ECE 4213/5213
Test 1

Monday, October 28, 2019
4:30 PM - 5:45 PM

Fall 2019 Name: SOLUTION

Dr. Havlicek Student Num: ______

Directions: This test is open book. You may also use a calculator, a clean copy of the course notes, and a clean copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed: ______ Date: ______

⇒ This is homework problem 4.3c) from the textbook.

1. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system H are related by

$$y[n] = \ln\left(1 + |x[n]|\right).$$

(a) **8/6 pts.** Is the system H linear? *Justify your answer.*

Let the input be $x_1[n]$. Then the output is $y_1[n] = \ln(1 + |x_1[n]|)$.
Let the input be $x_2[n]$. Then the output is $y_2[n] = \ln(1 + |x_2[n]|)$

Let $a, b \in \mathbb{C}$ be constants.

Then $a y_1[n] + b y_2[n] = a \ln(1 + |x_1[n]|) + b \ln(1 + |x_2[n]|)$.

Now let $x_3[n] = a x_1[n] + b x_2[n]$.

Then
$$\begin{aligned} y_3[n] &= \ln(1 + |x_3[n]|) \\ &= \ln(1 + |a x_1[n] + b x_2[n]|) \\ &\neq a y_1[n] + b y_2[n] \quad \text{in general.} \end{aligned}$$

$\boxed{\text{NOT LINEAR}}$

Problem 1, cont...

(b) **8/6 pts.** Is the system H causal? *Justify your answer.*

At time $n = n_0$, we have $y[n_0] = \ln(1 + |x[n_0]|)$.

This depends on the current value of the input signal $x[n_0]$, but <u>not</u> on any <u>future</u> values of the input signal for $n > n_0$.

Therefore, the system <u>IS CAUSAL</u>

(c) **9/8 pts.** Is the system H BIBO stable? *Justify your answer.*

Let $x[n]$ be a bounded input signal. Then $\exists B \in \mathbb{R}$, $B > 0$, s.t. $|x[n]| \leq B \;\forall n \in \mathbb{Z}$.

The system output is given by $y[n] = \ln(1 + |x[n]|)$.

So
$$
\begin{aligned}
|y[n]| &= |\ln(1 + |x[n]|)| \\
&\leq |\ln(1+B)| \quad \leftarrow \{\text{because } \ln x \text{ is monotonically increasing in } x \text{ for } x \geq 1. \\
&= \ln(1+B) \quad \leftarrow \{\text{because } \ln x \geq 0 \text{ for } x \geq 1.
\end{aligned}
$$

$\Rightarrow$ $y[n]$ is bounded by $\ln(1+B)$ when $x[n]$ is bounded by B.

$\Rightarrow$ Every bounded input signal produces a bounded output signal.

$\boxed{\text{STABLE}}$

2. **25/20 pts.** A a discrete-time LTI system H has impulse response

$$h[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3].$$

(a) **5/4 pts.** Find the system frequency response $H(e^{j\omega})$.

Table : $\underline{\underline{H(e^{j\omega}) = 1 + 2e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}}}$

(b) **5/4 pts.** Find the system magnitude response $A(\omega)$ and phase response $\theta(\omega)$ such that $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$.

Factor out "half the highest power":

$$H(e^{j\omega}) = e^{-j\frac{3}{2}\omega}e^{j\frac{3}{2}\omega} + 2e^{-j\frac{3}{2}\omega}e^{j\frac{1}{2}\omega} + 2e^{-j\frac{3}{2}\omega}e^{-j\frac{1}{2}\omega} + e^{-j\frac{3}{2}\omega}e^{-j\frac{3}{2}\omega}$$

$$= \left[e^{j\frac{3}{2}\omega} + 2e^{j\frac{1}{2}\omega} + 2e^{-j\frac{1}{2}\omega} + e^{-j\frac{3}{2}\omega}\right]e^{-j\frac{3}{2}\omega}$$

$$= \left[\left(e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}\right) + 2\left(e^{j\frac{1}{2}\omega} + e^{-j\frac{1}{2}\omega}\right)\right]e^{-j\frac{3}{2}\omega}$$

$$= \underbrace{\left[2\cos\left(\tfrac{3}{2}\omega\right) + 4\cos\left(\tfrac{1}{2}\omega\right)\right]}_{A(\omega)}\underbrace{e^{-j\frac{3}{2}\omega}}_{e^{j\theta(\omega)}}$$

$$\boxed{\begin{aligned} A(\omega) &= 2\cos\tfrac{3}{2}\omega + 4\cos\tfrac{1}{2}\omega \\ \theta(\omega) &= -\tfrac{3}{2}\omega \end{aligned}}$$

Problem 2, cont...

(c) **5/4 pts.** Does the system H have linear phase? *(Justify your answer).*

$\boxed{\text{YES}}$ because $\theta(\omega) = -\frac{3}{2}\omega$ is a linear function of ω.

(d) **5/4 pts.** Find the system phase delay.

NOTES p. 3-119:

$$\text{phase delay} = \frac{-\theta(\omega)}{\omega} = \frac{\frac{3}{2}\omega}{\omega} = \boxed{\frac{3}{2}}$$

(e) **5/4 pts.** Find the system group delay.

NOTES p. 3-120:

$$\text{group delay} = \tau_g(\omega) = -\frac{d}{d\omega}\theta(\omega)$$

$$= -\frac{d}{d\omega}\left(-\frac{3}{2}\omega\right)$$

$$= \boxed{\frac{3}{2}}$$

3. **25/20 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(-\frac{1}{3}\right)^n u[n] - 3^n u[-n-1].$$

(a) **4/3 pts.** Is the system H causal? *Justify your answer.*

NOT CAUSAL because $h[-1] = -\frac{1}{3} \neq 0$.

(b) **8/7 pts.** Find the transfer function $H(z)$ and give a pole-zero plot. Don't forget to specify the ROC.

Table: $\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \ |z| > \frac{1}{3}$

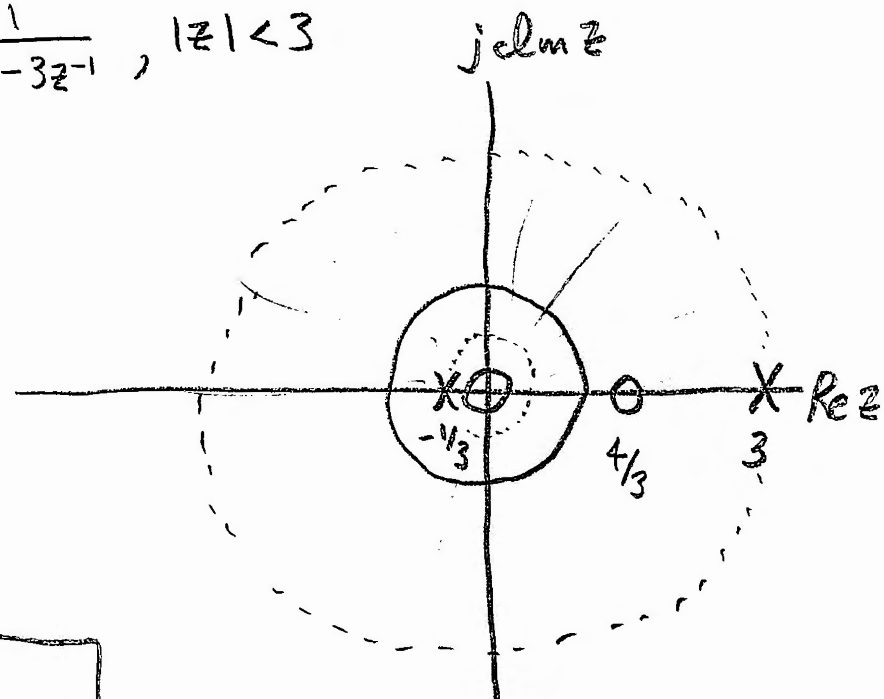
$-3^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-3z^{-1}}, \ |z| < 3$

$$H(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-3z^{-1}}$$

$$= \frac{1-3z^{-1}+1+\frac{1}{3}z^{-1}}{(1+\frac{1}{3}z^{-1})(1-3z^{-1})}$$

$$= \frac{2-\frac{8}{3}z^{-1}}{(1+\frac{1}{3}z^{-1})(1-3z^{-1})}$$

$$\boxed{H(z) = \frac{2(1-\frac{4}{3}z^{-1})}{(1+\frac{1}{3}z^{-1})(1-3z^{-1})}, \ \frac{1}{3} < |z| < 3}$$



ROC: $\frac{1}{3} < |z| < 3$

$$H(z) \cdot \frac{z^2}{z^2} = \frac{2z(z-\frac{4}{3})}{(z+\frac{1}{3})(z-3)} \ :$$

zeros: $0, 4/3$

poles: $-1/3, 3$

Problem 3, cont...

(c) **5/3 pts.** Is the system H BIBO stable? *Justify your answer.*

$\boxed{\text{YES}}$ Because the ROC of $H(z)$ includes the unit circle.

(d) **8/7 pts.** The system input is given by $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Use the z-transform to find the system output $y[n]$.

Table: $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

$$Y(z) = H(z)X(z) = \frac{2\left(1-\frac{4}{3}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-3z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad \frac{1}{2} < |z| < 3$$

$$= \underbrace{\frac{A}{1+\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{B}{1-3z^{-1}}}_{|z|<3} + \underbrace{\frac{C}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

$$A = \left.\frac{2\left(1-\frac{4}{3}\theta\right)}{(1-3\theta)\left(1-\frac{1}{2}\theta\right)}\right|_{\theta=-3} = \frac{2(1+4)}{(1+9)\left(1+\frac{3}{2}\right)} = \frac{10}{10\left(5/2\right)} = \frac{1}{5/2} = \frac{2}{5}$$

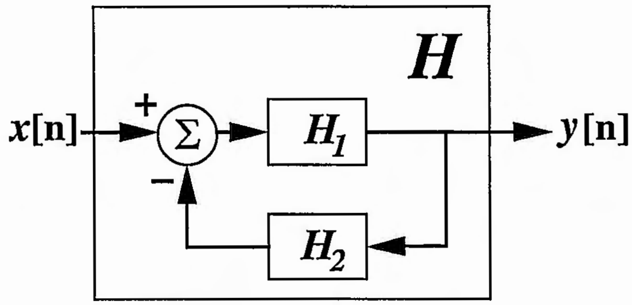
$$B = \left.\frac{2\left(1-4/3\,\theta\right)}{\left(1+\frac{1}{3}\theta\right)\left(1-\frac{1}{2}\theta\right)}\right|_{\theta=\frac{1}{3}} = \frac{2\left(1-4/9\right)}{\left(1+\frac{1}{9}\right)\left(1-\frac{1}{6}\right)} = \frac{2\left(5/9\right)}{\frac{10}{9}\cdot 5/6} = \frac{2}{10/6} = \frac{2}{5/3} = \frac{6}{5}$$

$$C = \left.\frac{2\left(1-4/3\,\theta\right)}{\left(1+\frac{1}{3}\theta\right)(1-3\theta)}\right|_{\theta=2} = \frac{2\left(1-8/3\right)}{\left(1+\frac{2}{3}\right)(1-6)} = \frac{2\left(-5/3\right)}{5/3\,(-5)} = \frac{2}{5}$$

$$Y(z) = \underbrace{\frac{2/5}{1+\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{6/5}{1-3z^{-1}}}_{|z|<3} + \underbrace{\frac{2/5}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}, \quad \frac{1}{2} < |z| < 3$$

Table: $\boxed{y[n] = \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n] - \frac{6}{5}(3)^n u[-n-1] + \frac{2}{5}\left(\frac{1}{2}\right)^n u[n]}$

4. **25/20 pts.** The **causal** discrete-time system H is formed by connecting two discrete-time LTI systems H_1 and H_2 in a negative feedback configuration as shown in the figure below.



The impulse response of H_1 is given by

$$h_1[n] = \left(\frac{1}{\sqrt{2}}\right)^n u[n].$$

The impulse response of H_2 is given by

$$h_2[n] = \left(-\frac{1}{\sqrt{2}}\right)^n u[n].$$

(a) **2/2 pts.** Find the transfer function $H_1(z)$. Be sure to specify the ROC.

Table: $H_1(z) = \dfrac{1}{1 - \frac{1}{\sqrt{2}} z^{-1}}$, $|z| > \frac{1}{\sqrt{2}}$

(b) **2/2 pts.** Find the transfer function $H_2(z)$. Be sure to specify the ROC.

Table: $H_2(z) = \dfrac{1}{1 + \frac{1}{\sqrt{2}} z^{-1}}$, $|z| > \frac{1}{\sqrt{2}}$

Problem 4, cont...

(c) **6/5 pts.** Find the transfer function $H(z)$ of the overall system H.

$$H(z) = \frac{H_1(z)}{1+H_1(z)H_2(z)} = \frac{\frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}}}{1+\frac{1}{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})}} \cdot \overbrace{\frac{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})}{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})}}^{\text{ONE}}$$

$$= \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})+1} = \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{1-\frac{1}{2}z^{-2}+1} = \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{2-\frac{1}{2}z^{-2}}$$

$$= \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{2(1-\frac{1}{4}z^{-2})} = \frac{\frac{1}{2}(1+\frac{1}{\sqrt{2}}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{2}$$

(d) **3/3 pts.** Give a pole-zero plot for $H(z)$ and specify the ROC. **Note:** it was given above that the system H is **causal**.

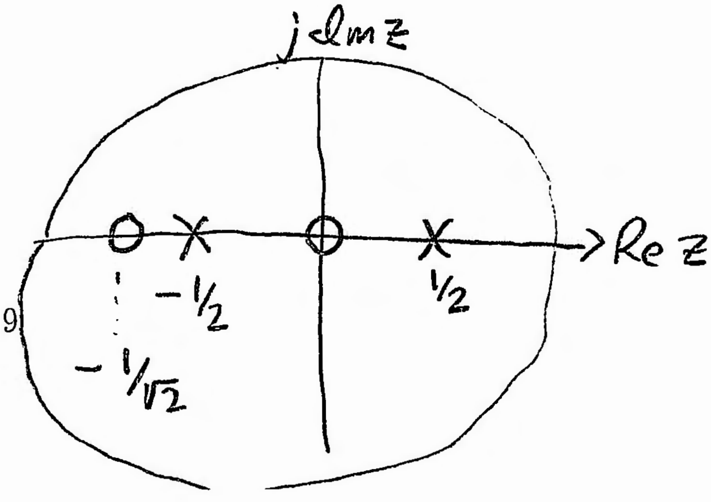
$$H(z)\cdot\frac{z^2}{z^2} = \frac{\frac{1}{2}z(z+1/\sqrt{2})}{(z-\frac{1}{2})(z+\frac{1}{2})}$$

Zeros: $0, -\frac{1}{\sqrt{2}}$

Poles: $\pm\frac{1}{2}$

→ Since H is causal, the ROC must be exterior.

$$\boxed{\text{ROC}: |z| > \frac{1}{2}}$$



Problem 4, cont...

(e) **2/2 pts.** Is the system H BIBO stable? Justify your answer.

$\boxed{\text{YES}}$ Because the ROC of $H(z)$ includes the unit circle.

Alternate answer: Yes, because H is causal and all poles are inside the unit circle.

(f) **10/6 pts.** Find the overall system impulse response $h[n]$.

$$H(z) = \frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\theta\right)}{1+\frac{1}{2}\theta}\right|_{\theta=2} = \frac{\frac{1}{2}\left(1+\frac{2}{\sqrt{2}}\right)}{1+1} = \frac{\frac{1}{2}(1+\sqrt{2})}{2} = \frac{1}{4}(1+\sqrt{2})$$

$$B = \left.\frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\theta\right)}{1-\frac{1}{2}\theta}\right|_{\theta=-2} = \frac{\frac{1}{2}\left(1-\frac{2}{\sqrt{2}}\right)}{1+1} = \frac{\frac{1}{2}(1-\sqrt{2})}{2} = \frac{1}{4}(1-\sqrt{2})$$

$$H(z) = \underbrace{\frac{\frac{1}{4}(1+\sqrt{2})}{1-\frac{1}{2}z^{-1}}}_{|z|>1/2} + \underbrace{\frac{\frac{1}{4}(1-\sqrt{2})}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}, \quad |z|>\frac{1}{2}$$

Table: $\boxed{h[n] = \frac{1}{4}(1+\sqrt{2})\left(\frac{1}{2}\right)^n u[n] + \frac{1}{4}(1-\sqrt{2})\left(-\frac{1}{2}\right)^n u[n]}$

5. **25/20 pts.** On page 3-50 of the course lecture notes we established the distributional Fourier transform pair

$$\mathcal{F}\{\operatorname{sgn} t\} = \frac{2}{j\Omega}$$

using the **fact** that, *in the sense of distributions*,

$$\int_0^\infty \sin \Omega t \, dt = \frac{1}{\Omega}.$$

However, this **fact** was not proved anywhere in the notes.

Use the Riemann-Lebesgue Lemma (RLL) to show that, in the sense of distributions,

$$\int_0^\infty \sin \Omega t \, dt = \frac{1}{\Omega}.$$

Hint:

- Write the improper integral as

$$\int_0^\infty \sin \Omega t \, dt = \lim_{A \to \infty} \int_0^A \sin \Omega t \, dt$$

- Use Euler's formula to write the integrand ($\sin \Omega t$) as a sum of two complex exponentials.
- Break the sum into two integrals. Work each integral using the fundamental theorem of calculus with the antiderivative formula $\int e^{ax} \, dx = \frac{1}{a} e^{ax}$.
- Apply the RLL (notes p. 3-27) to evaluate $\lim_{A \to \infty}$.
  - In doing this, it's important to realize that, compared to the notes, the roles of t and Ω are **reversed!** The reason is that, in the notes, the RLL is stated for a time domain signal that is a function of t. But here, we are applying the RLL to a Fourier transform that is a function of Ω. So the roles of t and Ω are **reversed** compared to the notes.

$\longrightarrow$

More Workspace for Problem 5...

$$\int_0^\infty \sin \Omega t \, dt = \lim_{A \to \infty} \int_0^A \sin \Omega t \, dt$$

$$= \lim_{A \to \infty} \int_0^A \frac{1}{2j} e^{j\Omega t} - \frac{1}{2j} e^{-j\Omega t} \, dt$$

$$= \lim_{A \to \infty} \left\{ \frac{1}{2j} \int_0^A e^{j\Omega t} \, dt - \frac{1}{2j} \int_0^A e^{-j\Omega t} \, dt \right\}$$

$$= \lim_{A \to \infty} \left\{ \frac{1}{(2j)(j\Omega)} e^{j\Omega t} \Big|_{t=0}^{A} - \frac{1}{(2j)(-j\Omega)} e^{-j\Omega t} \Big|_{t=0}^{A} \right\}$$

$$= \lim_{A \to \infty} \left\{ \frac{-1}{2\Omega} \left[e^{jA\Omega} - 1 \right] - \frac{1}{2\Omega} \left[e^{-jA\Omega} - 1 \right] \right\}$$

$$= \lim_{A \to \infty} \left\{ \frac{-1}{2\Omega} \left[e^{jA\Omega} - 1 \right] - \frac{1}{2\Omega} \left[\cos A\Omega - j \sin A\Omega - 1 \right] \right\}$$

$$= \frac{-1}{2\Omega} \Big[\underbrace{\lim_{A \to \infty} e^{jA\Omega}}_{\text{zero by RLL}} - 1 + \underbrace{\lim_{A \to \infty} \cos A\Omega}_{\text{zero by RLL}} - \underbrace{j \lim_{A \to \infty} \sin A\Omega}_{\text{zero by RLL}} - 1 \Big]$$

$$= -\frac{1}{2\Omega} \left[-1 - 1 \right]$$

$$= \frac{-1}{2\Omega}(-2) = \frac{1}{\Omega} \checkmark$$

QED