ECE 4213/5213
Test 1

Thursday, November 5, 2009
12:00 PM - 1:15 PM

Fall 2009 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, Circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25/20 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

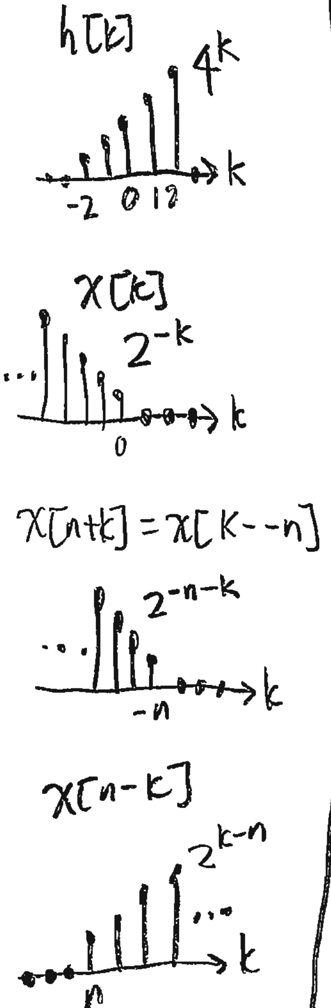
$$h[n] = \begin{cases} 4^n, & -2 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

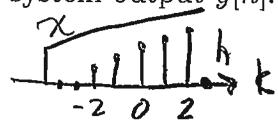
The system input is given by

$$x[n] = 2^{-n} u[-n].$$

Use time domain convolution to find the system output $y[n]$.



Case I) $n \le -2$:

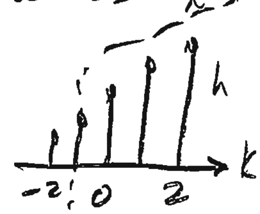


$$y[n] = \sum_{k=-2}^{2} 4^k 2^{k-n} = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{2} 4^k 2^k = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{2} 8^k$$

$$= \left(\tfrac{1}{2}\right)^n \left[\frac{8^{-2} - 8^3}{1-8}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{\frac{1}{64} - 512}{-7}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{1 - 32{,}768}{(-7)(64)}\right]$$

$$= \left(\tfrac{1}{2}\right)^n \left[\frac{32{,}767}{(7)(64)}\right] = \frac{4681}{64}\left(\tfrac{1}{2}\right)^n.$$

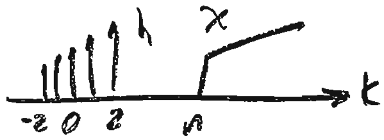
Case II) $n > -2$ and $n \le 2$; $-2 < n \le 2$



$$y[n] = \sum_{k=n}^{2} 4^k 2^{k-n} = \left(\tfrac{1}{2}\right)^n \sum_{k=n}^{2} 8^k = \left(\tfrac{1}{2}\right)^n \left[\frac{8^n - 8^3}{1-8}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{512 - 8^n}{7}\right]$$

$$= \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{1}{7}\left(\tfrac{1}{2}\right)^n 8^n = \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{1}{7}4^n.$$

case III) $n > 2$



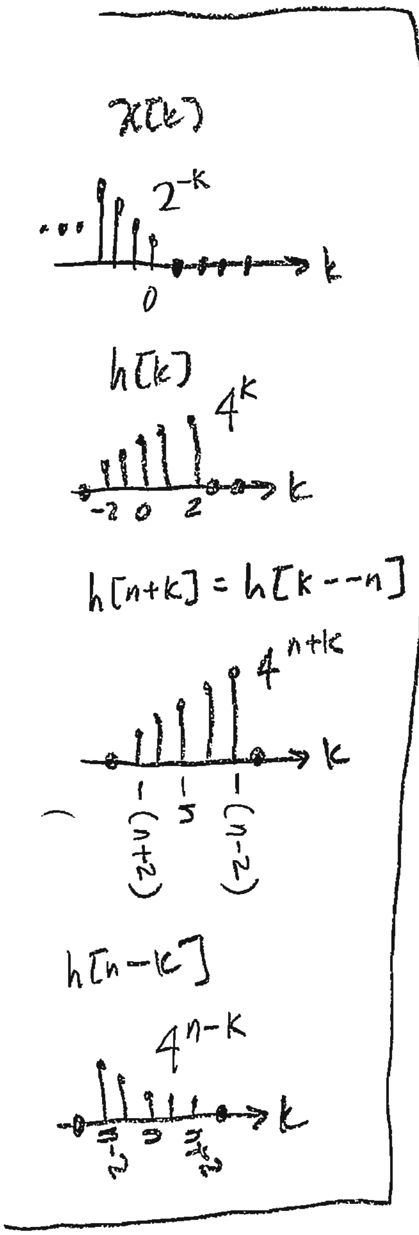
$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0.$$

ALL TOGETHER:

$$\boxed{y[n] = \begin{cases} \frac{4681}{64}\left(\tfrac{1}{2}\right)^n, & n \le -2 \\ \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{4^n}{7}, & -2 < n \le 2 \\ 0, & n > 2 \end{cases}}$$

OTHER WAY:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



Case I) $n+2 \le 0$: $n \le -2$: overlap from $k = n-2$ to $k = n+2$

$$y[n] = \sum_{k=n-2}^{n+2} 2^{-k} 4^{n-k} = 4^n \sum_{k=n-2}^{n+2} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^k$$

$$= 4^n \sum_{k=n-2}^{n+2} \left(\tfrac{1}{8}\right)^k = 4^n \left[\frac{\left(\frac{1}{8}\right)^{n-2} - \left(\frac{1}{8}\right)^{n+3}}{1-\frac{1}{8}}\right]$$

$$= 4^n \left[\frac{\left(\frac{1}{8}\right)^{n-2} - \left(\frac{1}{8}\right)^{n-2}\left(\frac{1}{8}\right)^5}{7/8}\right] = 4^n \left(\tfrac{1}{8}\right)^{n-2} \frac{8}{7}\left[1 - \left(\tfrac{1}{8}\right)^5\right]$$

$$= 4^n \left(\tfrac{1}{8}\right)^n \left(\tfrac{1}{8}\right)^{-2} 8\, \tfrac{1}{7}\left[\frac{8^5-1}{8^5}\right] = \left(\tfrac{1}{2}\right)^n \frac{8^3}{7}\left[\frac{8^5-1}{8^5}\right]$$

$$= \left(\tfrac{1}{2}\right)^n \frac{512}{7}\left[\frac{32{,}767}{32{,}768}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{4{,}681 \cdot 7 \cdot 512}{7 \cdot 32{,}768}\right]$$

$$= \left(\tfrac{1}{2}\right)^n \left[\frac{4681 \cdot 512}{64 \cdot 512}\right] = \frac{4681}{64}\left(\tfrac{1}{2}\right)^n.$$

Case II) $n+2 > 0$ and $n-2 \le 0$: $n > -2$ and $n \le 2$

: $-2 < n \le 2$: overlap from $k = n-2$ to $k = 0$:

$$y[n] = \sum_{k=n-2}^{0} 2^{-k} 4^{n-k} = 4^n \sum_{k=n-2}^{0} 8^{-k} = 4^n \sum_{k=n-2}^{0} \left(\tfrac{1}{8}\right)^k$$

$$= 4^n \left[\frac{\left(\frac{1}{8}\right)^{n-2} - \frac{1}{8}}{1-\frac{1}{8}}\right] = 4^n \left[\frac{64\left(\frac{1}{8}\right)^n - \frac{1}{8}}{7/8}\right] = \frac{4^n}{7} 8 \left[64\left(\tfrac{1}{8}\right)^n - \tfrac{1}{8}\right]$$

$$= \frac{64 \cdot 8}{7}\left(\tfrac{1}{2}\right)^n - \frac{4^n}{7} = \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{4^n}{7}.$$

Case III) $n-2 > 0$: $n > 2$: no overlap : $y[n] = 0$.

$\longrightarrow$ Same Answer as the other way.

2. **25/20 pts.** A continuous-time LTI system H has impulse response $h(t) = e^{-3t}u(t)$ and input $x(t) = e^{-4t}u(t)$. Use the Fourier transform to find the system output $y(t)$.

Table: $H(\omega) = \frac{1}{3+j\omega}$ Table: $X(\omega) = \frac{1}{4+j\omega}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(3+j\omega)(4+j\omega)} = \frac{A}{3+j\omega} + \frac{B}{4+j\omega}$$

$$A = \frac{1}{4+\theta}\Big|_{\theta=-3} = \frac{1}{1} = 1$$

$$B = \frac{1}{3+\theta}\Big|_{\theta=-4} = \frac{1}{-1} = -1$$

$$Y(\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega}$$

Table: $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$

3. **25/20 pts.** H is a discrete-time LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$y[n] - \frac{1}{3}y[n-1] = x[n] - \frac{3}{2}x[n-1] - x[n-2].$$

An inverse system G is desired to "undo" the action of H.

(a) **13/10 pts.** Assume that the inverse system G is required to be **causal**. Find the impulse response $g[n]$ of the inverse system.

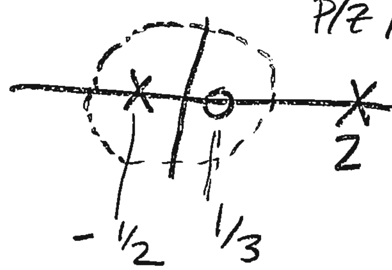
Z-transform: $Y(z) - \frac{1}{3}z^{-1}Y(z) = X(z) - \frac{3}{2}z^{-1}X(z) - z^{-2}X(z)$

$$Y(z)\left[1-\tfrac{1}{3}z^{-1}\right] = X(z)\left[1-\tfrac{3}{2}z^{-1} - z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{3}{2}z^{-1}-z^{-2}}{1-\frac{1}{3}z^{-1}} = \frac{(1-2z^{-1})(1+\frac{1}{2}z^{-1})}{1-\frac{1}{3}z^{-1}}$$

$$G(z) = \frac{1}{H(z)} = \frac{1-\frac{1}{3}z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})}$$

P/Z plot



$$= \frac{A}{1-2z^{-1}} + \frac{B}{1+1/2\,z^{-1}}$$

$$A = \left.\frac{1-\frac{1}{3}\theta}{1+\frac{1}{2}\theta}\right|_{\theta=\frac{1}{2}} = \frac{1-\frac{1}{6}}{1+\frac{1}{4}} = \frac{5/6}{5/4} = \frac{4}{6} = \frac{2}{3}$$

$$B = \left.\frac{1-\frac{1}{3}\theta}{1-2\theta}\right|_{\theta=-2} = \frac{1+2/3}{1+4} = \frac{5/3}{5} = \frac{1}{3}$$

$$G(z) = \frac{2/3}{1-2z^{-1}} + \frac{1/3}{1+1/2\,z^{-1}} \quad (*)$$

$\Rightarrow$ For causal, the ROC must be exterior: ROC: $|z|>2$.

Table: $g[n] = \frac{2}{3}2^n u[n] + \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n]$

(b) **12/10 pts.** Now, instead of causal, assume that the inverse system G is required to be **BIBO stable**. Find the impulse response $g[n]$ of the inverse system.

For stability, the ROC must include the unit circle of the z-plane. From the P-Z plot on p. 5, this means the ROC must be $\frac{1}{2} < |z| < 2$.

So (*) on p. 5 becomes

$$G(z) = \underbrace{\frac{2}{3} \cdot \frac{1}{1-2z^{-1}}}_{|z|<2} + \underbrace{\frac{1}{3} \cdot \frac{1}{1+\frac{1}{2}z^{-1}}}_{|z|>1/2}$$

Table

$-\frac{2}{3}2^n u[-n-1]$

Table

$\frac{1}{3}\left(-\frac{1}{2}\right)^n u[n]$

$$g[n] = -\frac{2}{3}2^n u[-n-1] + \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n]$$

4. **25/20 pts.** Consider a 3-point discrete-time signal

$$x[n] = [2\ 2\ -1] = 2\delta[n] + 2\delta[n-1] - \delta[n-2],\ 0 \le n \le 2.$$

The signal $x[n]$ is input to a discrete-time LTI FIR filter H with a 2-point impulse response given by

$$h[n] = [1\ -1] = \delta[n] - \delta[n-1],\ 0 \le n \le 1.$$

Use pointwise multiplication of DFT's to find the **linear** convolution $y[n] = x[n] * h[n]$.

Length $x[n] = 3$. Length $h[n] = 2$. Minimum DFT length $= 3+2-1 = 4$

$$W_4 = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

$$x_4[n] = [x[n]\ 0] = [2\ 2\ -1\ 0]$$

$$h_4[n] = [h[n]\ 0\ 0] = [1\ -1\ 0\ 0]$$

$$X_4[k] = DFT_4\{x_4[n]\} = \sum_{n=0}^{3} x_4[n] W_4^{nk} = 2 + 2W_4^k - W_4^{2k}$$

$$H_4[k] = DFT_4\{h_4[n]\} = \sum_{n=0}^{3} h_4[n] W_4^{nk} = 1 - W_4^k$$

$$Y_4[k] = X_4[k] H_4[k] = (2 + 2W_4^k - W_4^{2k})(1 - W_4^k)$$

$$= 2 + 2W_4^k - W_4^{2k}$$
$$- 2W_4^k - 2W_4^{2k} + W_4^{3k}$$

$$= 2 - 3W_4^{2k} + W_4^{3k}.$$

At this point, $y[n]$ can be deduced by observing that $Y_4[k] = \sum_{n=0}^{3} y_4[n] W_4^{nk} = y_4[0] + y_4[1]W_4^k + y_4[2]W_4^{2k} + y_4[3]W_4^{3k}$

$\Rightarrow$ So $y_4[0] = 2$, $y_4[1] = 0$, $y_4[2] = -3$, $y_4[3] = 1$

$\Rightarrow$ $y[n] = [2\ \ 0\ \ -3\ \ 1]$ $\longrightarrow$

But let's go ahead and work out the inverse DFT formally....

$$Y_4[k] = 2 - 3W_4^{2k} + W_4^{3k} = 2 + 3(-j)^{2k} + (-j)^{3k}$$
$$= 2 - 3(-1)^k + j^k$$

k	$Y_4[k]$
0	$2-3+1=0$
1	$2+3+j=5+j$
2	$2-3-1=-2$
3	$2+3-j=5-j$

$$Y_4[k] = [0 \quad 5+j \quad -2 \quad 5-j]$$

$$y[n] = \mathrm{DFT}_4^{-1}\{Y_4[k]\} = \frac{1}{4}\sum_{k=0}^{3} Y_4[k] W_4^{-nk}$$

$$= \frac{1}{4}\{(5+j)W_4^{-n} - 2W_4^{-2n} + (5-j)W_4^{-3n}\}$$

$$= \frac{1}{4}\{(5+j)(-j)^{-n} - 2(-j)^{-2n} + (5-j)(-j)^{3n}\}$$

$$= \frac{1}{4}\{(5+j)j^n - 2(j)^{2n} + (5-j)j^{3n}\} = \frac{1}{4}\{5j^n + j^{n+1} - 2(-1)^n + (5-j)j^{3n}\}$$

$$= \frac{1}{4}\{5j^n + j^{n+1} - 2(-1)^n + (5-j)(-1)^n j^n\}$$

$$y[0] = \frac{1}{4}\{5+j-2+(5-j)\} = \frac{1}{4}\cdot 8 = 2$$

$$y[1] = \frac{1}{4}\{5j - 1 + 2 - (5-j)j\} = \frac{1}{4}\{5j + 1 - 5j - 1\} = 0$$

$$y[2] = \frac{1}{4}\{-5-j-2+(5-j)(-1)\} = \frac{1}{4}\{-7-j-5+j\} = \frac{-12}{4} = -3$$

$$y[3] = \frac{1}{4}\{-5j+1+2+(5-j)j\} = \frac{1}{4}\{-5j+3+5j+1\} = \frac{4}{4} = 1$$

$$y[n] = [2 \quad 0 \quad -3 \quad 1]$$

$$= 2\delta[n] - 3\delta[n-2] + \delta[n-3], \quad 0 \le n \le 3.$$

5. **25/20 pts.** Consider a 4-point discrete-time signal $x_4[n]$ given by

$$x_4[n] = [4 \;\; 3 \;\; 2 \;\; 1] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3], \;\; 0 \le n \le 3.$$

The signal $x_4[n]$ is zero padded on the right to a length of $N = 6$ to obtain a new 6-point signal $x_6[n]$ given by

$$x_6[n] = [x_4[n] \;\; 0 \;\; 0] = [4 \;\; 3 \;\; 2 \;\; 1 \;\; 0 \;\; 0], \;\; 0 \le n \le 5.$$

Let $X_6[k] = \mathrm{DFT}_6\{x_6[n]\}$.

(a) **13/10 pts.** Find the 6-point time sequence

NOTES P. 5.43 : Time shift property : $W_N^{kn_0} X[k] \overset{DFT}{\longleftrightarrow} x[\langle n-n_0\rangle_N]$

$$y[n] = \mathrm{DFT}_6^{-1}\{W_6^{4k} X_6[k]\}.$$

So $y[n] = x_6[\langle n-4\rangle_6]$.

$y[0] = x_6[\langle -4\rangle_6] = x_6[2] = 2$

$y[1] = x_6[\langle -3\rangle_6] = x_6[3] = 1$

$y[2] = x_6[\langle -2\rangle_6] = x_6[4] = 0$

$y[3] = x_6[\langle -1\rangle_6] = x_6[5] = 0$

$y[4] = x_6[\langle 0\rangle_6] = x_6[0] = 4$

$y[5] = x_6[\langle 1\rangle_6] = x_6[1] = 3$

$$\boxed{\begin{aligned} y[n] &= [2 \;\; 1 \;\; 0 \;\; 0 \;\; 4 \;\; 3] \\ &= 2\delta[n] + \delta[n-1] + 4\delta[n-4] + 3\delta[n-5], \\ &\qquad 0 \le n \le 5 \end{aligned}}$$

Problem 5, cont...

(b) 12/10 pts. Let $V[k] = \text{Re}\{X_6[k]\}$. Find the 6-point time sequence $v[n]$.

Notes p. 5.56: $x_{pe}[n] \xleftrightarrow{DFT} \text{Re}\{X[k]\}$

Since $V[k] = \text{Re}\{X_6[k]\}$, this implies

$$v[n] = x_{6pe}[n] = \frac{1}{2}\{x_6[n] + x_6[\langle -n \rangle_6]\}$$

$$v[0] = \frac{1}{2}\{x_6[0] + x_6[\langle -0 \rangle_6]\} = \frac{1}{2}\{x_6[0] + x_6[0]\} = \frac{1}{2}\cdot 2x_6[0]$$
$$= x_6[0] = 4$$

$$v[1] = \frac{1}{2}\{x_6[1] + x_6[\langle -1 \rangle_6]\} = \frac{1}{2}\{x_6[1] + x_6[5]\} = \frac{1}{2}\{3+0\} = \frac{3}{2}$$

$$v[2] = \frac{1}{2}\{x_6[2] + x_6[\langle -2 \rangle_6]\} = \frac{1}{2}\{x_6[2] + x_6[4]\} = \frac{1}{2}\{2+0\} = 1$$

$$v[3] = \frac{1}{2}\{x_6[3] + x_6[\langle -3 \rangle_6]\} = \frac{1}{2}\{x_6[3] + x_6[3]\} = x_6[3] = 1$$

$$v[4] = \frac{1}{2}\{x_6[4] + x_6[\langle -4 \rangle_6]\} = \frac{1}{2}\{x_6[4] + x_6[2]\} = \frac{1}{2}\{0+2\} = 1$$

$$v[5] = \frac{1}{2}\{x_6[5] + x_6[\langle -5 \rangle_6]\} = \frac{1}{2}\{x_6[5] + x_6[1]\} = \frac{1}{2}\{0+3\} = \frac{3}{2}$$

$$\boxed{\begin{aligned} v[n] &= [4 \;\; \tfrac{3}{2} \;\; 1 \;\; 1 \;\; 1 \;\; \tfrac{3}{2}] \\ &= 4\delta[n] + \tfrac{3}{2}\delta[n-1] + \delta[n-2] + \delta[n-3] \\ &\quad + \delta[n-4] + \tfrac{3}{2}\delta[n-5], \quad 0 \le n \le 5 \end{aligned}}$$

"Quicker" solution:

$v[n]$ = periodically even part of $x_6[n]$

$$= \frac{1}{2}\left\{ x_6[n] + x_6[\langle -n\rangle_6] \right\}$$

$$x_6[n] = [4 \;\; 3 \;\; 2 \;\; 1 \;\; 0 \;\; 0]$$

$$x_6[\langle -n\rangle_6] = x_6[\langle 6-n\rangle_6]$$

$$= [x_6[0] \;\; x_6[5] \;\; x_6[4] \;\; x_6[3] \;\; x_6[2] \;\; x_6[1]]$$

$$= [4 \;\; 0 \;\; 0 \;\; 1 \;\; 2 \;\; 3].$$

So $v[n] = \frac{1}{2}\left\{ [4\,3\,2\,1\,0\,0] + [4\,0\,0\,1\,2\,3] \right\}$

$$= \frac{1}{2}\{8 \;\; 3 \;\; 2 \;\; 2 \;\; 2 \;\; 3\}$$

$$= \left[4 \;\; \tfrac{3}{2} \;\; 1 \;\; 1 \;\; 1 \;\; \tfrac{3}{2}\right]$$