ECE 4213/5213
Test 1

Monday, October 24, 2016
4:30 PM - 5:45 PM

Fall 2016

Name: SOLUTION

Dr. Havlicek

Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed: ______________ Date: ______________

1. **25/20 pts.** A discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by

$$y[n] + y[n-1] - \frac{4}{9}y[n-2] = x[n] - \frac{2}{3}x[n-1].$$

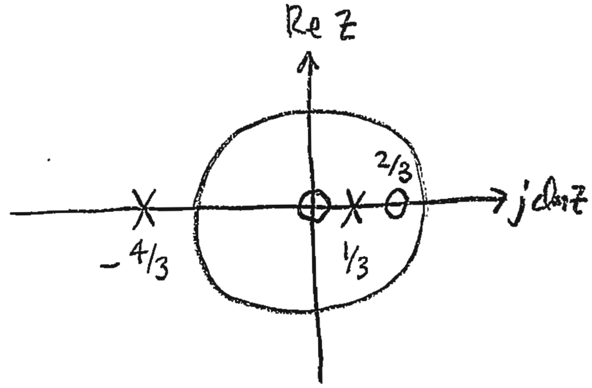
(a) **9/7 pts.** Find the transfer function $H(z)$ and give a pole-zero plot.

$$\mathcal{Z}: \quad Y(z) + z^{-1}Y(z) - \frac{4}{9}z^{-2}Y(z) = X(z) - \frac{2}{3}z^{-1}X(z)$$

$$Y(z)\left[1 + z^{-1} - \frac{4}{9}z^{-2}\right] = X(z)\left[1 - \frac{2}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{2}{3}z^{-1}}{1 + z^{-1} - \frac{4}{9}z^{-2}}$$

$$\boxed{H(z) = \frac{1 - \frac{2}{3}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + \frac{4}{3}z^{-1}\right)}}$$



$$\frac{z^2}{z^2}H(z) = \frac{z(z - 2/3)}{(z - 1/3)(z + 4/3)}$$

poles: $z = 1/3,\ -4/3$

zeros: $z = 0,\ 2/3$

(b) **8/7 pts.** Assume that the system is **causal** and find the impulse response $h[n]$.

Causal $\rightarrow$ ROC is exterior to the largest magnitude pole.

$\rightarrow$ ROC: $|z| > 4/3$

$$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + 4/3\, z^{-1}\right)} = \frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B}{1 + 4/3\, z^{-1}}$$

$$A = \left.\frac{1 - 2/3\,\theta}{1 + 4/3\,\theta}\right|_{\theta = 3} = \frac{1 - 2}{1 + 4} = -1/5$$

$$B = \left.\frac{1 - 2/3\,\theta}{1 - 1/3\,\theta}\right|_{\theta = -3/4} = \frac{1 + \frac{2}{3}\cdot\frac{3}{4}}{1 + \frac{1}{3}\cdot\frac{3}{4}} = \frac{1 + \frac{1}{2}}{1 + \frac{1}{4}} = \frac{3/2}{5/4} = \frac{4}{5}\cdot\frac{3}{2} = \frac{6}{5}$$

$$H(z) = \underbrace{\frac{6/5}{1 + 4/3\, z^{-1}}}_{|z| > 4/3} - \underbrace{\frac{1/5}{1 - 1/3\, z^{-1}}}_{|z| > 1/3}$$

Table: $\boxed{h[n] = \frac{6}{5}\left(-\frac{4}{3}\right)^n u[n] - \frac{1}{5}\left(\frac{1}{3}\right)^n u[n]}$

Problem 1, cont...

(c) **8/6 pts.** Now assume instead that the system is **stable** and find the impulse response $h[n]$.

Stable $\rightarrow$ ROC must include the unit circle.

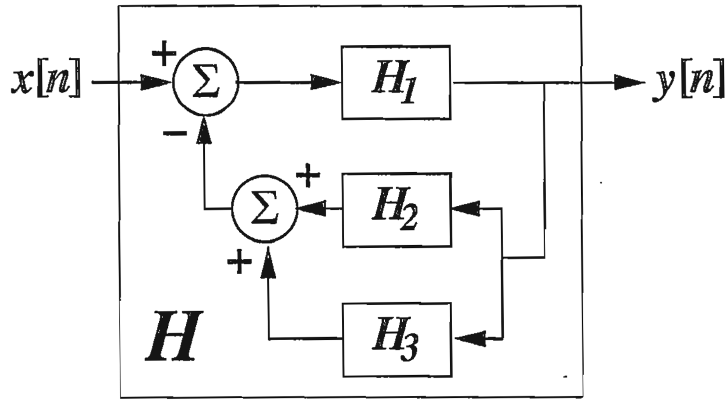
$\rightarrow$ ROC: $\frac{1}{3} < |z| < \frac{4}{3}$

$\Rightarrow$ The PFE is unchanged from part (b):

$$H(z) = \underbrace{\frac{6/5}{1 + \frac{4}{3}z^{-1}}}_{|z| < 4/3} - \underbrace{\frac{1/5}{1 - 1/3\, z^{-1}}}_{|z| > \frac{1}{3}}$$

Table: $\boxed{h[n] = -\frac{6}{5}\left(-\frac{4}{3}\right)^n u[-n-1] - \frac{1}{5}\left(\frac{1}{3}\right)^n u[n]}$

2. **25/20 pts.** The **stable, causal** discrete-time LTI system H is formed by interconnecting three discrete-time LTI systems H_1, H_2, and H_3 as shown in the figure below.



The impulse responses $h_1[n]$, $h_2[n]$, and $h_3[n]$ are given by

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],$$
$$h_2[n] = \left(\frac{3}{32}\right)^n u[n],$$
$$h_3[n] = -\delta[n].$$

Table:

$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$
$$H_2(z) = \frac{1}{1-\frac{3}{32}z^{-1}}, \quad |z| > \frac{3}{32}$$
$$H_3(z) = -1, \quad \text{all } z.$$

(a) **10/8 pts.** Find the transfer function $H(z)$ and specify the ROC.

$$H(z) = \frac{H_1(z)}{1 + H_1(z)\left[H_2(z) + H_3(z)\right]} = \frac{\frac{1}{1-\frac{1}{2}z^{-1}}}{1 + \frac{1}{1-\frac{1}{2}z^{-1}}\left[\frac{1}{1-\frac{3}{32}z^{-1}} - 1\right]} \cdot \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1} + \frac{1}{1-\frac{3}{32}z^{-1}} - 1} = \frac{1}{-\frac{1}{2}z^{-1} + \frac{1}{1-\frac{3}{32}z^{-1}}} \cdot \frac{1-\frac{3}{32}z^{-1}}{1-\frac{3}{32}z^{-1}}$$

$$= \frac{1-\frac{3}{32}z^{-1}}{-\frac{1}{2}z^{-1}\left(1-\frac{3}{32}z^{-1}\right) + 1} = \frac{1-\frac{3}{32}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{3}{64}z^{-2}} = \frac{1-\frac{3}{32}z^{-1}}{\left(1-\frac{1}{8}z^{-1}\right)\left(1-\frac{3}{8}z^{-1}\right)}$$

Causal:
ROC must be exterior to the largest magnitude pole.

$$\boxed{H(z) = \frac{1-\frac{3}{32}z^{-1}}{\left(1-\frac{1}{8}z^{-1}\right)\left(1-\frac{3}{8}z^{-1}\right)}, \quad |z| > \frac{3}{8}}$$

Problem 2, cont...

(b) **5/4 pts.** Find the difference equation that relates the system input $x[n]$ and output $y[n]$.

$$H(z) = \frac{1 - \frac{3}{32}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{3}{64}z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) + \frac{3}{64}z^{-2}Y(z) = X(z) - \frac{3}{32}z^{-1}X(z)$$

$$\mathcal{Z}^{-1}: \boxed{y[n] - \frac{1}{2}y[n-1] + \frac{3}{64}y[n-2] = x[n] - \frac{3}{32}x[n-1]}$$

(c) **10/8 pts.** Find the system impulse response $h[n]$.

$$H(z) = \frac{1 - \frac{3}{32}z^{-1}}{(1 - \frac{1}{8}z^{-1})(1 - \frac{3}{8}z^{-1})} = \frac{A}{1 - \frac{1}{8}z^{-1}} + \frac{B}{1 - \frac{3}{8}z^{-1}}$$

$$A = \left.\frac{1 - \frac{3}{32}\theta}{1 - \frac{3}{8}\theta}\right|_{\theta=8} = \frac{1 - \frac{3 \cdot 8}{32}}{1 - 3} = \frac{1 - \frac{3}{4}}{-2} = \frac{\frac{1}{4}}{-2} = -\frac{1}{8}$$

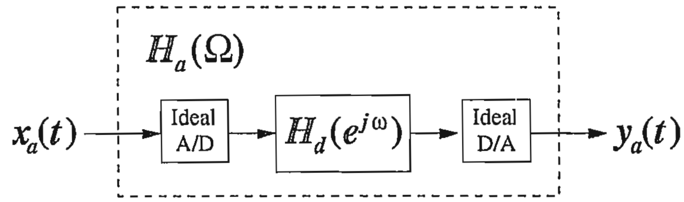
$$B = \left.\frac{1 - \frac{3}{32}\theta}{1 - \frac{1}{8}\theta}\right|_{\theta=\frac{8}{3}} = \frac{1 - \frac{3}{32}\cdot\frac{8}{3}}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{4}}{2/3} = \frac{3}{2}\cdot\frac{3}{4} = \frac{9}{8}$$

$$H(z) = \underbrace{\frac{\frac{9}{8}}{1 - \frac{3}{8}z^{-1}}}_{|z| > 3/8} - \underbrace{\frac{1/8}{1 - \frac{1}{8}z^{-1}}}_{|z| > 1/8}$$

Table:

$$\boxed{\begin{aligned} h[n] &= \frac{9}{8}\left(\frac{3}{8}\right)^n u[n] - \frac{1}{8}\left(\frac{1}{8}\right)^n u[n] \\ &= 3\left(\frac{3}{8}\right)^{n+1} u[n] - \left(\frac{1}{8}\right)^{n+1} u[n] \end{aligned}}$$

3. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



The sampling interval of the A/D and D/A converters is $T = 1$ sec, so the sampling frequency is given by $\Omega_T = 2\pi/T = 2\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2 = \pi$ rad/sec (this simply ensures that the overall structure shown in the figure will be an LTI system).

The impulse response of the discrete-time filter H_d is given by

$$h_d[n] = \frac{1}{2}\delta[n] + \delta[n-2] + \frac{1}{2}\delta[n-4].$$

(a) **6/5 pts.** Find the discrete-time frequency response $H_d(e^{j\omega})$.

$$H_d(e^{j\omega}) = \frac{1}{2} + e^{-j2\omega} + \frac{1}{2}e^{-j4\omega}$$

$$= e^{-j2\omega}\left[\frac{1}{2}e^{j2\omega} + 1 + \frac{1}{2}e^{-j2\omega}\right]$$

$$= \left[1 + \frac{e^{j2\omega} + e^{-j2\omega}}{2}\right]e^{-j2\omega}$$

$$= \left[1 + \cos 2\omega\right] e^{-j2\omega}$$

Problem 3, cont...

(b) **3/3 pts.** Find the discrete-time magnitude response $A(\omega)$ and phase response $\theta(\omega)$ such that $H_d(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$.

From part (a),

$$\boxed{\begin{aligned} A(\omega) &= 1 + \cos 2\omega \\ \theta(\omega) &= -2\omega \end{aligned}}$$

(c) **3/2 pts.** Does the discrete-time filter $H_d(e^{j\omega})$ have linear phase? *(Justify your answer).*

YES. The phase response $\theta(\omega) = -2\omega$ is linear in ω.

(d) **4/3 pts.** Find the group delay of the discrete-time filter $H_d(e^{j\omega})$.

$$\tau_g(\omega) = -\frac{d}{d\omega}\theta(\omega) = -\frac{d}{d\omega}(-2\omega) = 2$$

$$\boxed{\tau_g(\omega) = 2}$$

Problem 3, cont...

(e) **6/5 pts.** Find the continuous-time frequency response $H_a(\Omega)$.

$$H_a(\Omega) = \begin{cases} H_d(e^{j\Omega T}) & , \ |\Omega| \le \frac{\Omega_T}{2} = \pi \text{ rad/sec} \\ 0 & , \ |\Omega| > \frac{\Omega_T}{2} = \pi \text{ rad/sec} \end{cases}$$

$T = 1$

$$\boxed{H_a(\Omega) = \begin{cases} (1+\cos 2\Omega)\, e^{-j2\Omega} & , \ |\Omega| \le \pi \text{ rad/sec} \\ 0 & , \ |\Omega| > \pi \text{ rad/sec} \end{cases}}$$

$$= \left[(1+\cos 2\Omega)\, e^{-j2\Omega}\right]\left[u(\Omega+\pi) - u(\Omega-\pi)\right]$$

$$= (1+\cos 2\Omega)\left[u(\Omega+\pi) - u(\Omega-\pi)\right] e^{-j2\Omega}$$

(f) **3/2 pts.** Does the continuous-time filter $H_a(\Omega)$ have linear phase? *(Justify your answer).*

Magnitude Response: $A(\Omega) = \begin{cases} 1+\cos 2\Omega & , \ |\Omega| \le \pi \\ 0 & , \ |\Omega| > \pi \end{cases}$

Phase Response: $\theta(\Omega) = -2\Omega$

$\boxed{\text{YES, The phase is linear in } \Omega.}$

4. **25/20 pts.** A continuous-time (analog) filter has frequency response $H(\Omega) = A(\Omega)e^{j\theta(\Omega)}$, where the magnitude response is given by

$$A(\Omega) = (1+\cos 2\Omega)\,[u(\Omega+\pi) - u(\Omega-\pi)] = \begin{cases} 1+\cos 2\Omega, & |\Omega| \le \pi \text{ rad/sec}, \\ 0, & |\Omega| > \pi \text{ rad/sec}, \end{cases}$$

and the phase response is given by $\theta(\Omega) = -2\Omega$.

Use the inverse Fourier transform to find the impulse response $h(t)$.

Hint: use the definition of the inverse Fourier transform to invert $A(\Omega)$ first. Then use the Fourier transform time shift property to account for the phase response.

$$a(t) = \mathcal{F}^{-1}[A(\Omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1+\cos 2\Omega) e^{j\Omega t}\, d\Omega$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\, d\Omega}_{\alpha} + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 2\Omega\, e^{j\Omega t}\, d\Omega}_{\beta}$$

$$\alpha = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\left[\frac{1}{jt} e^{j\Omega t}\right]_{\Omega=-\pi}^{\pi}$$

$$= \frac{1}{2\pi j t}\left[e^{j\pi t} - e^{-j\pi t}\right] = \frac{1}{\pi t}\,\frac{e^{j\pi t} - e^{-j\pi t}}{2j}$$

$$= \frac{\sin \pi t}{\pi t} \quad //$$

Formula Sheet: $\int e^{ax} \cos bx\, dx = e^{ax}\,\frac{a\cos bx + b \sin bx}{a^2+b^2}$

For the 2nd integral (β), we have:

$$\begin{cases} x = \Omega \\ a = jt \\ b = 2 \end{cases}$$

More Workspace for Problem 4...

$$\beta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\cos 2\Omega \, d\Omega = \frac{1}{2\pi}\left[e^{j\Omega t} \, \frac{jt\cos 2\Omega + 2\sin 2\Omega}{(jt)^2 + 4} \right]_{\Omega=-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[e^{j\pi t} \, \frac{jt\cos 2\pi + 2\sin 2\pi}{4 - t^2} - e^{-j\pi t} \, \frac{jt\cos(-2\pi) + 2\sin(-2\pi)}{4 - t^2} \right]$$

$$= \frac{1}{2\pi(4-t^2)}\left[e^{j\pi t}\left(jt(1) + 0 \right) - e^{-j\pi t}\left(jt(1) + 0 \right) \right]$$

$$= \frac{jt}{2\pi(4-t^2)}\left[e^{j\pi t} - e^{-j\pi t} \right] \cdot \frac{j}{j}$$

$$= \frac{-t}{\pi(4-t^2)} \, \frac{e^{j\pi t} - e^{-j\pi t}}{2j} = -\frac{t\sin\pi t}{\pi(4-t^2)} = \frac{t\sin\pi t}{\pi(t^2-4)}$$

$$a(t) = \alpha + \beta = \frac{\sin\pi t}{\pi t} + \frac{t\sin\pi t}{\pi(t^2-4)}$$

$$H(\Omega) = A(\Omega)e^{-j2\Omega}$$

$\longrightarrow$ Time Shift property: $h(t) = a(t-2)$

$$h(t) = \frac{\sin[\pi(t-2)]}{\pi(t-2)} + \frac{(t-2)\sin[\pi(t-2)]}{\pi[(t-2)^2 - 4]}$$

$$= \frac{\sin[\pi(t-2)]}{\pi(t-2)} + \frac{(t-2)\sin[\pi(t-2)]}{\pi(t^2 - 4t + 4 - 4)}$$

$$\boxed{h(t) = \frac{\sin[\pi(t-2)]}{\pi(t-2)} + \frac{(t-2)\sin[\pi(t-2)]}{\pi(t^2-4t)} \quad // \\ = \frac{\sin\pi t}{\pi(t-2)} + \frac{(t-2)\sin\pi t}{\pi(t^2-4t)}}$$

5. **25/20 pts**. Let $h[n]$ and $x[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} h[n] &= [1 \quad -3 \quad 2] \\ &= \delta[n] - 3\delta[n-1] + 2\delta[n-2],\ 0 \le n \le 2, \end{aligned}$$

and

$$\begin{aligned} x[n] &= [8 \ 1 \ 4 \ 7] \\ &= 8\delta[n] + \delta[n-1] + 4\delta[n-2] + 7\delta[n-3],\ 0 \le n \le 3. \end{aligned}$$

Use the DFT to find the **linear** convolution $y_\ell[n] = h[n] \circledast x[n]$.

$N_1 = 3$, $N_2 = 4$, zero pad to length $N = N_1 + N_2 - 1$

$= 3 + 4 - 1 = 6$ //

$h_6[n] = [1 \ -3 \ 2 \ 0 \ 0 \ 0]$

$H_6[k] = \sum_{n=0}^{5} h_6[n] W_6^{nk}$

$= 1 - 3W_6^k + 2W_6^{2k}$

$x_6[n] = [8 \ 1 \ 4 \ 7 \ 0 \ 0]$

$X_6[k] = \sum_{n=0}^{5} x_6[n] W_6^{nk}$

$= 8 + W_6^k + 4W_6^{2k} + 7W_6^{3k}$

$Y_\ell[k] = H_6[k] X_6[k] = (1 - 3W_6^k + 2W_6^{2k})(8 + W_6^k + 4W_6^{2k} + 7W_6^{3k})$

$$\begin{aligned} = 8 &+ W_6^k + 4W_6^{2k} + 7W_6^{3k} \\ &- 24W_6^k - 3W_6^{2k} - 12W_6^{3k} - 21W_6^{4k} \\ &+ 16W_6^{2k} + 2W_6^{3k} + 8W_6^{4k} + 14W_6^{5k} \end{aligned}$$

$Y_\ell[k] = 8 - 23W_6^k + 17W_6^{2k} - 3W_6^{3k} - 13W_6^{4k} + 14W_6^{5k}$

$\longrightarrow$

More Workspace for Problem 5...

By definition,

$$Y_\ell[n] = \sum_{n=0}^{5} y_\ell[n] W_6^{nk}$$

$$= y_\ell[0] + y_\ell[1] W_6^{k} + y_\ell[2] W_6^{2k} + \dots + y_\ell[5] W_6^{5k}$$

- Comparing this to the result on page 11 and "picking off" the values $y_\ell[n]$, we have

$$\boxed{\begin{aligned} y_\ell[n] &= [8 \quad -23 \quad 17 \quad -3 \quad -13 \quad 14] \\ &= 8\delta[n] - 23\delta[n-1] + 17\delta[n-2] - 3\delta[n-3] \\ &\quad - 13\delta[n-4] + 14\delta[n-5], \\ &\qquad 0 \le n \le 5 \end{aligned}}$$