ECE 4213/5213
Test 1

Wednesday, October 21, 2015
4:30 PM - 5:45 PM

Fall 2015 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed:________________ Date:________________

1. **25/20 pts**. A **stable** discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by

$$y[n] - y[n-1] - \frac{3}{4}y[n-2] = x[n] - 2x[n-1].$$

(a) **6/5 pts**. Find the transfer function $H(z)$.

$$\mathcal{Z}: \quad Y(z) - z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) = X(z) - 2z^{-1}X(z)$$

$$Y(z)\left[1 - z^{-1} - \tfrac{3}{4}z^{-2}\right] = X(z)\left[1 - 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-z^{-1}-\frac{3}{4}z^{-2}} = \frac{1-2z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{2}z^{-1})} \quad //$$

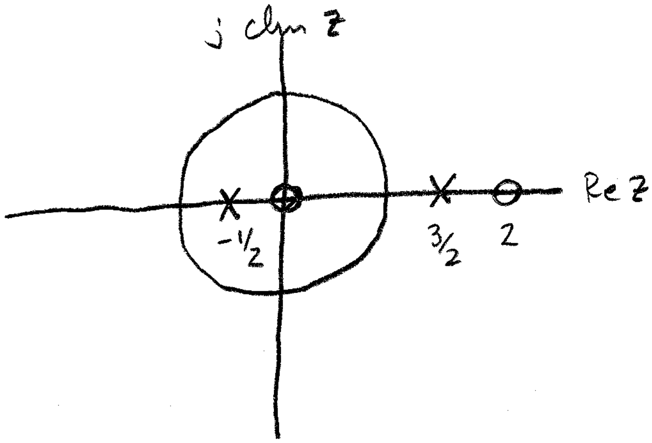
$$\text{Note: } \frac{z^2}{z^2}H(z) = \frac{z(z-2)}{(z+\frac{1}{2})(z-\frac{3}{2})} \qquad (*)$$

(b) **5/4 pts**. Give a pole-zero plot for $H(z)$ and specify the ROC.

From $(*)$ above:

poles: $z = -1/2$, $z = 3/2$

zeros: $z = 0$, $z = 2$



Because H is stable, the ROC must include the unit circle.

ROC: $\frac{1}{2} < |z| < \frac{3}{2}$

(c) **2/2 pts**. Is the system H causal? *Justify your answer.*

Because the ROC is an annulus, the impulse response must be two-sided $\Rightarrow$ can't be causal.

Problem 1, cont...

(d) **12/9 pts**. Find the impulse response $h[n]$.

$$H(z) = \frac{1-2z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{2}z^{-1})} = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{3}{2}z^{-1}}, \quad \frac{1}{2}<|z|<\frac{3}{2}$$

$$A = \left.\frac{1-2\theta}{1-\frac{3}{2}\theta}\right|_{\theta=-2} = \frac{1+4}{1+3} = \frac{5}{4}$$

$$B = \left.\frac{1-2\theta}{1+\frac{1}{2}\theta}\right|_{\theta=\frac{2}{3}} = \frac{1-\frac{4}{3}}{1+\frac{1}{3}} = \frac{-\frac{1}{3}}{4/3} = -\frac{3}{4}\cdot\frac{1}{3} = -\frac{1}{4}$$

$$H(z) = \underbrace{\frac{5/4}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} - \underbrace{\frac{1/4}{1-\frac{3}{2}z^{-1}}}_{|z|<3/2}$$

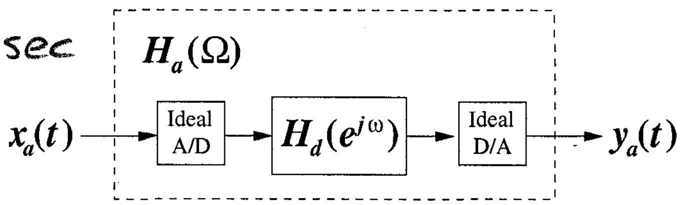
Table $\quad$ Table

$$\frac{5}{4}\left(-\frac{1}{2}\right)^n u[n] \qquad -\frac{1}{4}\left(\frac{3}{2}\right)^n u[-n-1]$$

$$\boxed{h[n] = \frac{5}{4}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{4}\left(\frac{3}{2}\right)^n u[-n-1]}$$

2. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.

$T = \frac{1}{5000} = 0.0002$ sec



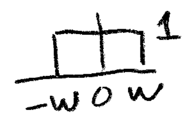
The sampling frequency of the A/D and D/A converters is $F_T = 5$ kHz, so $\Omega_T = 10,000\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this simply ensures that the overall structure shown in the figure will be an LTI system).

The impulse response of the continuous-time filter H_a is given by $\quad \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$

$W = 2000\pi$

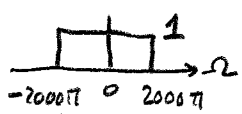
$$h_a(t) = \delta(t) - \frac{\sin 2000\pi t}{\pi t}.$$

$\frac{\sin Wt}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$



(a) **8/6 pts.** Find the continuous-time frequency response $H_a(\Omega)$.

Table: $H_a(\Omega) = 1 -$



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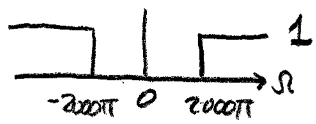


$$H_a(\Omega) = \begin{cases} 1, & |\Omega| \geq 2000\pi \\ 0, & |\Omega| < 2000\pi \end{cases}$$

(b) **8/7 pts.** Find the discrete-time frequency response $H_d(e^{j\omega})$.

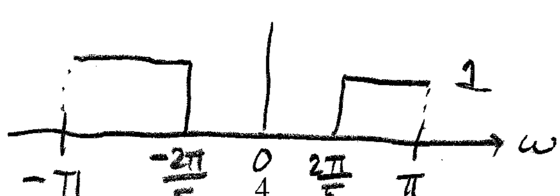
$\omega = \Omega T = \frac{\Omega}{5000}$

$H_d(e^{j\omega}) = H_a\left(\frac{\omega}{T}\right), \quad |\omega| < \pi$

$\therefore \quad H_d(e^{j\omega}) = \begin{cases} 1, & \frac{2\pi}{5} \leq |\omega| \leq \pi \\ 0, & |\omega| < \frac{2\pi}{5} \end{cases}$

(2π-periodic)

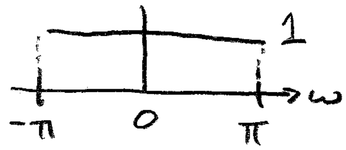
$=$



Problem 2, cont...

(c) **9/7 pts**. Find the discrete-time impulse response $h_d[n]$.

$H_d(e^{j\omega}) =$



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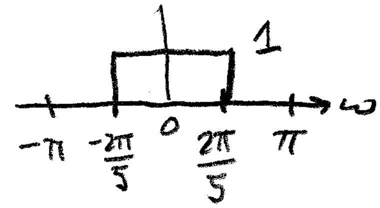


Table $\updownarrow$ $\delta[n]$

Table $\updownarrow$ $\dfrac{\sin \frac{2\pi}{5} n}{\pi n}$

$$\boxed{h_d[n] = \delta[n] - \frac{\sin \frac{2\pi}{5} n}{\pi n}}$$

3. **25/20 pts.** A discrete-time LTI system H has impulse response

$$h[n] = 3^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$$

The input is given by $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

(a) **3/2 pts.** Is the system H causal? *Justify your answer.*

The system is causal because $h[n] = 0 \;\; \forall n < 0$.

(b) **8/7 pts.** Find the transfer function $H(z)$ and give a pole-zero plot. Don't forget to specify the ROC.

Table: $3^n u[n] \overset{Z}{\longleftrightarrow} \dfrac{1}{1-3z^{-1}}, \quad |z|>3$

$\left(-\frac{1}{3}\right)^n u[n] \overset{Z}{\longleftrightarrow} \dfrac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$

$$H(z) = \frac{1}{1-3z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$$

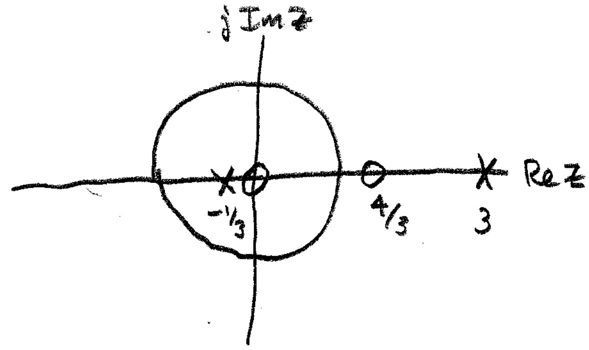
$$= \frac{1+\frac{1}{3}z^{-1} + 1 - 3z^{-1}}{(1-3z^{-1})(1+\frac{1}{3}z^{-1})}$$

$$= \frac{2-\frac{8}{3}z^{-1}}{(1-3z^{-1})(1+\frac{1}{3}z^{-1})}$$

$$= \frac{2(1-\frac{4}{3}z^{-1})}{(1-3z^{-1})(1+\frac{1}{3}z^{-1})} \quad //$$

$$\left\{ = \frac{2z(z-\frac{4}{3})}{(z-3)(z+\frac{1}{3})} \right\}$$

poles: $z=3$, $z=-\frac{1}{3}$
zeros: $z=0$, $z=\frac{4}{3}$



Because H is causal, the ROC must be exterior to the largest pole.

ROC: $|z|>3$ //

(c) **3/2 pts.** Is the system H BIBO stable? *Justify your answer.*

The system is not stable because the ROC does not include the unit circle.

Alternate Answer: not stable because H is causal and the pole at $z=3$ is outside the unit circle.

Problem 3, cont...

(d) **3/2 pts.** Could the discrete-time Fourier transform (DTFT) be used to find the system output $y[n]$? Briefly explain why or why not.

The DTFT could not be used because $H(e^{j\omega})$ does not exist (the DTFT of $h[n]$ diverges).

The reason is that $H(e^{j\omega})$ is equal to $H(z)$ evaluated on the unit circle, but in this case the unit circle is not in the ROC of $H(z)$.

(e) **8/7 pts.** Use the z-transform to find the system output $y[n]$.

Table: $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

$$Y(z) = X(z)H(z) = \frac{2(1-\frac{4}{3}z^{-1})}{(1-3z^{-1})(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z|>3$$

$$= \frac{A}{1-3z^{-1}} + \frac{B}{1+\frac{1}{3}z^{-1}} + \frac{C}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{2(1-\frac{4}{3}\theta)}{(1+\frac{1}{3}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=\frac{1}{3}} = \frac{2(1-\frac{4}{9})}{(1+\frac{1}{9})(1-\frac{1}{6})} = \frac{2(\frac{5}{9})}{\frac{10}{9}\cdot\frac{5}{6}} = \frac{10}{10\cdot\frac{5}{6}} = \frac{6}{5}$$

$$B = \left.\frac{2(1-\frac{4}{3}\theta)}{(1-3\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=-3} = \frac{2(1+4)}{(1+9)(1+\frac{3}{2})} = \frac{2\cdot 5}{10(\frac{5}{2})} = \frac{2}{5}$$

$$C = \left.\frac{2(1-\frac{4}{3}\theta)}{(1-3\theta)(1+\frac{1}{3}\theta)}\right|_{\theta=2} = \frac{2(1-\frac{8}{3})}{(1-6)(1+\frac{2}{3})} = \frac{2(-\frac{5}{3})}{(-5)(\frac{5}{3})} = \frac{2}{5}$$

$$Y(z) = \underbrace{\frac{6/5}{1-3z^{-1}}}_{|z|>3} + \underbrace{\frac{2/5}{1+\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{2/5}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}, \quad |z|>3$$

Table $\updownarrow$ $\frac{6}{5}3^n u[n]$ Table $\updownarrow$ $\frac{2}{5}(-\frac{1}{3})^n u[n]$ Table $\updownarrow$ $\frac{2}{5}(\frac{1}{2})^n u[n]$

$$\boxed{y[n] = \frac{6}{5}3^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n] + \frac{2}{5}\left(\frac{1}{2}\right)^n u[n]}$$

4. **25/20 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(\frac{1}{2}\right)^n \sin\left[\frac{\pi}{4}(n-1)\right] u[n-1].$$

(a) **11/9 pts.** Find the transfer function $H(z)$. Don't forget to specify the ROC.

$$h[n] = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \sin\left[\frac{\pi}{4}(n-1)\right] u[n-1]$$

Let $g[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] \longmapsto h[n] = \frac{1}{2} g[n-1]$

$\rightarrow$ Then $H(z) = \frac{1}{2} z^{-1} G(z)$

Table $\left(r = \frac{1}{2}, \omega_0 = \frac{\pi}{4}\right)$: $$G(z) = \frac{\left[\frac{1}{2}\sin\frac{\pi}{4}\right] z^{-1}}{1 - \left[2\left(\frac{1}{2}\right)\cos\frac{\pi}{4}\right] z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2}}$$

$$= \frac{\frac{1}{2}\frac{\sqrt{2}}{2} z^{-1}}{1 - \frac{\sqrt{2}}{2} z^{-1} + \frac{1}{4} z^{-2}} = \frac{\frac{1}{2\sqrt{2}} z^{-1}}{1 - \frac{1}{\sqrt{2}} z^{-1} + \frac{1}{4} z^{-2}}, \quad |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{2} z^{-1} G(z) = \frac{\frac{1}{4\sqrt{2}} z^{-2}}{1 - \frac{1}{\sqrt{2}} z^{-1} + \frac{1}{4} z^{-2}}, \quad |z| > 1/2$$

Problem 4, cont…

(b) **3/2 pts**. Find the frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{\frac{1}{4\sqrt{2}} e^{-j2\omega}}{1 - \frac{1}{\sqrt{2}} e^{-j\omega} + \frac{1}{4} e^{-j2\omega}} \quad //$$

(c) **11/9 pts**. The system input is given by $x[n] = \cos \frac{\pi}{2} n$. Find the output $y[n]$.

The input is a sum of two eigenfunctions.

Notes p. 3-114: $y[n] = \left|H(e^{j\pi/2})\right| \cos\left[\frac{\pi}{2} n + \arg H(e^{j\pi/2})\right]$

$$H(e^{j\pi/2}) = \frac{\frac{1}{4\sqrt{2}} e^{-j\pi}}{1 - \frac{1}{\sqrt{2}} e^{-j\pi/2} + \frac{1}{4} e^{-j\pi}} = \frac{-\frac{1}{4\sqrt{2}}}{1 + j\frac{1}{\sqrt{2}} - \frac{1}{4}}$$

$e^{-j\pi} = -1$

$e^{-j\pi/2} = -j$

$$= \frac{-\frac{1}{4\sqrt{2}}}{\frac{3}{4} + j\frac{1}{\sqrt{2}}}$$

$$\left|H(e^{j\pi/2})\right| = \left|\frac{-\frac{1}{4\sqrt{2}}}{3/4 + j\,1/\sqrt{2}}\right| = \frac{\frac{1}{4\sqrt{2}}}{\left|3/4 + j\frac{1}{\sqrt{2}}\right|} = \frac{\frac{1}{4\sqrt{2}}}{\sqrt{(3/4)^2 + (1/\sqrt{2})^2}} = \frac{1/4\sqrt{2}}{\sqrt{9/16 + 1/2}}$$

$$= \frac{1/4\sqrt{2}}{\sqrt{9/16 + 8/16}} = \frac{1/4\sqrt{2}}{\sqrt{17/16}} = \frac{\frac{1}{4\sqrt{2}}}{\sqrt{17}/4} = \frac{4}{\sqrt{17}} \cdot \frac{1}{4\sqrt{2}}$$

$$= \frac{1}{\sqrt{17}\sqrt{2}} = \frac{1}{\sqrt{34}} \approx 0.17150$$

$$\arg H(e^{j\pi/2}) = \arg \frac{1}{4\sqrt{2}} \frac{-1}{3/4 + j\,1/\sqrt{2}} = \arg \frac{-1}{3/4 + j\,1/\sqrt{2}} = \arg(-1) - \arg\left(\frac{3}{4} + j\frac{1}{\sqrt{2}}\right)$$

$$= \pi - \arctan \frac{1/\sqrt{2}}{3/4} = \pi - \arctan \frac{4}{3\sqrt{2}}$$

$$\approx \pi - 0.75597 = 2.38562$$

$$\boxed{y[n] = 0.17150 \cos\left[\frac{\pi}{2} n + 2.38562\right]}$$

5. **25/20 pts**. Let $x[n]$ and $h[n]$ be 4-point discrete-time signals given by

$$\begin{aligned} x[n] &= [3\ \ 1\ \ 4\ \ 1] \\ &= 3\delta[n] + \delta[n-1] + 4\delta[n-2] + \delta[n-3],\ 0 \le n \le 3, \end{aligned}$$

and

$$\begin{aligned} h[n] &= [5\ \ -9\ \ \ 2\ \ -6] \\ &= 5\delta[n] - 9\delta[n-1] + 2\delta[n-2] - 6\delta[n-3],\ 0 \le n \le 3. \end{aligned}$$

(a) **13/10 pts**. Use the DFT to find the 4-point circular convolution $y_c[n] = x[n] \textcircled{4} h[n]$.

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{nk} = 3W_4^{0k} + W_4^{k} + 4W_4^{2k} + W_4^{3k}$$

$$= 3 + W_4^{k} + 4W_4^{2k} + W_4^{3k}, \quad 0 \le k \le 3$$

$$H[k] = \sum_{n=0}^{3} h[n] W_4^{nk} = 5 - 9W_4^{k} + 2W_4^{2k} - 6W_4^{3k}, \quad 0 \le k \le 3$$

$$Y_c[k] = X[k]H[k] = (3 + W_4^{k} + 4W_4^{2k} + W_4^{3k})(5 - 9W_4^{k} + 2W_4^{2k} - 6W_4^{3k})$$

$$\begin{aligned} = 15 - 27W_4^{k} &+ 6W_4^{2k} - 18W_4^{3k} \\ + 5W_4^{k} &- 9W_4^{2k} + 2W_4^{3k} - 6W_4^{4k} \\ &+ 20W_4^{2k} - 36W_4^{3k} + 8W_4^{4k} - 24W_4^{5k} \\ &+ 5W_4^{3k} - 9W_4^{4k} + 2W_4^{5k} - 6W_4^{6k} \end{aligned}$$

$$Y_c[k] = 15 - 22W_4^{k} + 17W_4^{2k} - 47W_4^{3k} - 7W_4^{4k} - 22W_4^{5k} - 6W_4^{6k}$$

But $W_4 = e^{-j\frac{2\pi}{4}}$. So $W_4^{4k} = e^{-j\frac{2\pi}{4}4k} = e^{-j2\pi k} = 1$

$$W_4^{5k} = W_4^{4k} W_4^{k} = W_4^{k}$$

$$W_4^{6k} = W_4^{4k} W_4^{2k} = W_4^{2k}$$

$$\Rightarrow Y_c[k] = 15 - 22W_4^{k} + 17W_4^{2k} - 47W_4^{3k} - 7 - 22W_4^{k} - 6W_4^{2k}$$

$$= 8 - 44W_4^{k} + 11W_4^{2k} - 47W_4^{3k}, \quad 0 \le k \le 3$$

Problem 5, cont... (more workspace for part 5(a))

From the last page:

$$Y_c[k] = 8 - 44W_4^{k} + 11W_4^{2k} - 47W_4^{3k}$$
$$= 8W_4^{0k} - 44W_4^{k} + 11W_4^{2k} - 47W_4^{3k}, \quad 0 \le k \le 3 \quad (*)$$

By definition,

$$Y_c[k] = \sum_{n=0}^{3} y_c[n] W_4^{nk} = y_c[0]W_4^{0k} + y_c[1]W_4^{k} + y_c[2]W_4^{2k} + y_c[3]W_4^{3k} \quad (**)$$

Comparing $(*)$ and $(**)$,

$y_c[0] = 8 \quad y_c[1] = -44 \quad y_c[2] = 11 \quad y_c[3] = -47$

$$\boxed{\begin{aligned} y_c[n] &= [8 \; -44 \; 11 \; -47] \\ &= 8\delta[n] - 44\delta[n-1] + 11\delta[n-2] - 47\delta[n-3], \quad 0 \le n \le 3 \end{aligned}}$$

Problem 5, cont...

(b) **12/10 pts.** Now use the DFT to find the **linear** convolution $y_\ell[n] = x[n] \circledast h[n]$.

$N_1 = N_2 = 4 \Rightarrow N = N_1 + N_2 - 1 = 7 \Rightarrow$ zero pad both sequences to length $N = 7$.

$x_7[n] = [3\ 1\ 4\ 1\ 0\ 0\ 0]$

$$X_7[k] = \sum_{n=0}^{6} x_7[n] W_7^{nk}$$
$$= 3 + W_7^k + 4W_7^{2k} + W_7^{3k}, \quad 0 \le k \le 6$$

$h_7[n] = [5\ -9\ 2\ -6\ 0\ 0\ 0]$

$$H_7[k] = \sum_{n=0}^{6} h_7[n] W_7^{nk}$$
$$= 5 - 9W_7^k + 2W_7^{2k} - 6W_7^{3k}, \quad 0 \le k \le 6$$

$$Y_\ell[k] = X_7[k] H_7[k] = (3 + W_7^k + 4W_7^{2k} + W_7^{3k})(5 - 9W_7^k + 2W_7^{2k} - 6W_7^{3k})$$

$$\begin{aligned} &= 15 - 27W_7^k + 6W_7^{2k} - 18W_7^{3k} \\ &\quad + 5W_7^k - 9W_7^{2k} + 2W_7^{3k} - 6W_7^{4k} \\ &\quad + 20W_7^{2k} - 36W_7^{3k} + 8W_7^{4k} - 24W_7^{5k} \\ &\quad + 5W_7^{3k} - 9W_7^{4k} + 2W_7^{5k} - 6W_7^{6k} \end{aligned}$$

$$Y_\ell[k] = 15 - 22W_7^k + 17W_7^{2k} - 47W_7^{3k} - 7W_7^{4k} - 22W_7^{5k} - 6W_7^{6k}, \quad 0 \le k \le 6$$

(This could have been deduced from the work in part (a))

By definition, $Y_\ell[k] = \sum_{n=0}^{6} y_\ell[n] W_7^{nk}$

$$= y_\ell[0] + y_\ell[1]W_7^k + y_\ell[2]W_7^{2k} + y_\ell[3]W_7^{3k} + y_\ell[4]W_7^{4k} + y_\ell[5]W_7^{5k} + y_\ell[6]W_7^{6k}, \quad 0 \le k \le 6$$

Comparing this to the result above and "picking off" the $y_\ell[n]$ values,

$$\boxed{\begin{aligned} y_\ell[n] &= [15\ -22\ 17\ -47\ -7\ -22\ -6] \\ &= 15\delta[n] - 22\delta[n-1] + 17\delta[n-2] - 47\delta[n-3] - 7\delta[n-4] \\ &\quad - 22\delta[n-5] - 6\delta[n-6], \quad 0 \le n \le 6 \end{aligned}}$$