ECE 4213/5213
Test 1

Monday, November 6, 2017
4:30 PM - 5:45 PM

Fall 2017 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed:________________ Date:________________

1. **25/20 pts.** A **stable** discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by

$$y[n] + \frac{3}{2}y[n-1] - y[n-2] = 2x[n] + \frac{3}{2}x[n-1].$$

(a) **12/10 pts.** Find the transfer function $H(z)$ and give a pole-zero plot. Be sure to specify the ROC.

$$\mathcal{Z}: \quad Y(z) + \frac{3}{2}z^{-1}Y(z) - z^{-2}Y(z) = 2X(z) + \frac{3}{2}z^{-1}X(z)$$

$$Y(z)\left[1 + \frac{3}{2}z^{-1} - z^{-2}\right] = X(z)\left[2 + \frac{3}{2}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{3}{2}z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}} = \underline{\underline{\frac{2(1 + \frac{3}{4}z^{-1})}{(1+2z^{-1})(1-\frac{1}{2}z^{-1})}}}$$

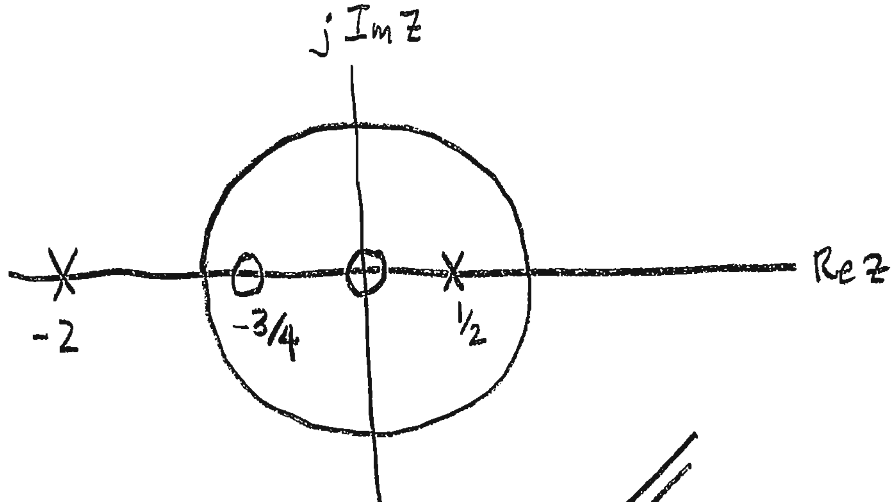
To make zeros explicit, multiply $H(z)$ by $\frac{z^2}{z^2}$ to convert from "z^{-1}" to "z":

$$H(z) \cdot \frac{z^2}{z^2} = \frac{2z(z + \frac{3}{4})}{(z+2)(z-\frac{1}{2})}$$

poles: $z = -2, \frac{1}{2}$

zeros: $z = 0, -3/4$

P/Z plot:



ROC: because H is stable, ROC must include the unit circle:

$$\underline{\underline{\text{ROC: } \frac{1}{2} < |z| < 2}}$$

Problem 1, cont...

(b) **13/10 pts.** Find the impulse response $h[n]$.

PFE on $H(z)$: $\dfrac{2(1+\frac{3}{4}\theta)}{(1+2\theta)(1-\frac{1}{2}\theta)} = \dfrac{A}{1+2\theta} + \dfrac{B}{1-\frac{1}{2}\theta}$

$$A = \left.\frac{2(1+\frac{3}{4}\theta)}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{2}} = \frac{2(1-3/8)}{1+1/4} = \frac{2\cdot 5/8}{5/4} = \frac{5/4}{5/4} = 1$$

$$B = \left.\frac{2(1+\frac{3}{4}\theta)}{1+2\theta}\right|_{\theta=2} = \frac{2(1+\frac{3}{2})}{1+4} = \frac{2(5/2)}{5} = \frac{5}{5} = 1$$

$$H(z) = \underbrace{\frac{1}{1+2z^{-1}}}_{\substack{|z|<2 \\ \alpha=-2}} + \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{\substack{|z|>\frac{1}{2} \\ \alpha=1/2}}$$

Table: $\boxed{h[n] = -(-2)^n u[-n-1] + (\tfrac{1}{2})^n u[n]}$

2. **25/20 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n].$$

The system input is given by

$$x[n] = \delta[n] + \frac{4}{3}\delta[n-1] - \frac{1}{2}\delta[n-2] - \frac{1}{6}\delta[n-3].$$

(a) **12/10 pts.** Find $Y(z)$, the z-transform of the output signal. Be sure to specify the ROC.

Table: $H(z) = \underbrace{\frac{3/5}{1-\frac{1}{2}z^{-1}}}_{|z|>1/2} + \underbrace{\frac{2/5}{1+\frac{1}{3}z^{-1}}}_{|z|>1/3} = \frac{\frac{3}{5}\left(1+\frac{1}{3}z^{-1}\right) + \frac{2}{5}\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$

$$= \frac{\frac{3}{5} + \frac{1}{5}z^{-1} + \frac{2}{5} - \frac{1}{5}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-1} - \frac{1}{6}z^{-2}} = \frac{5/5}{1 - \frac{3}{6}z^{-1} + \frac{2}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

$$= \frac{1}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}, \quad |z| > 1/2$$

Table + Timeshift: $X(z) = 1 + \frac{4}{3}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{6}z^{-3}, \quad |z| > 0$

$$Y(z) = X(z)H(z) = \frac{1 + \frac{4}{3}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{6}z^{-3}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}, \quad |z| > 1/2$$

Problem 2, cont...

(b) 13/10 pts. Find the ouptput signal $y[n]$.

$\Rightarrow$ Can not apply PFE directly b/c $Y(z)$ is an improper fraction.
$\Rightarrow$ Use long division; b/c ROC is exterior, $y[n]$ is right-sided.

Need quotient to be in terms of z^{-1}, not z.
($y[n]$ is right-sided b/c ROC is exterior)

$$\begin{array}{r|l} & z^{-1} + 2 \\ \hline -\frac{1}{6}z^{-2} - \frac{1}{6}z^{-1} + 1 & -\frac{1}{6}z^{-3} - \frac{1}{2}z^{-2} + \frac{4}{3}z^{-1} + 1 \\ & -\frac{1}{6}z^{-3} - \frac{1}{6}z^{-2} + z^{-1} \\ \hline & -\frac{2}{6}z^{-2} + \frac{1}{3}z^{-1} + 1 \\ & -\frac{2}{6}z^{-2} - \frac{2}{6}z^{-1} + 2 \\ \hline & \frac{4}{6}z^{-1} - 1 = \frac{2}{3}z^{-1} - 1 \end{array}$$

$$Y(z) = 2 + z^{-1} + \frac{-1 + \frac{2}{3}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = 2 + z^{-1} + \frac{-1 + \frac{2}{3}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

PFE on Fraction: $\dfrac{-1 + \frac{2}{3}\theta}{(1 - \frac{1}{2}\theta)(1 + \frac{1}{3}\theta)} = \dfrac{A}{1 - \frac{1}{2}\theta} + \dfrac{B}{1 + \frac{1}{3}\theta}$

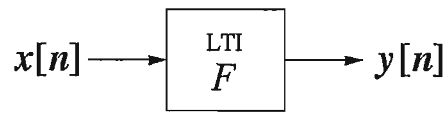
$$A = \left.\frac{-1 + \frac{2}{3}\theta}{1 + \frac{1}{3}\theta}\right|_{\theta = 2} = \frac{-1 + \frac{4}{3}}{1 + 2/3} = \frac{1/3}{5/3} = \frac{1}{5}$$

$$B = \left.\frac{-1 + \frac{2}{3}\theta}{1 - \frac{1}{2}\theta}\right|_{\theta = -3} = \frac{-1 - 2}{1 + 3/2} = \frac{-3}{5/2} = -\frac{6}{5}$$

$$Y(z) = \underbrace{2}_{\text{all } z} + \underbrace{z^{-1}}_{|z| > 0} + \underbrace{\frac{1/5}{1 - \frac{1}{2}z^{-1}}}_{|z| > 1/2} - \underbrace{\frac{6/5}{1 + \frac{1}{3}z^{-1}}}_{|z| > 1/3}$$

Table: $\boxed{y[n] = 2\delta[n] + \delta[n-1] + \frac{1}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{6}{5}\left(-\frac{1}{3}\right)^n u[n]}$

3. **25/20 pts.** The **causal** discrete-time LTI system F shown below has input $x[n]$ and output $y[n]$ related by the difference equation $y[n] - 4y[n-1] = x[n]$.



(a) **8/6 pts.** Find the transfer function $F(z)$. Be sure to specify the ROC.

$\mathcal{Z}$: $Y(z) - 4z^{-1}Y(z) = X(z) \Rightarrow Y(z)[1-4z^{-1}] = X(z)$

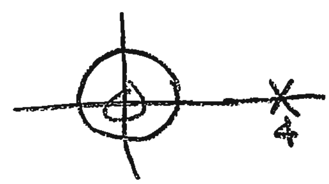
$$F(z) = \frac{Y(z)}{X(z)} = \underline{\underline{\frac{1}{1-4z^{-1}}}} \; ; \; F(z)\cdot\frac{z}{z} = \frac{z}{z-4}$$

pole: $z=4$

zero: $z=0$

Because F is causal, f[n] must be right-sided → ROC must be exterior.

ROC: $\underline{\underline{|z| > 4}}$



(b) **5/4 pts.** Is the system F stable? *Justify your answer.*

- The system F is NOT stable because the ROC does not include the unit circle.

- Alternatively, the system F is NOT stable because it is causal, but there is a pole outside the unit circle.

Problem 3, cont...

(c) **12/10 pts.** As shown in the figure below, a new causal discrete-time LTI system H is formed by adding negative feedback to the system F from part (a):

From part (a):

$F(z) = \frac{1}{1-4z^{-1}}, |z|>4$

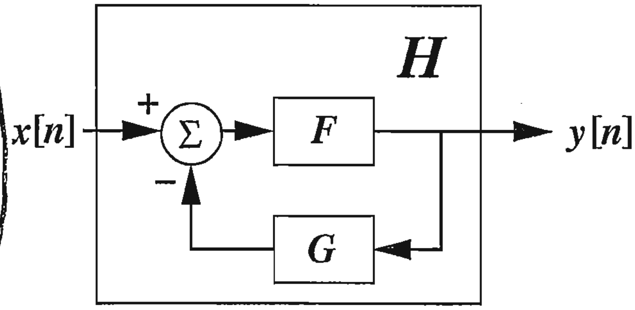


Table:

$G(z) = K$, all z

The impulse response of LTI system G is given by $g[n] = K\delta[n]$, where $K \in \mathbb{R}$ is a real constant.

For what values of the constant K is the overall system H BIBO stable?

$$H(z) = \frac{F(z)}{1+F(z)G(z)} = \frac{\frac{1}{1-4z^{-1}}}{1+\frac{K}{1-4z^{-1}}} \cdot \frac{1-4z^{-1}}{1-4z^{-1}} = \frac{1}{1-4z^{-1}+K}$$

$$= \frac{1}{(1+K)-4z^{-1}} \cdot \frac{\frac{1}{K+1}}{\frac{1}{K+1}} = \frac{\frac{1}{K+1}}{1-\frac{4}{K+1}z^{-1}}$$

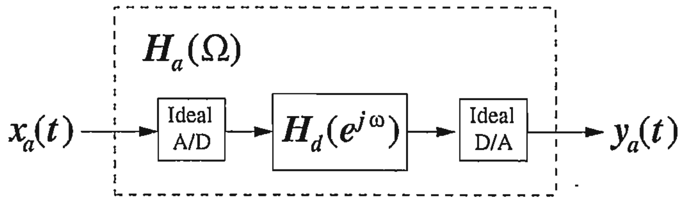
→ H has one pole at $z = \frac{4}{K+1}$.

→ Since H is LTI and causal, it is stable iff this pole is <u>inside</u> the unit circle. So, for stability we need $\left|\frac{4}{K+1}\right| < 1 \Rightarrow \frac{4}{|K+1|} < 1 \Rightarrow \frac{1}{|K+1|} < \frac{1}{4}$

$\Rightarrow |K+1| > 4 \Rightarrow K+1 < -4$ or $K+1 > 4$

$\Rightarrow \boxed{K < -5 \text{ or } K > 3}$

4. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



The sampling frequency of the A/D and D/A converters is $F_T = 44$ kHz. H_d is causal and has input/output relation

$$y[n] - \frac{2}{3}y[n-1] + \frac{1}{9}y[n-2] = x[n].$$

Assume that all input signals $x_a(t)$ are band limited with $|X_a(\Omega)| = 0 \ \forall \ |\Omega| > \pi/T$ so that aliasing does not occur.

(a) **9/8 pts.** Find the frequency response $H_d(e^{j\omega})$ of the digital filter.

DTFT: $Y(e^{j\omega}) - \frac{2}{3}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{9}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})$

$$Y(e^{j\omega})\left[1 - \frac{2}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega}\right] = X(e^{j\omega})$$

$$H_d(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{2}{3}e^{-j\omega} + \frac{1}{9}e^{-j2\omega}}$$

$$= \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

$$\boxed{H_d(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}e^{-j\omega})^2}}$$

Problem 4, cont...

(b) **8/6 pts.** Find the impulse response $h_d[n]$ of the digital filter.

Table: $\boxed{h_d[n] = (n+1)\left(\frac{1}{3}\right)^n u[n]}$

(c) **8/6 pts.** Find the frequency response $H_a(\Omega)$ of the overall analog system with input $x_a(t)$ and output $y_a(t)$.

Formula Sheet page 3: $H_a(\Omega) = \begin{cases} H_d(e^{j\Omega T}), & |\Omega| < \frac{\Omega_T}{2} \\ 0, & |\Omega| > \frac{\Omega_T}{2} \end{cases}$

$F_T = 44\text{ kHz} = 44{,}000\text{ Hz}.$

$\Omega_T = 2\pi F_T = 2\pi \frac{\text{rad}}{\text{cycle}} \cdot 44{,}000 \frac{\text{cycle}}{\text{sec}} = 88{,}000\pi \frac{\text{rad}}{\text{sec}}$

$\Omega_T = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\Omega_T} = \frac{2\pi}{88{,}000\pi} = \frac{1}{44{,}000}\text{ sec}$

$\frac{\Omega_T}{2} = \frac{88{,}000\pi}{2} = 44{,}000\pi\text{ rad/sec}$; $\Omega \cdot T = \Omega/44{,}000$

$$\boxed{H_a(\Omega) = \begin{cases} \dfrac{1}{\left(1 - \frac{1}{3}e^{-j\Omega/44{,}000}\right)^2}, & |\Omega| < 44{,}000\pi \\ 0, & |\Omega| > 44{,}000\pi \end{cases}}$$

5. **25/20 pts.** Let $h[n]$ and $x[n]$ be finite-length discrete-time signals given by

$$\begin{aligned} h[n] &= [-2 \quad 4 \quad -2] \\ &= -2\delta[n] + 4\delta[n-1] - 2\delta[n-2], \; 0 \le n \le 2, \end{aligned}$$

and

$$\begin{aligned} x[n] &= [3 \;\; 2 \;\; -1] \\ &= 3\delta[n] + 2\delta[n-1] - \delta[n-2], \; 0 \le n \le 2. \end{aligned}$$

Use the DFT to find the **linear** convolution $y_\ell[n] = h[n] \circledast x[n]$.

$N_1 = N_2 = 3$. For linear convolution, zero pad to $N = N_1 + N_2 - 1 = 3+3-1 = 5$.

$h_5[n] = [-2 \;\; 4 \;\; -2 \;\; 0 \;\; 0]$ $\qquad$ $x_5[n] = [3 \;\; 2 \;\; -1 \;\; 0 \;\; 0]$

$H_5[k] = \sum_{n=0}^{4} h_5[n] W_5^{nk}$ $\qquad$ $X_5[k] = \sum_{k=0}^{4} x_5[n] W_5^{nk}$

$= -2 + 4W_5^k - 2W_5^{2k}$ $\qquad$ $= 3 + 2W_5^k - W_5^{2k}$

$$Y_\ell[k] = H_5[k] X_5[k] = \left(-2 + 4W_5^k - 2W_5^{2k}\right)\left(3 + 2W_5^k - W_5^{2k}\right)$$

$$\begin{aligned} = -6 - 4W_5^k &+ 2W_5^{2k} \\ +12W_5^k &+ 8W_5^{2k} - 4W_5^{3k} \\ &- 6W_5^{2k} - 4W_5^{3k} + 2W_5^{4k} \end{aligned}$$

$$Y_\ell[k] = -6 + 8W_5^k + 4W_5^{2k} - 8W_5^{3k} + 2W_5^{4k}$$

$$Y_\ell[k] = \sum_{n=0}^{4} y_\ell[n] W_5^{nk} = y_\ell[0] + y_\ell[1] W_5^k + y_\ell[2] W_5^{2k} + y_\ell[3] W_5^{3k} + y_\ell[4] W_5^{4k}$$

$$\boxed{\begin{aligned} y_\ell[n] &= [-6 \;\; 8 \;\; 4 \;\; -8 \;\; 2] \\ &= -6\delta[n] + 8\delta[n-1] + 4\delta[n-2] - 8\delta[n-3] + 2\delta[n-4], \; 0 \le n \le 4 \end{aligned}}$$