ECE 4213/5213
Test 1

Wednesday, November 9, 2011
4:30 PM - 5:45 PM

Fall 2011 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, Circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________

2. (25/20) ________

3. (25/20) ________

4. (25/20) ________

5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________________ Date: ________________

1. **25/20 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(-\frac{1}{6}\right)^n u[n].$$

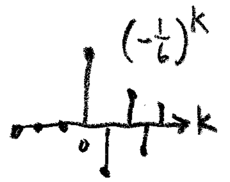
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

The system input is given by

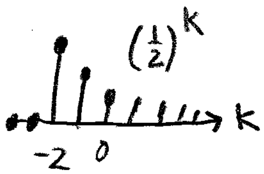
$$x[n] = \left(\frac{1}{2}\right)^n u[n+2].$$

Use time domain convolution to find the system output $y[n]$.

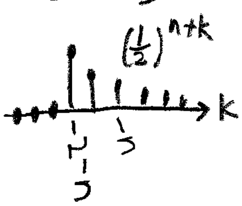
$h[k]$



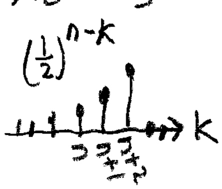
$x[k]$



$x[n+k]$

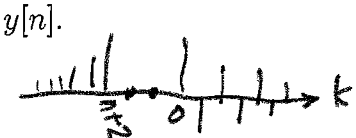


$x[n-k]$

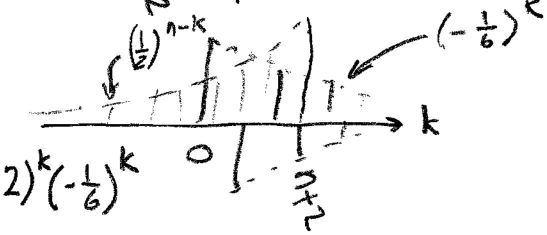


Case I) $n+2<0$; $n<-2$:

$$y[n] = \sum_{k=-\infty} 0 = 0.$$



Case II) $n+2 \geq 0$: $n \geq -2$;



$$y[n] = \sum_{k=0}^{n+2} \left(\tfrac{1}{2}\right)^{n-k}\left(-\tfrac{1}{6}\right)^k = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n+2} (2)^k \left(-\tfrac{1}{6}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n+2}\left(-\tfrac{1}{3}\right)^k = \left(\tfrac{1}{2}\right)^n \frac{\left(-\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{n+3}}{1-\left(-\frac{1}{3}\right)}$$

$$= \left(\tfrac{1}{2}\right)^n \frac{1-\left(-\frac{1}{3}\right)^3\left(-\frac{1}{3}\right)^n}{4/3} = \frac{3}{4}\left(\tfrac{1}{2}\right)^n \left[1 + \frac{1}{27}\left(-\tfrac{1}{3}\right)^n\right]$$

$$= \frac{3}{4}\left(\tfrac{1}{2}\right)^n + \frac{3}{4}\left(\tfrac{1}{2}\right)^n \frac{1}{27}\left(-\tfrac{1}{3}\right)^n$$

$$= 3\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^n + \underbrace{3\left(\tfrac{1}{2}\right)^2}_{3/4}\underbrace{\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)^2}_{1/27}\left(\tfrac{1}{2}\right)^n\left(-\tfrac{1}{3}\right)^n$$

$$= 3\left(\tfrac{1}{2}\right)^{n+2} + \left(\tfrac{1}{6}\right)^2\left(-\tfrac{1}{6}\right)^n = 3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^2\left(-\tfrac{1}{6}\right)^n$$

$$= 3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}$$

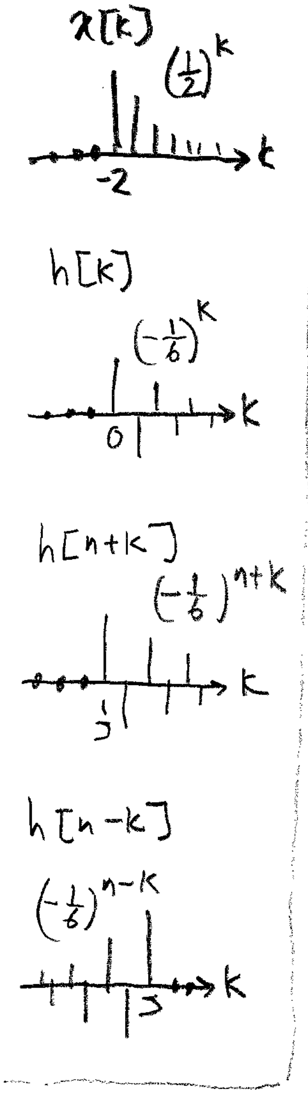
All Together: $$y[n] = \begin{cases} 0, & n<-2 \\ 3\left(\frac{1}{2}\right)^{n+2} + \left(-\frac{1}{6}\right)^{n+2}, & n \geq -2 \end{cases}$$

$$\boxed{y[n] = \left[3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}\right] u[n+2]}$$

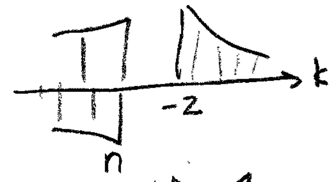
OTHER WAY

More Workspace for Problem 1...

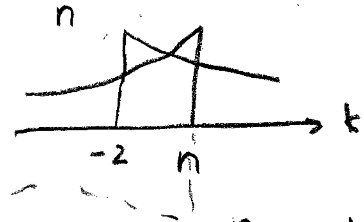
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$



case I) $n < -2$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$



case II) $n \geq -2$

$$y[n] = \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^k \left(-\tfrac{1}{6}\right)^{n-k} = \left(-\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^k (-6)^k$$

$$= \left(-\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} (-3)^k = \left(-\tfrac{1}{6}\right)^n \frac{(-3)^{-2} - (-3)^{n+1}}{1-(-3)}$$

$$= \left(-\tfrac{1}{6}\right)^n \frac{\left(-\frac{1}{3}\right)^2 + 3(-3)^n}{4}$$

$$= \tfrac{1}{4}\left(-\tfrac{1}{3}\right)^2\left(-\tfrac{1}{6}\right)^n + 3\left(\tfrac{1}{4}\right)\left(-\tfrac{1}{6}\right)^n(-3)^n$$

$$= \left(\tfrac{1}{2}\right)^2\left(-\tfrac{1}{3}\right)^2\left(-\tfrac{1}{6}\right)^n + 3\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^n$$

$$= \left(-\tfrac{1}{6}\right)^2\left(-\tfrac{1}{6}\right)^n + 3\left(\tfrac{1}{2}\right)^{n+2}$$

$$= 3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}$$

All Together:
$$y[n] = \begin{cases} 0, & n < -2 \\ 3\left(\frac{1}{2}\right)^{n+2} + \left(-\frac{1}{6}\right)^{n+2}, & n \geq -2 \end{cases}$$

$$\boxed{y[n] = \left[3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}\right] u[n+2]}$$

2. **25/20 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(-\frac{1}{6}\right)^n u[n].$$

The system input is given by

$$x[n] = \left(\frac{1}{2}\right)^n u[n+2].$$

Use the z-transform to find the system output $y[n]$.

Table: $H(z) = \dfrac{1}{1+\frac{1}{6}z^{-1}}, \quad |z| > \frac{1}{6}$

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n+2] = \underbrace{\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{-2}}_{\text{"one"}}\left(\tfrac{1}{2}\right)^n u[n+2] = 4\left(\tfrac{1}{2}\right)^{n+2} u[n+2]$$

Table: $\left(\tfrac{1}{2}\right)^n u[n] \xleftrightarrow{Z} \dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

Time Shift Property: $\left(\tfrac{1}{2}\right)^{n+2} u[n+2] \xleftrightarrow{Z} \dfrac{z^2}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

Linearity: $x[n] = 4\left(\tfrac{1}{2}\right)^{n+2} u[n+2] \xleftrightarrow{Z} X(z) = \dfrac{4z^2}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

$$Y(z) = X(z)H(z) = \frac{4z^2}{(1+\frac{1}{6}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{6}$$

Let $\tilde{Y}(z) = \dfrac{1}{(1+\frac{1}{6}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{6}.$

Then $Y(z) = 4z^2\tilde{Y}(z)$ and $y[n] = 4\tilde{y}[n+2]$ (*)

$$\tilde{Y}(z) = \frac{1}{(1+\frac{1}{6}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A}{1+\frac{1}{6}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{1}{1-\frac{1}{2}\theta}\right|_{\theta=-6} = \frac{1}{1+3} = \frac{1}{4}$$

$$B = \left.\frac{1}{1+\frac{1}{6}\theta}\right|_{\theta=2} = \frac{1}{1+\frac{1}{3}} = \frac{1}{4/3} = 3/4 \longrightarrow$$

More Workspace for Problem 2...

$$\tilde{Y}(z) = \underbrace{\frac{1/4}{1+\frac{1}{6}z^{-1}}}_{|z|>\frac{1}{6}} + \underbrace{\frac{3/4}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

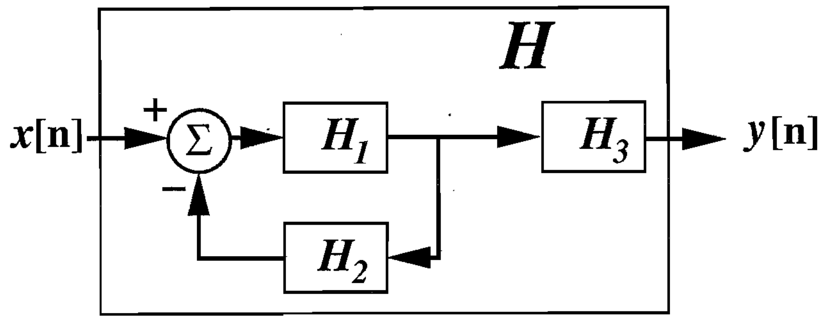
Table: $\tilde{y}[n] = \frac{1}{4}\left(-\frac{1}{6}\right)^n u[n] + \frac{3}{4}\left(\frac{1}{2}\right)^n u[n]$

$(*)$: $y[n] = 4\tilde{y}[n+2] = \left(-\frac{1}{6}\right)^{n+2} u[n+2] + 3\left(\frac{1}{2}\right)^{n+2} u[n+2]$

$$\boxed{y[n] = \left[3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}\right] u[n+2]}$$

NOTE: this agrees with the solution to problem 1.

3. **25/20 pts.** The causal discrete-time LTI system H is formed by interconnecting three causal discrete-time LTI systems H_1, H_2, and H_3 as shown in the figure below.



The input-output relation for for H_1 is given by

$$y_1[n] - 4y_1[n-1] = x_1[n].$$

The impulse responses $h_2[n]$ and $h_3[n]$ are given by

$$h_2(n) = \alpha\delta[n],$$

$$h_3(n) = \delta[n] - \left(\tfrac{1}{2} + \beta\right)\delta[n-1] + \tfrac{1}{2}\beta\delta[n-2],$$

where $\alpha,\beta \in \mathbb{R}$ are real constants.

Determine values for the real constants α and β so that H is both causal and stable and also the inverse system $G(z) = 1/H(z)$ is both causal and stable.

- It is given that H is causal. For H to be causal and stable, all of the poles of $H(z)$ must be inside the unit circle.
- For G to be causal and stable, all of the poles of $G(z)$ must be inside the unit circle. Since $G(z) = 1/H(z)$, this means that all of the zeros of $H(z)$ must be inside the unit circle.

$\Rightarrow$ We must choose α and β so that all of the poles and all of the zeros of $H(z)$ are inside the unit circle.

H_1 : $y_1[n] - 4y_1[n-1] = x_1[n]$

$$Y_1(z) - 4z^{-1}Y_1(z) = X_1(z) \qquad H_1(z) = \frac{Y_1(z)}{X_1(z)} = \frac{1}{1-4z^{-1}}$$

$$[1-4z^{-1}]\,Y_1(z) = X_1(z)$$

$\longrightarrow$

$h_2[n] = \alpha\delta[n] \rightarrow$ Table: $H_2(z) = \alpha$, all z.

$h_3[n] = \delta[n] - (\tfrac{1}{2}-\beta)\delta[n-1] + \tfrac{1}{2}\beta\delta[n-2]$

Table: $H_3(z) = 1 - (\tfrac{1}{2}+\beta)z^{-1} + \tfrac{1}{2}\beta z^{-2}$

$$= (1 - \beta z^{-1})(1 - \tfrac{1}{2}z^{-1})$$

Feedback and Series Connections: $H(z) = \dfrac{H_1(z)}{1+H_1(z)H_2(z)} \cdot H_3(z)$

$$= \frac{\frac{1}{1-4z^{-1}}}{1+\frac{\alpha}{1-4z^{-1}}}(1-\beta z^{-1})(1-\tfrac{1}{2}z^{-1}) = \frac{1}{1-4z^{-1}+\alpha}(1-\beta z^{-1})(1-\tfrac{1}{2}z^{-1})$$

$$= \frac{1}{(1+\alpha)-4z^{-1}} \cdot \underbrace{\frac{\frac{1}{1+\alpha}}{\frac{1}{1+\alpha}}}_{\text{"one"}} \cdot (1-\beta z^{-1})(1-\tfrac{1}{2}z^{-1})$$

$$H(z) = \frac{\frac{1}{1+\alpha}(1-\beta z^{-1})(1-\tfrac{1}{2}z^{-1})}{1-\frac{4}{1+\alpha}z^{-1}}$$

There are two zeros at $z = \frac{1}{2}$ and $z = \beta$

There is one pole at $z = \frac{4}{1+\alpha}$

For these to all be inside the unit circle, we need $|\beta| < 1$ and $\left|\frac{4}{1+\alpha}\right| = \frac{4}{|1+\alpha|} < 1 \Rightarrow |1+\alpha| > 4 \Rightarrow \alpha > 3$ or $\alpha < -5$.

☆ I pick: $\boxed{\beta = \tfrac{1}{3},\quad \alpha = 4}$ (These choices are not unique)

4. **25/20 pts.** H is a BIBO stable discrete-time LTI system with input-output relation

$$5y[n] - 4y[n-1] = x[n] - \frac{5}{6}x[n-1] + \frac{1}{6}x[n-2].$$

(a) **10/8 pts.** Find the transfer function $H(z)$ and be sure to specify the ROC. Give a pole zero plot. Is the system H causal?

$\mathcal{Z}$: $5Y(z) - 4z^{-1}Y(z) = X(z) - \frac{5}{6}z^{-1}X(z) + \frac{1}{6}z^{-2}X(z)$

$[5 - 4z^{-1}]Y(z) = [1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}]X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}{5 - 4z^{-1}} = \left(\frac{1}{5}\right)\frac{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 - \frac{4}{5}z^{-1}}$$

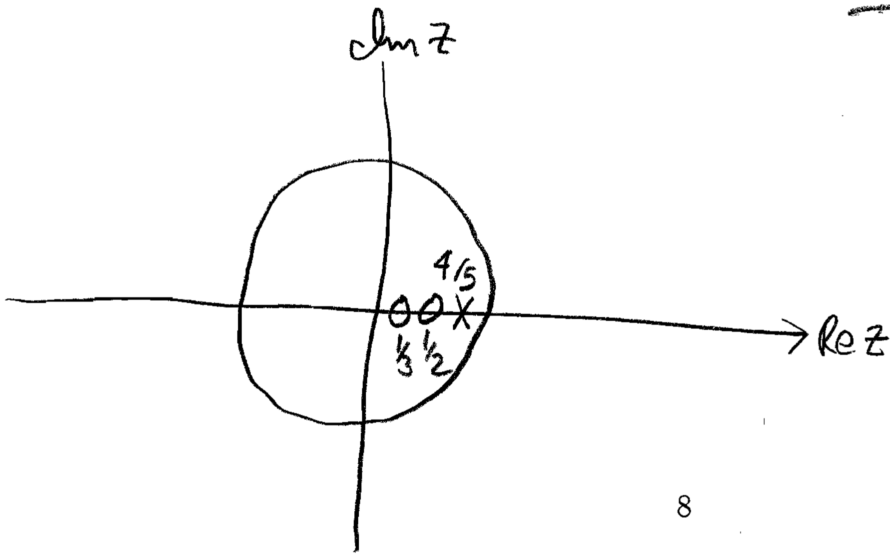
$$= \frac{1}{5}\,\frac{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})}{1 - 4/5\, z^{-1}}$$

- one pole at $z = \frac{4}{5}$
- Two zeros at $z = \frac{1}{3}$, $z = \frac{1}{2}$

- Because H is stable, the ROC must include the unit circle.
- The pole is inside the unit circle.

$\Rightarrow$ The ROC must be exterior $\Rightarrow$ The ROC is $|z| > \frac{4}{5}$

- Because the ROC is exterior, H is causal



Problem 4, cont...

(b) **15/12 pts**. Find the impulse response $h[n]$.

$$H(z) = \frac{1}{5}\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)\frac{1}{1-\frac{4}{5}z^{-1}}$$

$$= \underbrace{\frac{1}{5}\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)}_{\substack{H_1(z) \\ \text{ROC: } |z|>0 \\ \text{(all } z \text{ except } 0)}}\underbrace{\frac{1}{1-\frac{4}{5}z^{-1}}}_{\substack{H_2(z) \\ \text{ROC: } |z|>\frac{4}{5}}} = H_1(z)\,H_2(z).$$

So $h[n] = h_1[n] * h_2[n]$

$$H_2(z) = \frac{1}{1-\frac{4}{5}z^{-1}},\ |z|>\frac{4}{5} \ \text{: Table: } h_2[n] = \left(\frac{4}{5}\right)^n u[n]$$

$$H_1(z) = \frac{1}{5} - \frac{1}{6}z^{-1} + \frac{1}{30}z^{-2},\ |z|>0$$

Table: $h_1[n] = \frac{1}{5}\delta[n] - \frac{1}{6}\delta[n-1] + \frac{1}{30}\delta[n-2]$

Convolution property: $h[n] = h_1[n] * h_2[n] = \left(\frac{1}{5}\delta[n] - \frac{1}{6}\delta[n-1] + \frac{1}{30}\delta[n-2]\right) * h_2[n]$

$$= \frac{1}{5}h_2[n] - \frac{1}{6}h_2[n-1] + \frac{1}{30}h_2[n-2]$$

$$\boxed{h[n] = \frac{1}{5}\left(\frac{4}{5}\right)^n u[n] - \frac{1}{6}\left(\frac{4}{5}\right)^{n-1} u[n-1] + \frac{1}{30}\left(\frac{4}{5}\right)^{n-2} u[n-2]}$$

5. **25/20 pts.** Consider a length-4 discrete-time signal $x[n]$ given by

$$\begin{aligned} x[n] &= [1\ 0\ 2\ 1] \\ &= \delta[n] + 2\delta[n-2] + \delta[n-3]. \end{aligned}$$

Let $X[k]$ be the 4-point DFT of $x[n]$ and let

$$Y[k] = X[k] \times X[k] = X^2[k].$$

Find the length-4 time signal $y[n]$.

$$X(k) = \sum_{n=0}^{3} x[n] W_4^{nk} = 1W_4^{0k} + 2W_4^{2k} + W_4^{3k} = 1 + 2W_4^{2k} + W_4^{3k}$$

$$Y(k) = X(k)X(k) = (1 + 2W_4^{2k} + W_4^{3k})(1 + 2W_4^{2k} + W_4^{3k})$$

$$\begin{aligned} = {} & 1 + 0W_4^{k} + 2W_4^{2k} + W_4^{3k} \\ & + 0W_4^{k} + 2W_4^{2k} + 0W_4^{3k} + 4W_4^{4k} + 2W_4^{5k} \\ & + 0W_4^{2k} + W_4^{3k} + 0W_4^{4k} + 2W_4^{5k} + W_4^{6k} \end{aligned}$$

$$Y(k) = 1 + 4W_4^{2k} + 2W_4^{3k} + 4W_4^{4k} + 4W_4^{5k} + W_4^{6k}$$

But $W_4^{4k} = (e^{-j2\pi/4})^{4k} = e^{-j2\pi k} = 1$

$W_4^{5k} = W_4^{4k} W_4^{k} = W_4^{k}$; $W_4^{6k} = W_4^{4k} W_4^{2k} = W_4^{2k}$

$$\text{So } Y(k) = 5 + 4W_4^{k} + 5W_4^{2k} + 2W_4^{3k}$$

$$= 5W_4^{0k} + 4W_4^{1k} + 5W_4^{2k} + 2W_4^{3k}$$

$$= \sum_{n=0}^{3} y[n] W_4^{nk} = y[0]W_4^{0k} + y[1]W_4^{k} + y[2]W_4^{2k} + y[3]W_4^{3k}$$

$$\boxed{\Rightarrow y[n] = [5\ 4\ 5\ 2] = 5\delta[n] + 4\delta[n-1] + 5\delta[n-2] + 2\delta[n-3], \quad 0 \le n \le 3}$$