ECE 4213/5213
Test 1

Monday, October 15, 2018
4:30 PM - 5:45 PM

Fall 2018 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed:__________ Date:__________

1. **25/20 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

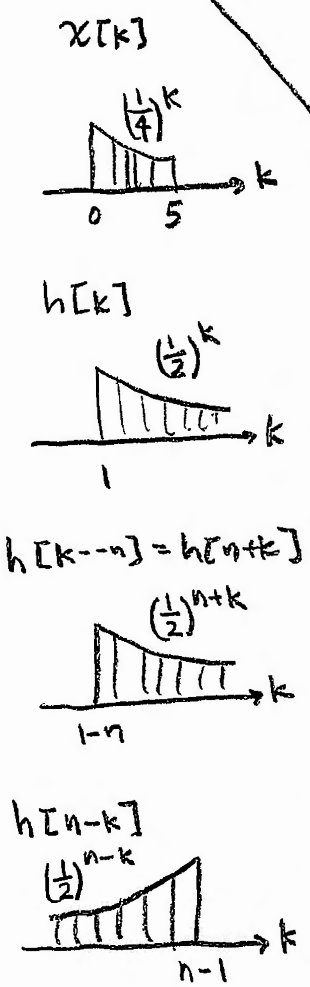
$$h[n] = \left(\frac{1}{2}\right)^n u[n-1].$$

The system input is given by

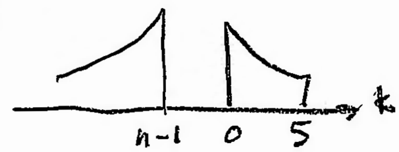
$$x[n] = \left(\frac{1}{4}\right)^n (u[n] - u[n-6]) = \begin{cases} \left(\frac{1}{4}\right)^n, & 0 \le n \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Use time domain convolution to find the system output $y[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



case I) $n-1 < 0$; $n < 1$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II) $n-1 \geq 0$ and $n-1 < 5$

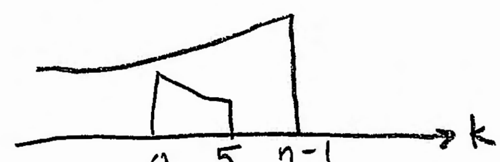
$1 \le n < 6$

$$y[n] = \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k 2^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{2}\right)^n \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^n}{1 - 1/2} = \left(\tfrac{1}{2}\right)^n \frac{1 - \left(\frac{1}{2}\right)^n}{1/2}$$

$$= 2\left[\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right]$$

case III) $n-1 \geq 5$: $n \geq 6$



$$y[n] = \sum_{k=0}^{5} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{5} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{5} \left(\tfrac{1}{4}\right)^k 2^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{5} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{2}\right)^n \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^6}{1 - 1/2} = \left(\tfrac{1}{2}\right)^n \frac{1 - \frac{1}{64}}{1/2}$$

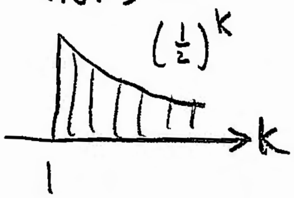
$$= 2 \cdot \frac{63}{64} \cdot \left(\tfrac{1}{2}\right)^n = \frac{63}{32}\left(\tfrac{1}{2}\right)^n$$

All Together:
$$y[n] = \begin{cases} 0, & n < 1 \\ 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right], & 1 \le n < 6 \\ \frac{63}{32}\left(\frac{1}{2}\right)^n, & n \geq 6 \end{cases}$$

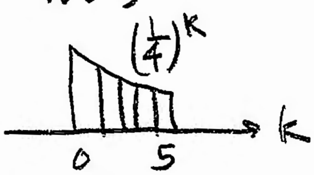
OTHER WAY: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

More Workspace for Problem 1...

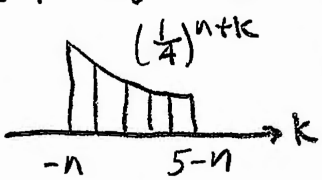
$h[k]$



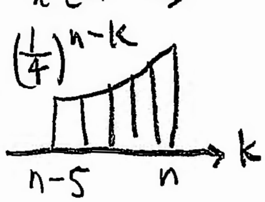
$x[k]$



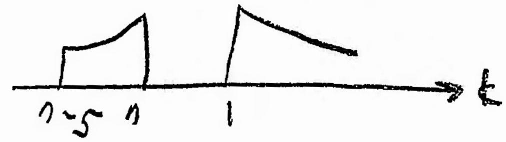
$x[k - -n] = x[n+k]$



$x[n-k]$

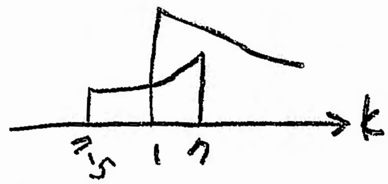


Case I) $n < 1$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq 1$ and $n - 5 < 1$



$1 \leq n < 6$

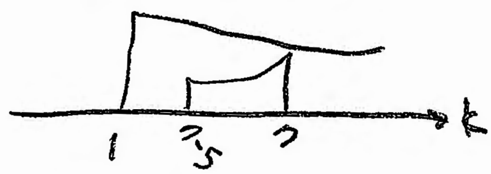
$$y[n] = \sum_{k=1}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{n-k} = (\tfrac{1}{4})^n \sum_{k=1}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{-k}$$

$$= (\tfrac{1}{4})^n \sum_{k=1}^{n} (\tfrac{1}{2})^k 4^k = (\tfrac{1}{4})^n \sum_{k=1}^{n} 2^k = (\tfrac{1}{4})^n \frac{2^1 - 2^{n+1}}{1-2}$$

$$= (\tfrac{1}{4})^n [2^{n+1} - 2] = 2(\tfrac{1}{4})^n [2^n - 1]$$

$$= 2[(\tfrac{1}{4})^n 2^n - (\tfrac{1}{4})^n] = 2[(\tfrac{1}{2})^n - (\tfrac{1}{4})^n]$$

Case III) $n \geq 6$



$$y[n] = \sum_{k=n-5}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{n-k}$$

$$= (\tfrac{1}{4})^n \sum_{k=n-5}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{-k} = (\tfrac{1}{4})^n \sum_{k=n-5}^{n} (\tfrac{1}{2})^k 4^k$$

$$= (\tfrac{1}{4})^n \sum_{k=n-5}^{n} 2^k = (\tfrac{1}{4})^n \frac{2^{n-5} - 2^{n+1}}{1-2} = (\tfrac{1}{4})^n [2 \cdot 2^n - \tfrac{1}{32} 2^n]$$

$$= 2^n (\tfrac{1}{4})^n [2 - \tfrac{1}{32}] = (\tfrac{1}{2})^n [\tfrac{64-1}{32}] = \tfrac{63}{32} (\tfrac{1}{2})^n$$

All Together:

$$y[n] = \begin{cases} 0 & , \; n < 1 \\ 2[(\frac{1}{2})^n - (\frac{1}{4})^n] & , \; 1 \leq n < 6 \\ \frac{63}{32}(\frac{1}{2})^n & , \; n \geq 6 \end{cases}$$

2. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system H are related by

$$y[n] = x[2-n] + 1.$$

(a) **6/5 pts.** Is the system H linear? *Justify your answer.*

Let $x_1[n] = 0 \ \forall n \longrightarrow y_1[n] = 1 \ \forall n$

Let $x_2[n] = 0 \ \forall n \longrightarrow y_2[n] = 1 \ \forall n$

Let $a = 2$ and $b = 3$.

Let $x_3[n] = ax_1[n] + bx_2[n] = 0 \ \forall n \rightarrow y_3[n] = 1 \ \forall n$.

But $ay_1[n] + by_2[n] = 2y_1[n] + 3y_2[n] = 2+3 = 5 \ \forall n \neq y_3[n]$.

Since $ay_1[n] + by_2[n] \neq y_3[n]$,

the system is NOT LINEAR.

Problem 2, cont...

(b) **6/5 pts.** Is the system H causal? *Justify your answer.*

Let $x[n]$ be the input signal.

When $n=0$, we have:

$$y[0] = x[2] + 1$$

which depends on the future input $x[2]$.

Therefore, the system is NOT CAUSAL

(c) **6/5 pts.** Is the system H BIBO stable? *Justify your answer.*

Let $x[n]$ be a bounded input signal.

Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|x[n]| \leq B \ \forall n \in \mathbb{Z}$.

Now, $|y[n]| = |x[2-n] + 1|$

$$\leq |x[2-n]| + 1$$

$$\leq B + 1 < \infty.$$

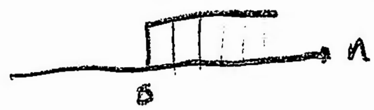
Therefore, $y[n]$ is bounded by $B+1$ and every bounded input signal produces a bounded output signal.

Therefore, the system IS BIBO STABLE.

Problem 2, cont...

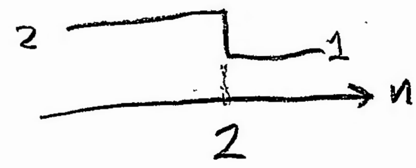
(d) 7/5 pts. Is the system H time invariant? *Justify your answer.*

Let $x_1[n] = u[n]$



Then $y_1[n] = x[2-n] + 1$

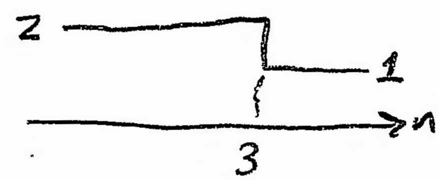
$= u[2-n] + 1$



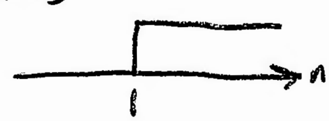
Let $n_0 = 1$. Then $y_1[n-n_0] = y_1[n-1] = u[2-\theta]\Big|_{\theta = n-1} + 1$

$= u[2-(n-1)] + 1$

$= u[3-n] + 1$



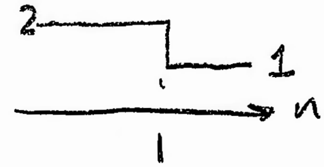
Now let $x_2[n] = x_1[n-n_0] = x_1[n-1] = u[n-1]$



$y_2[n] = x_2[2-n] + 1 = u[\theta - 1]\Big|_{\theta = 2-n} + 1$

$= u[2-n-1] + 1$

$= u[1-n] + 1$



Since $y_2[n] \neq y_1[n-n_0]$,

the system is <u>NOT TIME INVARIANT</u>

3. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by

$$y[n] = \frac{1}{4}x[n] + \frac{1}{2}x[n-1] + x[n-2] + \frac{1}{2}x[n-3] + \frac{1}{4}x[n-4].$$

Find the system group delay τ_g.

DTFT: $Y(e^{j\omega}) = X(e^{j\omega})\left[\frac{1}{4} + \frac{1}{2}e^{-j\omega} + e^{-j2\omega} + \frac{1}{2}e^{-j3\omega} + \frac{1}{4}e^{-j4\omega}\right]$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left[\frac{1}{4} + \frac{1}{2}e^{-j\omega} + e^{-j2\omega} + \frac{1}{2}e^{-j3\omega} + \frac{1}{4}e^{-j4\omega}\right]$$

$$= \left[\frac{1}{4}e^{j2\omega} + \frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}\right]e^{-j2\omega}$$

$$= \underbrace{\left[1 + \cos\omega + \frac{1}{2}\cos 2\omega\right]}_{|H(e^{j\omega})|}\underbrace{e^{-j2\omega}}_{\arg H(e^{j\omega})}$$

Spectral phase $= \arg H(e^{j\omega}) = \theta(\omega) = -2\omega$

group delay $= \tau_g = -\frac{d}{d\omega}\theta(\omega) = 2$

$$\boxed{\tau_g = 2}$$

4. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by

$$y[n] - \frac{10}{3}y[n-1] + y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

(a) **5/4 pts.** Find the transfer function $H(z)$ and give a pole zero plot.

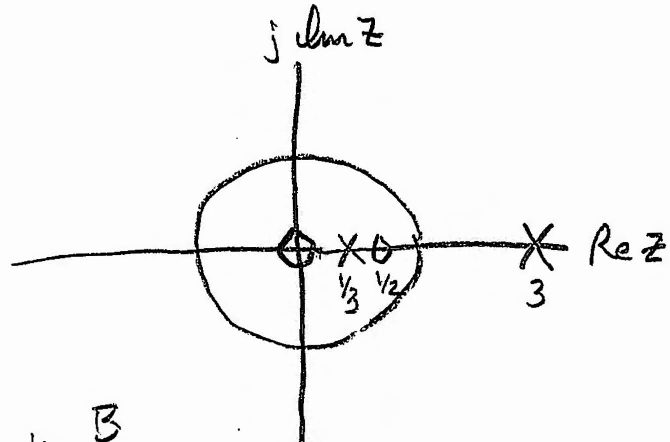
$$\mathcal{Z}: \quad Y(z)\left[1 - \frac{10}{3}z^{-1} + z^{-2}\right] = X(z)\left[1 - \frac{1}{2}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}} = \underline{\underline{\frac{1 - \frac{1}{2}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - 3z^{-1})}}}$$

$$H(z)\frac{z^2}{z^2} = \frac{z(z - \frac{1}{2})}{(z - \frac{1}{3})(z - 3)}$$

zeros: $z = 0$, $z = \frac{1}{2}$

poles: $z = \frac{1}{3}$, $z = 3$



PFE:

$$H(\theta) = \frac{1 - \frac{1}{2}\theta}{(1 - \frac{1}{3}\theta)(1 - 3\theta)} = \frac{A}{1 - \frac{1}{3}\theta} + \frac{B}{1 - 3\theta}$$

$$A = \left.\frac{1 - \frac{1}{2}\theta}{1 - 3\theta}\right|_{\theta = 3} = \frac{1 - \frac{3}{2}}{1 - 9} = \frac{-\frac{1}{2}}{-8} = \frac{1}{16}$$

$$B = \left.\frac{1 - \frac{1}{2}\theta}{1 - \frac{1}{3}\theta}\right|_{\theta = \frac{1}{3}} = \frac{1 - \frac{1}{6}}{1 - \frac{1}{9}} = \frac{5/6}{8/9} = \frac{9}{8} \cdot \frac{5}{6} = \frac{3 \cdot 5}{2 \cdot 8} = \frac{15}{16}$$

$$H(z) = \frac{1/16}{1 - \frac{1}{3}z^{-1}} + \frac{15/16}{1 - 3z^{-1}}$$

Problem 4, cont...

(b) **10/8 pts.** Assume that the system H is causal. Specify the ROC of $H(z)$ and find the impulse response $h[n]$. For this assumption, is the system H BIBO stable?

H causal means the ROC is exterior to the largest pole.

$$\boxed{\text{ROC: } |z| > 3}$$

Since the ROC does not contain the unit circle,

H is not BIBO STABLE.

$$H(z) = \underbrace{\frac{1/16}{1-\frac{1}{3}z^{-1}}}_{|z|>1/3} + \underbrace{\frac{15/16}{1-3z^{-1}}}_{|z|>3}$$

Table: $h[n] = \frac{1}{16}\left(\frac{1}{3}\right)^n u[n] + \frac{15}{16} 3^n u[n]$

(c) **10/8 pts.** Now assume instead that the system H is BIBO stable. Specify the ROC of $H(z)$ and find the impulse response $h[n]$. For this assumption, is the system H causal?

H stable means the ROC must include the unit circle.

$$\boxed{\text{ROC: } \tfrac{1}{3} < |z| < 3}$$

Since the ROC is annular, $h[n]$ is two-sided and the system is not causal.

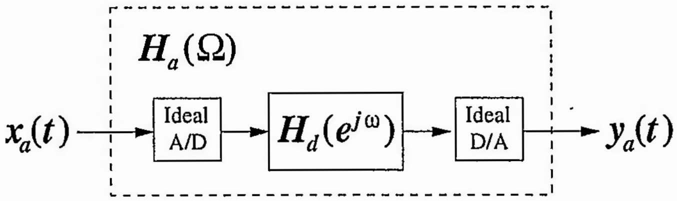
$$H(z) = \underbrace{\frac{1/16}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{15/16}{1-3z^{-1}}}_{|z|<3}$$

Table: $h[n] = \frac{1}{16}\left(\frac{1}{3}\right)^n u[n] - \frac{15}{16} 3^n u[-n-1]$

5. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.

$F_T = 44.1$ kHz $= 44{,}100$ Hz

$\Omega_T = 2\pi F_T = 88{,}200\pi \; \frac{\text{rad}}{\text{sec}}$



$T = \frac{2\pi}{\Omega_T} = \frac{1}{F_T} = \frac{1}{44{,}100}$ sec

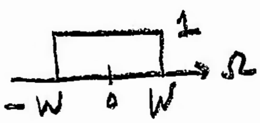
As in professional compact disc (CD) audio, the sampling frequency of the A/D and D/A converters is $F_T = 44.1$ kHz. So $\Omega_T = 88{,}200\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this simply ensures that the overall structure shown in the figure will be an LTI system).

The impulse response of the continuous-time filter H_a is given by

$$h_a(t) = \frac{\sin 7000\pi t}{\pi t}.$$

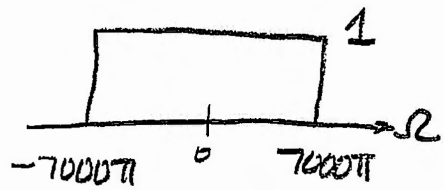
(a) **8/6 pts.** Find the continuous-time frequency response $H_a(\Omega)$.

Table: $\frac{\sin Wt}{\pi t} \longleftrightarrow$



$W = 7000\pi$

$$H_a(\Omega) = \begin{cases} 1, & |\Omega| \le 7000\pi \\ 0, & \text{otherwise} \end{cases} =$$

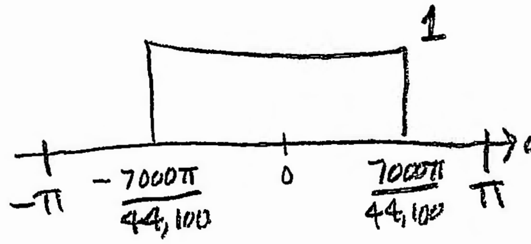


(b) **8/7 pts.** Find the discrete-time frequency response $H_d(e^{j\omega})$.

$$\omega = \Omega T = \frac{\Omega}{44{,}100}$$

$$H_d(e^{j\omega}) = H_a\left(\frac{\omega}{T}\right), \quad |\omega| < \pi$$

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{7000\pi}{44{,}100} \\ 0, & \frac{7000\pi}{44{,}100} \le |\omega| < \pi \end{cases} =$$



Problem 5, cont...

(c) 9/7 pts. Find the discrete-time impulse response $h_d[n]$.

Table: $h_d[n] = \dfrac{\sin\left(\frac{7000\pi}{44{,}160} n\right)}{\pi n} = \dfrac{\sin\left(\frac{70\pi}{441} n\right)}{\pi n}$