# ECE 4213/5213
# Test 1

Wednesday, October 22, 2014
Monday, October 27, 2014
4:30 PM - 5:45 PM

Fall 2014 | Name: SOLUTION

Dr. Havlicek | Student Num:

**Directions:** This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

**Students enrolled for undergraduate credit:** work any four problems. Each problem counts 25 points. Below, Circle the numbers of the four problems you wish to have graded.

**Students enrolled for graduate credit:** work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

*On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.*

Name: ________________ Date: ________________

1. **25/20 pts**. A discrete-time LTI system $H$ has impulse response $h[n]$ given by

$$h[n] = \left(-\frac{1}{3}\right)^n (u[n] - u[n-5]) = \begin{cases} \left(-\frac{1}{3}\right)^n, & 0 \le n \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

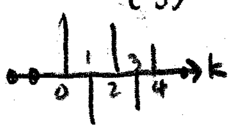
The system input is given by

$$x[n] = \left(\frac{1}{6}\right)^n u[n].$$

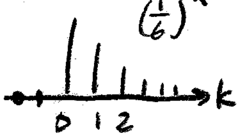
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Use time domain convolution to find the system output $y[n]$.

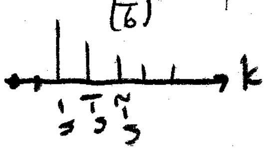
$h[k]$ $\left(-\frac{1}{3}\right)^k$



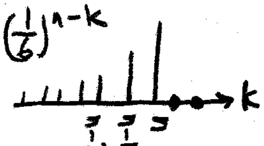

$x[k]$ $\left(\frac{1}{6}\right)^k$



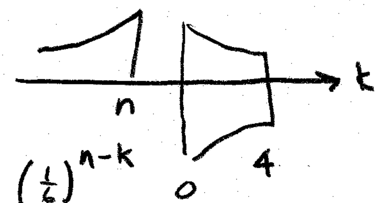

$x[k--n] = x[n+k]$ $\left(\frac{1}{6}\right)^{n+k}$




$x[n-k]$ $\left(\frac{1}{6}\right)^{n-k}$

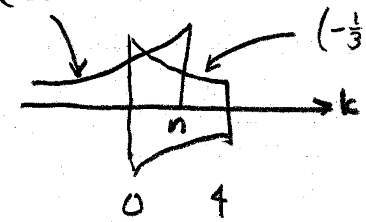



Case I) $n < 0$




$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

$\left(\frac{1}{6}\right)^{n-k}$ $\left(-\frac{1}{3}\right)^k$




Case II) $0 \le n < 5$

$$y[n] = \sum_{k=0}^{n} \left(\tfrac{1}{6}\right)^{n-k}\left(-\tfrac{1}{3}\right)^k$$

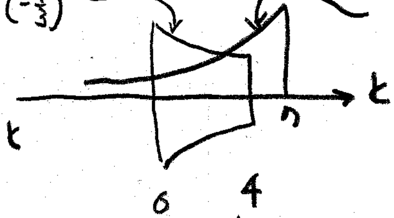
$$= \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{6}\right)^{-k}\left(-\tfrac{1}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{n} 6^k \left(-\tfrac{1}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{n} \left(-\tfrac{6}{3}\right)^k$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{n} (-2)^k = \left(\tfrac{1}{6}\right)^n \frac{(-2)^0 - (-2)^{n+1}}{1-(-2)} = \left(\tfrac{1}{6}\right)^n \frac{1+2(-2)^n}{3}$$

$$= \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n + \tfrac{2}{3}\left(\tfrac{1}{6}\right)^n(-2)^n = \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n + \tfrac{2}{3}\left(-\tfrac{1}{3}\right)^n$$

$\left(-\frac{1}{3}\right)^k$ $\left(\frac{1}{6}\right)^{n-k}$

Case III) $n \geq 5$




$$y[n] = \sum_{k=0}^{4} \left(\tfrac{1}{6}\right)^{n-k}\left(-\tfrac{1}{3}\right)^k$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{4} \left(\tfrac{1}{6}\right)^{-k}\left(-\tfrac{1}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{4} 6^k \left(-\tfrac{1}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=0}^{4} (-2)^k$$

$$= \left(\tfrac{1}{6}\right)^n \frac{(-2)^0 - (-2)^5}{1-(-2)} = \left(\tfrac{1}{6}\right)^n \frac{1-(-32)}{3} = \left(\tfrac{1}{6}\right)^n \frac{33}{3} = 11\left(\tfrac{1}{6}\right)^n$$

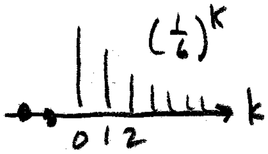
All Together:

$$\boxed{y[n] = \begin{cases} 0, & n < 0 \\ \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n + \tfrac{2}{3}\left(-\tfrac{1}{3}\right)^n, & 0 \le n < 5 \\ 11\left(\tfrac{1}{6}\right)^n, & n \geq 5 \end{cases}}$$

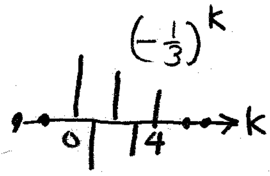
# OTHER WAY:

More Workspace for Problem 1... $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

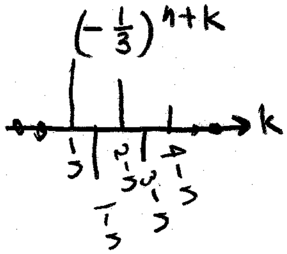
$x[k]$



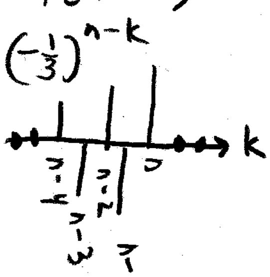

$h[k]$



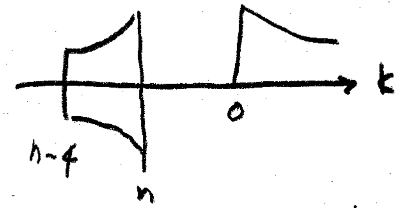

$h[k--n] = h[n+k]$




$h[n-k]$




case I) $n < 0$ :

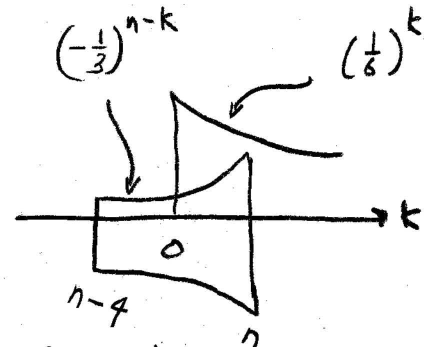



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II) $n \geq 0$ and $n-4 \leq 0$

$0 \leq n \leq 4$

$0 \leq n < 5$



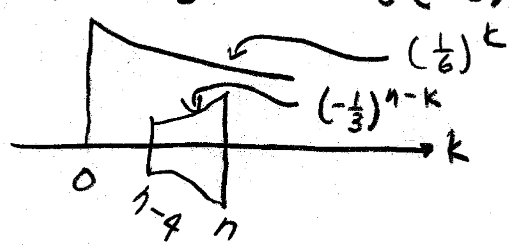

$$y[n] = \sum_{k=0}^{n} (-\tfrac{1}{3})^{n-k} (\tfrac{1}{6})^k$$

$$= (-\tfrac{1}{3})^n \sum_{k=0}^{n} (-\tfrac{1}{3})^{-k} (\tfrac{1}{6})^k = (-\tfrac{1}{3})^n \sum_{k=0}^{n} (-3)^k (\tfrac{1}{6})^k$$

$$= (-\tfrac{1}{3})^n \sum_{k=0}^{n} (-\tfrac{1}{2})^k = (-\tfrac{1}{3})^n \frac{1-(-\frac{1}{2})^{n+1}}{1-(-\frac{1}{2})} = (-\tfrac{1}{3})^n \frac{1+\frac{1}{2}(-\frac{1}{2})^n}{3/2}$$

$$= \tfrac{2}{3}(-\tfrac{1}{3})^n + \tfrac{1}{3}(-\tfrac{1}{3})^n(-\tfrac{1}{2})^n = \tfrac{1}{3}(\tfrac{1}{6})^n + \tfrac{2}{3}(-\tfrac{1}{3})^n$$

Case III) $n \geq 5$




$$y[n] = \sum_{k=n-4}^{n} (-\tfrac{1}{3})^{n-k} (\tfrac{1}{6})^k$$

$$= (-\tfrac{1}{3})^n \sum_{k=n-4}^{n} (-\tfrac{1}{3})^{-k} (\tfrac{1}{6})^k = (-\tfrac{1}{3})^n \sum_{k=n-4}^{n} (-3)^k (\tfrac{1}{6})^k$$

$$= (-\tfrac{1}{3})^n \sum_{k=n-4}^{n} (-\tfrac{1}{2})^k = (-\tfrac{1}{3})^n \frac{(-\frac{1}{2})^{n-4} - (-\frac{1}{2})^{n+1}}{1-(-\frac{1}{2})} = (-\tfrac{1}{3})^n \frac{(-\frac{1}{2})^n(-\frac{1}{2})^{-4} + \frac{1}{2}(-\frac{1}{2})^n}{3/2}$$

$$= (-\tfrac{1}{3})^n \frac{(-2)^4(-\frac{1}{2})^n + \frac{1}{2}(-\frac{1}{2})^n}{3/2} = \tfrac{2}{3}(-\tfrac{1}{3})^n \left[16 + \tfrac{1}{2}\right](-\tfrac{1}{2})^n = \tfrac{2}{3}(\tfrac{33}{2})(\tfrac{1}{6})^n$$

$$= 11(\tfrac{1}{6})^n.$$

$$\boxed{\text{All Together : } y[n] = \begin{cases} 0, & n < 0 \\ \frac{1}{3}(\frac{1}{6})^n + \frac{2}{3}(-\frac{1}{3})^n, & 0 \leq n < 5 \\ 11(\frac{1}{6})^n, & n \geq 5 \end{cases}}$$

2. **25/20 pts.** A **stable** discrete-time LTI system $H$ has input $x[n]$ and output $y[n]$ related by

$$y[n] + 2y[n-1] + \frac{3}{4}y[n-2] = x[n] - 2x[n-1].$$

(a) **7/5 pts.** Find the transfer function $H(z)$.

$$\mathcal{Z}: \quad Y(z) + 2z^{-1}Y(z) + \frac{3}{4}z^{-2}Y(z) = X(z) - 2z^{-1}X(z)$$

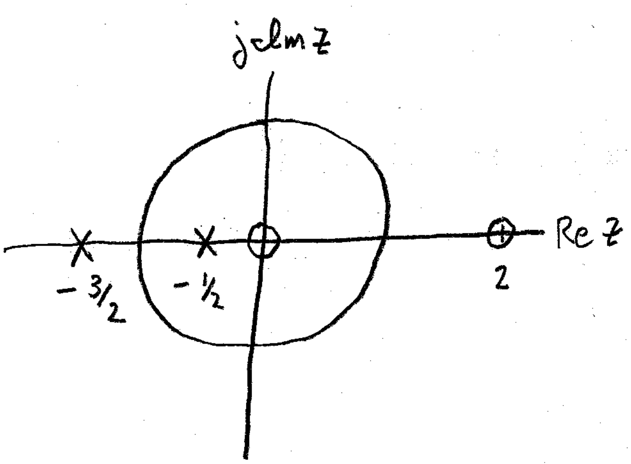
$$Y(z)\left[1 + 2z^{-1} + \frac{3}{4}z^{-2}\right] = X(z)\left[1 - 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \underline{\underline{\frac{1-2z^{-1}}{1+2z^{-1}+\frac{3}{4}z^{-2}}}} = \frac{1-2z^{-1}}{\left(1+\frac{3}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$

(b) **7/5 pts.** Give a pole-zero plot for $H(z)$ and specify the ROC.

- From $H(z)$ in part (a), there is a zero at $z=2$ and there are two poles at $z = -\frac{3}{2}, -\frac{1}{2}$
- Because denominator order (in $z^{-1}$) is one greater than numerator order, there is also a zero at $z=0$.

→ To see this, consider $H(z) \cdot \frac{z^2}{z^2} = \frac{z(z-2)}{(z+\frac{3}{2})(z+\frac{1}{2})}$ ← zeros at $z=0,2$




- Because the system is stable, the ROC must include the unit circle.

$$\underline{\underline{\text{ROC}: \ \frac{1}{2} < |z| < \frac{3}{2}}}$$

Problem 2, cont...

(c) **11/10 pts**. Find the impulse response $h[n]$.

$$H(z) = \frac{1-2z^{-1}}{(1+\frac{3}{2}z^{-1})(1+\frac{1}{2}z^{-1})} = \frac{A}{1+\frac{3}{2}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{1-2\theta}{1+\frac{1}{2}\theta}\right|_{\theta=-\frac{2}{3}} = \frac{1+\frac{4}{3}}{1-\frac{1}{3}} = \frac{7/3}{2/3} = \frac{7}{2}$$

$$B = \left.\frac{1-2\theta}{1+\frac{3}{2}\theta}\right|_{\theta=-2} = \frac{1+4}{1-3} = \frac{5}{-2} = -\frac{5}{2}$$

$$H(z) = \underbrace{\frac{7}{2}\frac{1}{1+\frac{3}{2}z^{-1}}}_{|z|<\frac{3}{2}} - \underbrace{\frac{5}{2}\frac{1}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

Table: $h[n] = \frac{7}{2}(-1)(-\frac{3}{2})^n u[-n-1] - \frac{5}{2}(-\frac{1}{2})^n u[n]$

$$= -\frac{5}{2}(-\frac{1}{2})^n u[n] - \frac{7}{2}(-\frac{3}{2})^n u[-n-1]$$

3. **25/20 pts**. A discrete-time LTI system $H$ has impulse response

$$h[n] = \alpha\delta[n] + \beta\delta[n-1],$$

where $\alpha, \beta \in \mathbb{R}$ are real constants. Find the system group delay $\tau_g(\omega)$.

$$H(e^{j\omega}) = \alpha + \beta e^{-j\omega} = [\alpha + \beta\cos\omega] - j\beta\sin\omega$$

$$\theta(\omega) = \arctan\frac{\mathrm{Im}[H(e^{j\omega})]}{\mathrm{Re}[H(e^{j\omega})]} = \arctan\frac{-\beta\sin\omega}{\alpha+\beta\cos\omega} = -\arctan\frac{\beta\sin\omega}{\alpha+\beta\cos\omega}$$

$$\tau_g(\omega) = -\frac{d}{d\omega}\theta(\omega) = \frac{d}{d\omega}\arctan\frac{\beta\sin\omega}{\alpha+\beta\cos\omega};\quad \text{use } \frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

$$= \left[1 + \frac{\beta^2\sin^2\omega}{(\alpha+\beta\cos\omega)^2}\right]^{-1}\frac{d}{d\omega}\beta\sin\omega\,(\alpha+\beta\cos\omega)^{-1}$$

$$= \left[1 + \frac{\beta^2\sin^2\omega}{(\alpha+\beta\cos\omega)^2}\right]^{-1}\Big[\beta\sin\omega\,(-1)(\alpha+\beta\cos\omega)^{-2}(-1)\beta\sin\omega + \beta\cos\omega\,(\alpha+\beta\cos\omega)^{-1}\Big]$$

$$= \frac{1}{1+\frac{\beta^2\sin^2\omega}{(\alpha+\beta\cos\omega)^2}}\left[\frac{\beta^2\sin^2\omega}{(\alpha+\beta\cos\omega)^2} + \frac{\beta\cos\omega}{\alpha+\beta\cos\omega}\right]\cdot\frac{(\alpha+\beta\cos\omega)^2}{(\alpha+\beta\cos\omega)^2}$$

$$= \frac{1}{(\alpha+\beta\cos\omega)^2+\beta^2\sin^2\omega}\left[\beta^2\sin^2\omega + \beta\cos\omega\,(\alpha+\beta\cos\omega)\right]$$

$$= \frac{1}{\alpha^2+2\alpha\beta\cos\omega+\beta^2\cos^2\omega+\beta^2\sin^2\omega}\left[\alpha\beta\cos\omega + \beta^2\sin^2\omega + \beta^2\cos^2\omega\right]$$

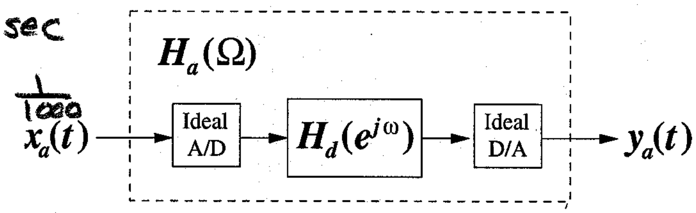
$$= \frac{1}{\alpha^2+\beta^2+2\alpha\beta\cos\omega}\left[\beta^2 + \alpha\beta\cos\omega\right] = \frac{\beta^2+\alpha\beta\cos\omega}{\alpha^2+\beta^2+2\alpha\beta\cos\omega}$$

4. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.

$T = \frac{1}{1000}$ sec $= 0.001$ sec

or: $T = \frac{2\pi}{\Omega_T} = \frac{2\pi}{2000\pi} = \frac{1}{1000}$

$\frac{\Omega_T}{2} = 1000\pi$




The sampling frequency of the A/D and D/A converters is $F_T = 1$ kHz, so $\Omega_T = 2000\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this simply ensures that the overall structure shown in the figure will be an LTI system).

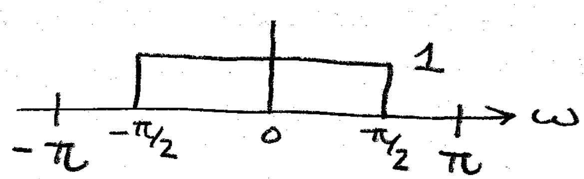
The impulse response of the discrete-time filter $H_d$ is given by

$$h_d[n] = \frac{\sin \frac{\pi}{2} n}{\pi n}.$$

(a) **7/6 pts.** Find the discrete-time frequency response $H_d(e^{j\omega})$.

Table: $H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$ ($2\pi$-periodic)

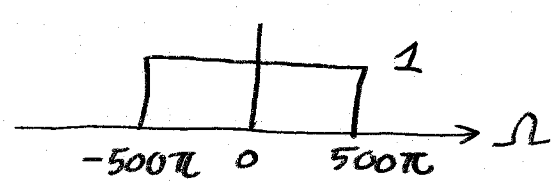
$=$




(b) **9/7 pts.** Find the continuous-time frequency response $H_a(\Omega)$. $\Omega = \frac{\omega}{T}$

$H_a(\Omega) = \begin{cases} 1, & |\Omega| \le \frac{\pi/2}{T} = 500\pi \\ 0, & |\Omega| > \frac{\pi/2}{T} = 500\pi \end{cases}$

$=$



Problem 4, cont...

(c) **9/7 pts**. The continuous-time input signal is given by

$$x_a(t) = \sin(400\pi t - \pi/8) + \cos(600\pi t + \pi/8).$$

Find the continuous-time output signal $y_a(t)$.

**Hint:** $Y_a(\Omega) = X_a(\Omega)H_a(\Omega)$.

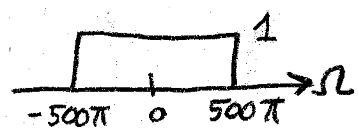
$H_a(\Omega)$ is an ideal low-pass filter. The cutoff frequency is $500\pi$ rad/sec. So it is immediate that $\sin(400\pi t - \pi/8)$ is in the passband and $\cos(600\pi t + \pi/8)$ is in the stopband: $y_a(t) = \sin(400\pi t - \pi/8)$.

But we can also show this:

$$x_a(t) = \sin\left[400\pi\left(t - \frac{1}{3200}\right)\right] + \cos\left[600\pi\left(t - \frac{1}{4800}\right)\right]$$

Table + time shift property:

$$X_a(\Omega) = e^{-j\Omega/3200} \cdot \frac{\pi}{j}\left[\delta(\Omega - 400\pi) - \delta(\Omega + 400\pi)\right]$$
$$+ e^{-j\Omega/4800}\, \pi\left[\delta(\Omega - 600\pi) + \delta(\Omega + 600\pi)\right]$$
$$= \frac{\pi}{j}\left[e^{-j\pi/800}\delta(\Omega - 400\pi) - e^{j\pi/800}\delta(\Omega + 400\pi)\right]$$
$$+ \pi\left[e^{-j\pi/800}\delta(\Omega - 600\pi) + e^{j\pi/800}\delta(\Omega + 600\pi)\right]$$

$Y_a(\Omega) = X_a(\Omega)H_a(\Omega) = X_a(\Omega) \times$ 


$$= \frac{\pi}{j}\left[e^{-j\pi/800}\delta(\Omega - 400\pi) - e^{j\pi/800}\delta(\Omega + 400\pi)\right]$$
$$= \frac{\pi}{j}\left[e^{-j\Omega/3200}\delta(\Omega - 400\pi) - e^{-j\Omega/3200}\delta(\Omega + 400\pi)\right]$$
$$= e^{-j\Omega/3200} \cdot \frac{\pi}{j}\left[\delta(\Omega - 400\pi) - \delta(\Omega + 400\pi)\right]$$

Table: $y_a(t) = \sin\left[400\pi\left(t - \frac{1}{3200}\right)^{8}\right] = \sin\left(400\pi t - \frac{\pi}{8}\right)$

5. **25/20 pts**. Let $x_1[n]$ be a length-$N$ discrete-time signal given by

$$x_1[n] = 1, \quad 0 \le n \le N-1.$$

Let $x_2[n]$ be a length-$N$ discrete-time signal given by

$$x_2[n] = 1 - \delta[n-(N-1)] = \begin{cases} 1, & 0 \le n \le N-2, \\ 0, & n = N-1. \end{cases}$$

**Note:** $x_1[n]$ is "all ones." $x_2[n]$ is the same, except that the last sample $x_2[N-1]$ is a zero instead of a one.

(a) **13/10 pts**. Find the $N$-point DFT $X_1[k]$.

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] W_N^{nk} = \sum_{n=0}^{N-1} e^{-j2\pi nk/N} \qquad (*)$$

when $k=0$, $(*) = \sum_{n=0}^{N-1} 1 = N$

when $1 \le k \le N-1$, $(*) = \dfrac{(e^{-j2\pi k/N})^0 - (e^{-j2\pi k/N})^N}{1 - e^{-j2\pi k/N}}$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j2\pi k/N}}$$

$$= \frac{1-1}{1-(\text{not } 1)} = \frac{0}{\text{not zero}} = 0.$$

So all together,

$$X_1[k] = \begin{cases} N, & k=0 \\ 0, & 1 \le k \le N-1 \end{cases}$$

**Problem 5, cont...**

(b) **12/10 pts.** Find the $N$-point DFT $X_2[k]$.

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n] W_N^{nk} = \sum_{n=0}^{N-2} W_N^{nk} = \sum_{n=0}^{N-2} e^{-j2\pi nk/N} \qquad (**)$$

when $k=0$, $\quad (**) = \sum_{n=0}^{N-2} 1 = N-1$

when $1 \leq k \leq N-1$,

$$(**) = \frac{\left(e^{-j2\pi k/N}\right)^0 - \left(e^{-j2\pi k/N}\right)^{N-1}}{1-e^{-j2\pi k/N}}$$

$$= \frac{1-e^{-j2\pi(N-1)k/N}}{1-e^{-j2\pi k/N}} = \frac{1-\overbrace{e^{-j2\pi kN/N}}^{\text{one}}e^{j2\pi k/N}}{1-e^{-j2\pi k/N}}$$

$$= \frac{1-e^{j2\pi k/N}}{1-e^{-j2\pi k/N}} = \frac{e^{j2\pi k/N}\left[e^{-j2\pi k/N}-1\right]}{1-e^{-j2\pi k/N}}$$

$$= \frac{-e^{j2\pi k/N}\left[1-e^{-j2\pi k/N}\right]}{1-e^{-j2\pi k/N}} = -e^{j2\pi k/N}$$

All together:

$$X_2[k] = \begin{cases} N-1, & k=0 \\ -e^{j2\pi k/N}, & 1 \leq k \leq N-1 \end{cases}$$