ECE 4213/5213
Test 1

Monday, October 19 Wednesday, October 21, 2020

Fall 2020 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet Other materials are not allowed. You have 48 hours to complete the test. All work must be your own. You may work the test on this test paper or you may use your own blank paper. Upload a scan or photograph of your test paper to the course Canvas page no later than midnight on Wednesday, October 21, 2020.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed: ______ Date: ______

1. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system H are related by

$$y[n] = x[n] + nx[n-1] + 1.$$

(a) **8/6 pts.** Is the system H linear? *Justify your answer.*

Let $x_1[n] = x_2[n] = \delta[n+1]$ and let $a = b = 1$.

Then $y_1[n] = H\{x_1[n]\} = \delta[n+1] + \underbrace{n\delta[n]}_{0} + 1 = \delta[n+1] + 1 = y_2[n]$.

Then $ay_1[n] + by_2[n] = y_1[n] + y_2[n] = 2y_1[n] = 2\delta[n+1] + 2$.

Now let $x_3[n] = ax_1[n] + bx_2[n] = x_1[n] + x_2[n] = 2\delta[n+1]$.

Then $y_3[n] = H\{x_3[n]\} = x_3[n] + nx_3[n-1] + 1$

$= 2\delta[n+1] + \underbrace{2n\delta[n]}_{0} + 1 = 2\delta[n+1] + 1 \neq ay_1[n] + by_2[n]$.

Since $y_3[n] \neq ay_1[n] + by_2[n]$, the system H is

NOT LINEAR

(b) **8/6 pts.** Is the system H time invariant? *Justify your answer.*

Let $x_1[n] = \delta[n+1]$.

Then $y_1[n] = \delta[n+1] + 1$, as in part (a).

Then $y_1[n-1] = \delta[n] + 1$.

Now let $x_2[n] = x_1[n-1] = \delta[n]$.

Then $y_2[n] = H\{x_2[n]\} = x_2[n] + nx_2[n-1] + 1 = \delta[n] + n\delta[n-1] + 1$

$= \delta[n] + \delta[n-1] + 1 \neq y_1[n-1]$.

$\Rightarrow$ Since $y_2[n] \neq y_1[n-1]$, the system H is

NOT TIME INVARIANT

Problem 1, cont...

(c) **9/8 pts.** Is the system H BIBO stable? *Justify your answer.*

Let $x[n] = u[n]$. Then $x[n]$ is a bounded input signal, since $|x[n]| \leq 1 \ \forall n \in \mathbb{Z}$.

Now suppose that $y[n] = H\{x[n]\}$ is also bounded.

Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|y[n]| \leq B \ \forall n \in \mathbb{Z}$.

When $n=0$, we have

$$|y[0]| = |u[0] + \underbrace{0u[-1]}_{0} + 1| = |1+1| = 2.$$

$\rightarrow$ Therefore, we know that $B \geq 2$.

Let $n_0 = \lceil B \rceil$. Then $n_0 \in \mathbb{Z}$, $n_0 \geq 2$, and $n_0 \geq B$.

But, when $n = n_0$ we have

$$\begin{aligned} |y[n_0]| &= |u[n_0] + n_0 \underbrace{u[n_0 - 1]}_{=1 \text{ b/c } n_0 \geq 2} + 1| \\ &= |1 + n_0 + 1| = |2 + n_0| \\ &= 2 + n_0 \quad (\text{because } n_0 \geq 2 > 0) \\ &\geq 2 + B \quad (\text{because } n_0 \geq B) \\ &> B \ /// . \end{aligned}$$

This contradicts the hypothesis that $y[n]$ is bounded by B. Therefore, no such B can exist and $y[n]$ is unbounded. Since a bounded input signal produced an unbounded output signal, the system H is NOT STABLE.

2. **25/20 pts**. The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by

$$y[n] = -x[n] + 2x[n-2] - x[n-4].$$

(a) **6/5 pts**. Find the system frequency response $H(e^{j\omega})$.

DTFT: $Y(e^{j\omega}) = -X(e^{j\omega}) + 2e^{-j2\omega}X(e^{j\omega}) - e^{-j4\omega}X(e^{j\omega})$
$= X(e^{j\omega})\left[-1 + 2e^{-j2\omega} - e^{-j4\omega}\right]$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = -1 + 2e^{-j2\omega} - e^{-j4\omega}}$$

(b) **7/5 pts**. Find the system magnitude response $A(\omega)$ and phase response $\theta(\omega)$ such that $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$

Use the linear phase "half power" trick (notes p. 3.126):

$$\begin{aligned}
H(e^{j\omega}) &= -1 + 2e^{-j2\omega} - e^{-j4\omega} \\
&= \left[-e^{j2\omega} + 2 - e^{-j2\omega}\right]e^{-j2\omega} \\
&= \left[2 - 2\cos(2\omega)\right]e^{-j2\omega} \\
&= A(\omega)e^{j\theta(\omega)}
\end{aligned}$$

$$\boxed{\begin{aligned} A(\omega) &= 2 - 2\cos(2\omega) \\ \theta(\omega) &= -2\omega \end{aligned}}$$

Problem 2, cont...

(c) **6/5 pts.** Does the system H have linear phase? *(Justify your answer).*

YES, because $\theta(\omega) = -2\omega$ is a linear function of ω.

(d) **6/5 pts.** Find the system group delay $\tau(\omega)$.

$$\tau(\omega) = -\frac{d}{d\omega}\theta(\omega)$$

$$= -\frac{d}{d\omega}\{-2\omega\}$$

$$= 2$$

3. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a **causal** discrete-time LTI system H are related by

$$y[n] + \frac{3}{2}y[n-1] - y[n-2] = 2x[n] + \frac{3}{2}x[n-1].$$

(a) **7/5 pts.** Find the transfer function $H(z)$, specify the ROC, and give a pole-zero plot.

$\mathcal{Z}$: $Y(z) + \frac{3}{2}z^{-1}Y(z) - z^{-2}Y(z) = 2X(z) + \frac{3}{2}z^{-1}X(z)$

$Y(z)\left[1 + \frac{3}{2}z^{-1} - z^{-2}\right] = X(z)\left[2 + \frac{3}{2}z^{-1}\right]$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{3}{2}z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}} = \frac{2\left(1 + \frac{3}{4}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1}\right)}}$$

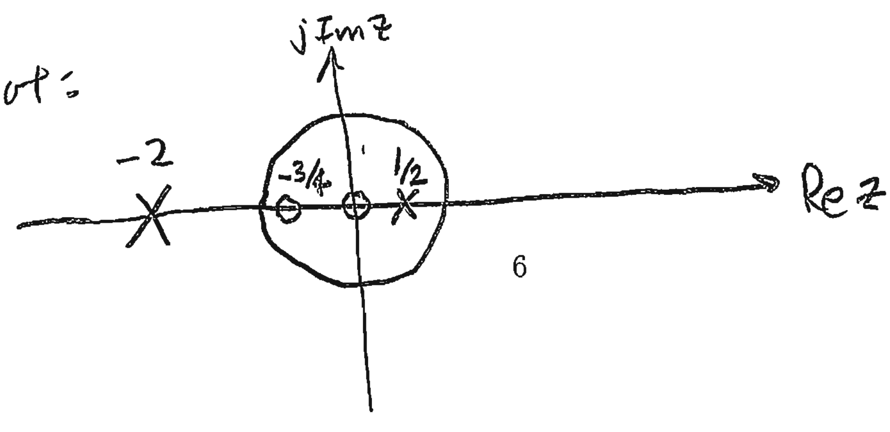
NOTE: $H(z)\cdot\frac{z^2}{z^2} = \frac{2z\left(z + \frac{3}{4}\right)}{\left(z - \frac{1}{2}\right)(z+2)}$ zeros: $z = 0, -\frac{3}{4}$

poles: $z = \frac{1}{2}, -2$

Since we are given that the system H is <u>causal</u>, the ROC of $H(z)$ must be the exterior of a circle that passes through the largest magnitude pole ($z = -2$).

$$\boxed{\text{ROC: } |z| > 2}$$

p/z-plot:



Problem 3, cont...

(b) **6/5 pts.** Is the system H BIBO stable? *Justify your answer.*

NO, because the ROC of $H(z)$ does not contain the unit circle of the z-plane.

(c) **6/5 pts.** Find the impulse response $h[n]$.

$$H(z) = \frac{2(1+\frac{3}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+2z^{-1}}$$

$$A = \left.\frac{2(1+\frac{3}{4}\theta)}{1+2\theta}\right|_{\theta=2} = \frac{2(1+\frac{3}{2})}{1+4} = \frac{2(5/2)}{5} = \frac{5}{5} = 1$$

$$B = \left.\frac{2(1+\frac{3}{4}\theta)}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{2}} = \frac{2(1+\frac{3}{4}(-\frac{1}{2}))}{1-\frac{1}{2}(-\frac{1}{2})} = \frac{2(1-\frac{3}{8})}{1+\frac{1}{4}}$$

$$= \frac{2(5/8)}{5/4} = \frac{5/4}{5/4} = 1$$

$$H(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{1}{1+2z^{-1}}}_{|z|>2}$$

Table: $\boxed{h[n] = (\tfrac{1}{2})^n u[n] + (-2)^n u[n]}$

Problem 3, cont...

(d) **6/5 pts.** The system input is given by $x[n] = \frac{1}{2}\delta[n] + \delta[n-1]$. Use the z-transform to find the system output $y[n]$.

Table: $X(z) = \frac{1}{2} + z^{-1}, \quad |z| > 0.$

$$Y(z) = X(z)H(z) = \left[\frac{1}{2} + z^{-1}\right] \cdot \frac{2(1+\frac{3}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})}$$

$$= \frac{2(\frac{1}{2}+z^{-1})(1+\frac{3}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})} = \frac{\cancel{(1+2z^{-1})}(1+\frac{3}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})\cancel{(1+2z^{-1})}}$$

$$= \frac{1+\frac{3}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \left[\frac{3/4}{1-\frac{1}{2}z^{-1}}\right]z^{-1}, \quad |z| > \frac{1}{2}$$

Table: $\frac{1}{1-\frac{1}{2}z^{-1}}, \ |z| > \frac{1}{2} \longleftrightarrow \left(\frac{1}{2}\right)^n u[n]$

Time-Shift plus linearity: $\left[\frac{3/4}{1-\frac{1}{2}z^{-1}}\right]z^{-1}, \ |z| > \frac{1}{2} \longleftrightarrow \frac{3}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]$

All Together: $\boxed{y[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{3}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]}$

4. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a causal discrete-time LTI system H are related by

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1].$$

The input signal is given by

$$x[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$$

and it is known that $y[-1] = 2$.

Use the unilateral z-transform to find $y[n]$ for $n \geq 0$.

Hint: when you apply the unilateral z-transform time-shift property to the term $y[n-1]$ on the left side of the difference equation, you will get a $y[-1]$ in the z-domain equation; that's fine, since $y[-1]$ is given. Similarly, the term $x[n-1]$ on the right side of the difference equation will give you an $x[-1]$ in the z-domain. But $x[-1]$ is not given. However, if you plug in $n = -1$ to the given expression for $x[n]$ it will show you that $x[-1] = 0$.

Table: $$\mathcal{X}_u(z) = \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} = \frac{3(1-\frac{1}{3}z^{-1}) - 2(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$= \frac{3 - z^{-1} - 2 + z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} \qquad (*)$$

$\mathcal{U}z$: $$\mathcal{Y}_u(z) - \frac{1}{2}\{z^{-1}\mathcal{Y}_u(z) + \underbrace{y[-1]}_{2}\} = \mathcal{X}_u(z) - \frac{1}{2}\{z^{-1}\mathcal{X}_u(z) + \underbrace{x[-1]}_{0}\}$$

$$\mathcal{Y}_u(z) - \frac{1}{2}z^{-1}\mathcal{Y}_u(z) - 1 = \mathcal{X}_u(z) - \frac{1}{2}z^{-1}\mathcal{X}_u(z)$$

$$\mathcal{Y}_u(z)\left[1 - \frac{1}{2}z^{-1}\right] = \mathcal{X}_u(z)\left[1 - \frac{1}{2}z^{-1}\right] + 1$$

$$\mathcal{Y}_u(z) = \frac{\mathcal{X}_u(z)\left[1-\frac{1}{2}z^{-1}\right] + 1}{1 - \frac{1}{2}z^{-1}} \qquad (**)$$

$\longrightarrow$

plug $(*) \to (**)$:

$$Y_u(z) = \frac{\frac{\cancel{1-\frac{1}{2}z^{-1}}}{\cancel{(1-\frac{1}{2}z^{-1})}(1-\frac{1}{3}z^{-1})} + 1}{1-\frac{1}{2}z^{-1}} = \frac{\frac{1}{1-\frac{1}{3}z^{-1}} + 1}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{\frac{1+1-\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}}{1-\frac{1}{2}z^{-1}} = \frac{\frac{2-\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}}{1-\frac{1}{2}z^{-1}}$$

$$Y_u(z) = \frac{2-\frac{1}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

$$A = \left.\frac{2-\frac{1}{3}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{2-2/3}{1-2/3} = \frac{4/3}{1/3} = 4$$

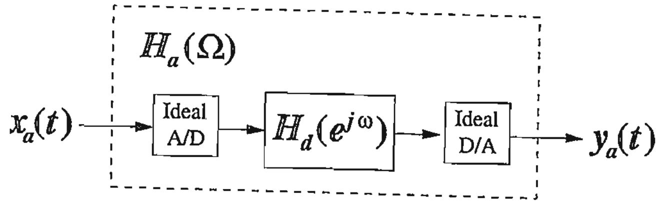
$$B = \left.\frac{2-\frac{1}{3}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=3} = \frac{2-1}{1-3/2} = \frac{1}{-1/2} = -2$$

$$Y_u(z) = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$$

Table:

$$\boxed{y[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{3}\right)^n u[n], \quad n \geq 0}$$

5. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



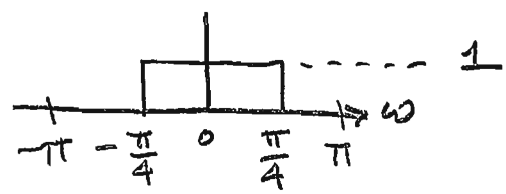
The sampling frequency of the A/D and D/A converters is $F_T = 2$ kHz, so $\Omega_T = 4,000\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this prevents any aliasing and ensures that H_a will be an LTI system).

The impulse response of the discrete-time filter H_d is given by

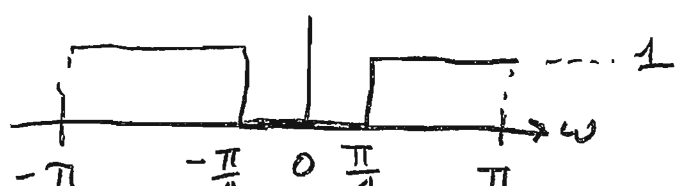
$$h_d[n] = \delta[n] - \frac{\sin \frac{\pi}{4} n}{\pi n}.$$

(a) **8/7 pts.** Find the discrete-time frequency response $H_d(e^{j\omega})$.

Table: $H_d(e^{j\omega}) = 1 -$



$H_d(e^{j\omega}) =$



Problem 5, cont ..

(b) **9/7 pts** Find the continuous-time frequency response $H_a(\Omega)$.

$\Omega_T = 4{,}000\pi$

$T = \frac{2\pi}{\Omega_T} = \frac{2\pi}{4{,}000\pi} = \frac{1}{2000}$

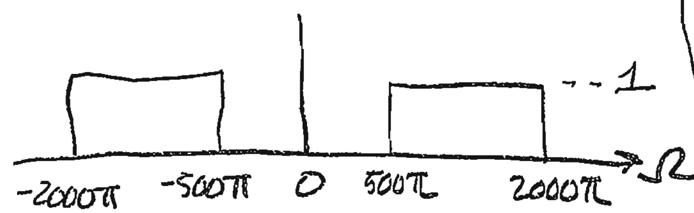
$\Omega = \frac{\omega}{T} = 2000\omega$

when $\omega = \frac{\pi}{4}$, $\Omega = 500\pi$

when $\omega = \pi$, $\Omega = 2000\pi$

$$H_a(\Omega) = \begin{cases} H_d(e^{j\Omega T}) & , |\Omega| < \frac{\Omega_T}{2} = 2000\pi \\ 0 & , |\Omega| > \frac{\Omega_T}{2} = 2000\pi \end{cases}$$

$H_a(\Omega) =$



(c) **8/6 pts.** Find the continuous-time impulse response $h_a(t)$.

Solution I: Assume that the antialiasing filter zeros out all analog frequencies $\Omega > 2000\pi$:

$H_a(\Omega) = 1 -$

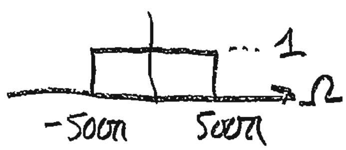
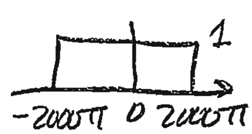


Table:

$$\boxed{h_a(t) = \delta(t) - \frac{\sin 500\pi t}{\pi t}}$$

Solution II: Assume there is no antialiasing filter. $H_a(\Omega)$ must explicitly zero out analog frequencies $\Omega > 2000\pi$.

$H_a(\Omega) =$



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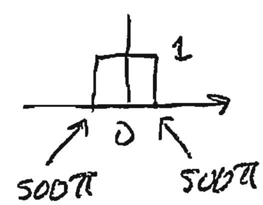

Table:

$$\boxed{h_a(t) = \frac{\sin 2000\pi}{\pi t} - \frac{\sin 500\pi t}{\pi t}}$$

$\Rightarrow$ Either solution is acceptable.