \frac{1}{2}+\frac{1}{2}j = -1$

$n=3: v[3] = \left(\frac{1}{2}+\frac{1}{2}j\right)(-j) + \left(\frac{1}{2}-\frac{1}{2}j\right)j = \frac{1}{2}-\frac{1}{2}j + \frac{1}{2}+\frac{1}{2}j = 1$

$$v[n] = [1 \ -1 \ -1 \ 1] = \delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3], \ 0 \le n \le 3.$$

Zero padding $x[n]$ and $v[n]$ to all of $\mathbb{Z}$,

we obtain

$$\hat{x}[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3], \quad n \in \mathbb{Z}$$

$$\hat{v}[n] = \delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3], \quad n \in \mathbb{Z}.$$

$$\begin{aligned}\hat{c}[n] &= \hat{x}[n] * \hat{v}[n] \\ &= \delta[n-1] * \hat{v}[n] + 2\delta[n-2] * \hat{v}[n] + 3\delta[n-3] * \hat{v}[n] \\ &= \hat{v}[n-1] + 2\hat{v}[n-2] + 3\hat{v}[n-3]\end{aligned}$$

$$\begin{aligned}= \delta[n-1] - \delta[n-2] - \delta[n-3] + \delta[n-4] \\ +2\delta[n-2] - 2\delta[n-3] - 2\delta[n-4] + 2\delta[n-5] \\ +3\delta[n-3] - 3\delta[n-4] - 3\delta[n-5] + 3\delta[n-6]\end{aligned}$$

$$= \delta[n-1] + \delta[n-2] - 4\delta[n-4] - \delta[n-5] + 3\delta[n-6], \quad n \in \mathbb{Z}$$

Finally,

$$\begin{aligned}c[n] &= \hat{c}[n], \quad 0 \leq n \leq 6 \\ &= \delta[n-1] + \delta[n-2] - 4\delta[n-4] - \delta[n-5] + 3\delta[n-6], \quad 0 \leq n \leq 6 \\ &= [0 \;\; 1 \;\; 1 \;\; 0 \;\; -4 \;\; -1 \;\; 3]\end{aligned}$$