ECE 4213/5213
Test 1

Monday, October 25, 2021
4:30 PM - 5:45 PM

Fall 2021 — Name: SOLUTION

Dr. Havlicek — Student Num: __________

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed: __________ Date: __________

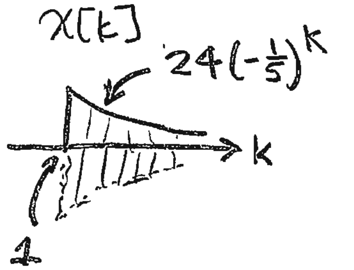
1. **25/20 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

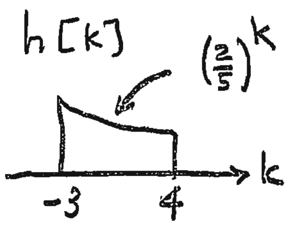
$$h[n] = \begin{cases} \left(\frac{2}{5}\right)^n, & -3 \le n \le 4, \\ 0, & \text{otherwise} \end{cases} \quad = \quad \left(\frac{2}{5}\right)^n \{u[n+3] - u[n-5]\}.$$

The system input is given by

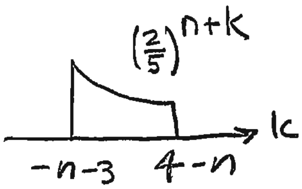
$$x[n] = 24\left(-\frac{1}{5}\right)^n u[n-1].$$

Use time domain convolution to find the system output $y[n]$.

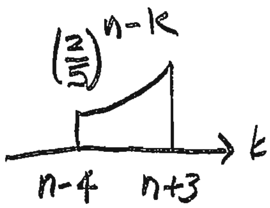




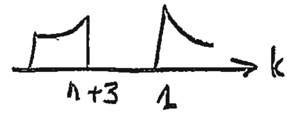
$h[k--n] = h[n+k]$



$h[-k--n] = h[n-k]$



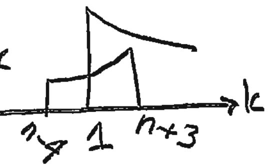
Case I) $n+3 < 1$: , $n < -2$



$y[n] = 0$

Case II) $n+3 \geq 1$ and $n-4 < 1$: $n \geq -2$ and $n < 5$

$-2 \le n < 5$



$$y[n] = \sum_{k=1}^{n+3} 24\left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{n-k}$$

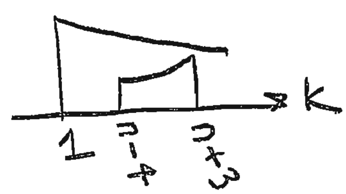
$$= 24\left(\tfrac{2}{5}\right)^n \sum_{k=1}^{n+3} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{-k} = 24\left(\tfrac{2}{5}\right)^n \sum_{k=1}^{n+3} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{5}{2}\right)^k$$

$$= 24\left(\tfrac{2}{5}\right)^n \sum_{k=1}^{n+3} \left(-\tfrac{1}{2}\right)^k = 24\left(\tfrac{2}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^1 - \left(-\frac{1}{2}\right)^{n+4}}{1 - \left(-\frac{1}{2}\right)}$$

$$= 24\left(\tfrac{2}{5}\right)^n \frac{-\frac{1}{2} - \left(\frac{1}{16}\right)\left(-\frac{1}{2}\right)^n}{3/2} = \left(\tfrac{2}{3}\right) 24 \left(\tfrac{2}{5}\right)^n \left[-\tfrac{1}{2} - \left(\tfrac{1}{16}\right)\left(-\tfrac{1}{2}\right)^n\right]$$

$$= 16\left[-\tfrac{1}{2}\left(\tfrac{2}{5}\right)^n - \left(\tfrac{1}{16}\right)\left(\tfrac{2}{5}\right)^n\left(-\tfrac{1}{2}\right)^n\right] = -8\left(\tfrac{2}{5}\right)^n - \left(-\tfrac{1}{5}\right)^n$$

Case III) $n-4 \geq 1$: $n \geq 5$



$$y[n] = \sum_{k=n-4}^{n+3} 24\left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{n-k}$$

$$= 24\left(\tfrac{2}{5}\right)^n \sum_{k=n-4}^{n+3} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{5}{2}\right)^k = 24\left(\tfrac{2}{5}\right)^n \sum_{k=n-4}^{n+3} \left(-\tfrac{1}{2}\right)^k$$

$$= 24\left(\tfrac{2}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^{n-4} - \left(-\frac{1}{2}\right)^{n+4}}{1 - \left(-\frac{1}{2}\right)} = 24\left(\tfrac{2}{5}\right)^n \frac{(-2)^4\left(-\frac{1}{2}\right)^n - \left(-\frac{1}{2}\right)^4\left(-\frac{1}{2}\right)^n}{3/2}$$

$$= \left(\tfrac{2}{3}\right) 24\left(\tfrac{2}{5}\right)^n \left[16\left(-\tfrac{1}{2}\right)^n - \tfrac{1}{16}\left(-\tfrac{1}{2}\right)^n\right] = 16^2\left(\tfrac{2}{5}\right)^n\left(-\tfrac{1}{2}\right)^n - \left(\tfrac{2}{5}\right)^n\left(-\tfrac{1}{2}\right)^n$$

$$= 16^2\left(-\tfrac{1}{5}\right)^n - \left(-\tfrac{1}{5}\right)^n$$

$$= 255\left(-\tfrac{1}{5}\right)^n$$

All Together:

$$\boxed{y[n] = \begin{cases} 0, & n < -2 \\ -8(2/5)^n - \left(-\frac{1}{5}\right)^n, & -2 \le n < 5 \\ 255\left(-\frac{1}{5}\right)^n, & n \geq 5 \end{cases}}$$

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OTHER WAY

1. **25/20 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

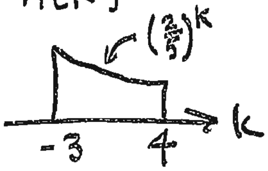
$$h[n] = \begin{cases} \left(\frac{2}{5}\right)^n, & -3 \le n \le 4, \\ 0, & \text{otherwise} \end{cases} = \left(\frac{2}{5}\right)^n \{u[n+3] - u[n-5]\}.$$

The system input is given by

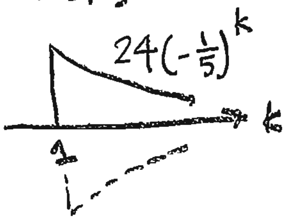
$$x[n] = 24\left(-\frac{1}{5}\right)^n u[n-1].$$

Use time domain convolution to find the system output $y[n]$.

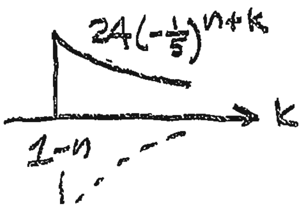
$h[k]$



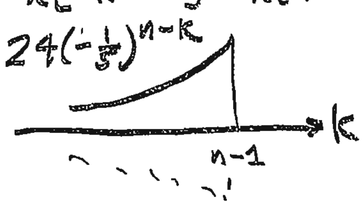
$x[k]$



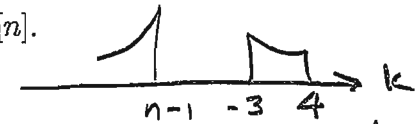
$x[k--n] = x[k+n]$



$x[-k--n] = x[n-k]$

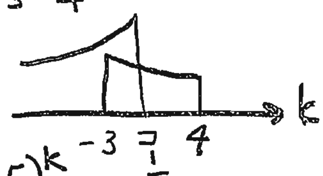


case I) $n-1 < -3 : \quad n < -2$



$y[n] = 0$

Case II) $n-1 \geq -3$ and $n-1 < 4 : \quad -2 \le n < 5$



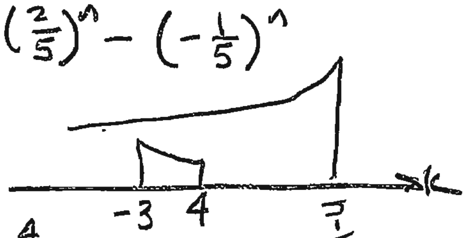
$$y[n] = \sum_{k=-3}^{n-1} \left(\tfrac{2}{5}\right)^k 24\left(-\tfrac{1}{5}\right)^{n-k} = 24\left(-\tfrac{1}{5}\right)^n \sum_{k=-3}^{n-1} \left(\tfrac{2}{5}\right)^k (-5)^k$$

$$= 24\left(-\tfrac{1}{5}\right)^n \sum_{k=-3}^{n-1} (-2)^k = 24\left(-\tfrac{1}{5}\right)^n \frac{(-2)^{-3} - (-2)^n}{1-(-2)}$$

$$= 24\left(-\tfrac{1}{5}\right)^n \frac{-\frac{1}{8} - (-2)^n}{3} = \left(\tfrac{1}{3}\right) 24\left(-\tfrac{1}{5}\right)^n \left[-\tfrac{1}{8} - (-2)^n\right]$$

$$= -\left(-\tfrac{1}{5}\right)^n - 8\left(-\tfrac{1}{5} \cdot -2\right)^n = -8\left(\tfrac{2}{5}\right)^n - \left(-\tfrac{1}{5}\right)^n$$

Case III) $n-1 > 4 : \quad n > 5$



$$y[n] = \sum_{k=-3}^{4} 24\left(-\tfrac{1}{5}\right)^{n-k} \left(\tfrac{2}{5}\right)^k$$

$$= 24\left(-\tfrac{1}{5}\right)^n \sum_{k=-3}^{4} (-5)^k \left(\tfrac{2}{5}\right)^k = 24\left(-\tfrac{1}{5}\right)^n \sum_{k=-3}^{4} (-2)^k$$

$$= 24\left(-\tfrac{1}{5}\right)^n \frac{(-2)^{-3} - (-2)^5}{1-(-2)} = 24\left(-\tfrac{1}{5}\right)^n \frac{-\frac{1}{8} - (-2)^5}{3}$$

$$= \left(\tfrac{1}{3}\right) 24\left(-\tfrac{1}{5}\right)^n \left[-\tfrac{1}{8} + 32\right] = \left[-1 + 8 \cdot 32\right]\left(-\tfrac{1}{5}\right)^n = 255\left(-\tfrac{1}{5}\right)^n$$

All Together:

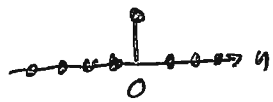
$$y[n] = \begin{cases} 0, & n < -2 \\ -8\left(\frac{2}{5}\right)^n - \left(-\frac{1}{5}\right)^n, & -2 \le n < 5 \\ 255\left(-\frac{1}{5}\right)^n, & n \geq 5 \end{cases}$$

2. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system are related by

$$y[n] = \begin{cases} nx[n+1], & n \geq 0, \\ x[n], & n < 0. \end{cases}$$

(a) **12/10 pts.** Is the system time invariant? Justify your answer.

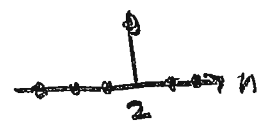
I guess "no" because of "n" multiplier for $n \geq 0$ and because the input $x[0]$ never makes it to the output signal $y[n]$. So I will try to construct a counterexample using these observations.

Let $x_1[n] = \delta[n]$ 

Then $y_1[n] = \begin{cases} n x_1[n+1], & n \geq 0 \\ x[n], & n < 0 \end{cases} = \begin{cases} n\delta[n+1], & n \geq 0 \\ \delta[n], & n < 0 \end{cases} = 0 \quad \forall n.$

Let $n_0 = 2$

Then $y_1[n-n_0] = y_1[n-2] = 0 \quad \forall n.$

Now let $x_2[n] = x_1[n-n_0] = x_1[n-2] = \delta[n-2]$ 

Then $y_2[n] = \begin{cases} n x_2[n+1], & n \geq 0 \\ x_2[n], & n < 0 \end{cases} = \begin{cases} n\delta[n-1], & n \geq 0 \\ \delta[n-2], & n < 0 \end{cases}$

– but $n\delta[n-1] = \delta[n-1]$ since it's only "on" at $n=1$
and $\delta[n-2] = 0$ for $n<0$.

So $y_2[n] = \begin{cases} \delta[n-1], & n \geq 0 \\ 0, & n < 0 \end{cases} = \delta[n-1] \neq y_1[n-n_0]$

$$\boxed{\text{NOT TIME INVARIANT}}$$

Problem 2, cont...

(b) **13/10 pts.** Is the system BIBO stable? Justify your answer.

I guess "no" because of the "n" multiplier for $n \geq 0$. So I will try to construct a bounded input signal that leads to an unbounded output signal. $u[n]$ should do the trick since it's bounded and also turned on where the "n" multiplier is in effect.

Let $x[n] = u[n]$. Then $x[n]$ is a bounded input signal because $|u[n]| \leq 1 \ \forall n \in \mathbb{Z}$.

$$\text{Then } y[n] = \begin{cases} n x[n+1], & n \geq 0 \\ x[n], & n < 0 \end{cases} = \begin{cases} n u[n+1], & n \geq 0 \\ u[n], & n < 0 \end{cases}$$

$$= \begin{cases} n u[n], & n \geq 0 \\ 0, & n < 0 \end{cases} = n u[n].$$

Now suppose $y[n]$ is bounded.
Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|y[n]| \leq B \ \forall n \in \mathbb{Z}$.

Let $n_0 = \lceil B \rceil + 1$. Note that this implies $n_0 \geq 2$.

We have $|y[n_0]| = |n_0 u[n_0]| = |n_0| = n_0 = \lceil B \rceil + 1 > B$.
This contradicts the hypothesis that $y[n]$ is bounded by B.
Therefore, no such B can exist and $y[n]$ is unbounded.

Since a bounded input signal produced an unbounded output signal, the system is

$$\boxed{\text{UNSTABLE}}$$

3. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by

$$y[n] = x[n] + 2x[n-1] - 2x[n-2] - x[n-3].$$

(a) **8/6 pts.** Find the system frequency response $H(e^{j\omega})$.

DTFT: $Y(e^{j\omega}) = X(e^{j\omega})\left[1 + 2e^{-j\omega} - 2e^{-j2\omega} - e^{-j3\omega}\right]$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + 2e^{-j\omega} - 2e^{-j2\omega} - e^{-j3\omega}}$$

(b) **9/7 pts.** Find functions $A(\omega)$ and $\theta(\omega)$ such that $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$.

using the "half-power" trick from Notes p. 3.126,

$$H(e^{j\omega}) = \left[e^{j\frac{3}{2}\omega} + 2e^{j\frac{1}{2}\omega} - 2e^{-j\frac{1}{2}\omega} - e^{-j\frac{3}{2}\omega}\right]e^{-j\frac{3}{2}\omega}$$

$(j = e^{j\frac{\pi}{2}})$ $$= 2j\left[\frac{e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}}{2j} + 2\,\frac{e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}}{2j}\right]e^{-j\frac{3}{2}\omega}$$

$$= 2\left[\sin\left(\tfrac{3}{2}\omega\right) + 2\sin\left(\tfrac{1}{2}\omega\right)\right]e^{j\frac{\pi}{2}}e^{-j\frac{3}{2}\omega}$$

$$= \left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right]e^{-j\left(\frac{3}{2}\omega - \frac{\pi}{2}\right)}$$

$$\boxed{\begin{aligned} A(\omega) &= 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) \\ \theta(\omega) &= -\tfrac{3}{2}\omega + \tfrac{\pi}{2} \end{aligned}}$$

Problem 3, cont...

(c) **8/7 pts**. Find the system group delay $\tau(\omega)$.

$$\tau(\omega) = -\frac{d}{d\omega}\theta(\omega)$$

$$= -\frac{d}{d\omega}\left[-\frac{3}{2}\omega + \frac{\pi}{2}\right]$$

$$\boxed{\tau(\omega) = +\frac{3}{2}}$$

4. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a **stable** discrete-time LTI system H are related by

$$y[n] - \frac{8}{3}y[n-1] - y[n-2] = x[n] - \frac{1}{3}x[n-1].$$

(a) **12/10 pts.** Find the transfer function $H(z)$, specify the ROC, and give a pole-zero plot.

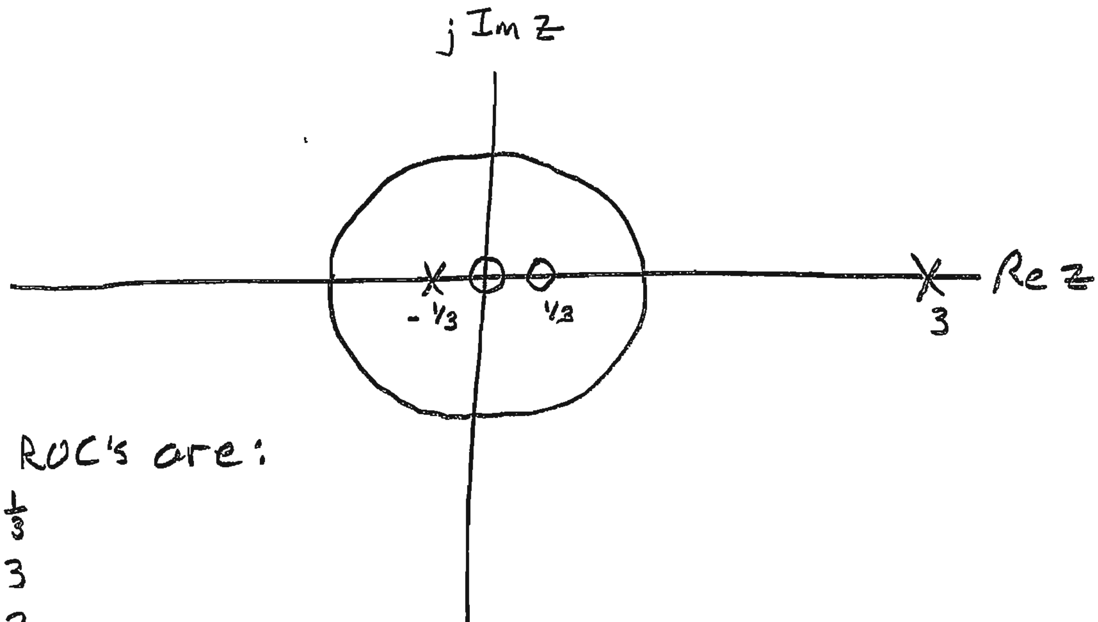
$\mathcal{Z}$: $Y(z)\left[1 - \frac{8}{3}z^{-1} - z^{-2}\right] = X(z)\left[1 - \frac{1}{3}z^{-1}\right]$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{8}{3}z^{-1} - z^{-2}} = \underline{\underline{\frac{1 - \frac{1}{3}z^{-1}}{(1+\frac{1}{3}z^{-1})(1-3z^{-1})}}}$$

$$H(z)\frac{z^2}{z^2} = \frac{z(z-\frac{1}{3})}{(z+\frac{1}{3})(z-3)}$$

zeros @ $z = 0, \frac{1}{3}$
poles @ $z = -\frac{1}{3}, 3$

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{z^2}{z^2} = \lim_{z\to\infty} 1 = 1 \Rightarrow$$ no additional pole or zero @ $z = \infty$.



– The possible ROC's are:
$|z| < \frac{1}{3}$
$\frac{1}{3} < |z| < 3$
$|z| > 3$

– Since the system is given to be stable, the ROC must include the unit circle.

$$\boxed{\text{ROC: } \frac{1}{3} < |z| < 3}$$

Problem 4, cont...

(b) **13/10 pts.** Find the impulse response $h[n]$.

$$H(\theta) = \frac{1-\frac{1}{3}\theta}{(1+\frac{1}{3}\theta)(1-3\theta)} = \frac{A}{1+\frac{1}{3}\theta} + \frac{B}{1-3\theta}$$

$$A = \left.\frac{1-\frac{1}{3}\theta}{1-3\theta}\right|_{\theta=-3} = \frac{1+1}{1+9} = \frac{2}{10} = \frac{1}{5}$$

$$B = \left.\frac{1-\frac{1}{3}\theta}{1+\frac{1}{3}\theta}\right|_{\theta=\frac{1}{3}} = \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{8/9}{10/9} = \frac{8}{10} = \frac{4}{5}$$

$$H(z) = \underbrace{\frac{1/5}{1+\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{4/5}{1-3z^{-1}}}_{|z|<3}$$

Table: $\boxed{h[n] = \frac{1}{5}\left(-\frac{1}{3}\right)^n u[n] - \frac{4}{5}(3)^n u[-n-1]}$

5. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a causal discrete-time LTI system H are related by

$$y[n] + \frac{1}{2}y[n-1] = x[n] + x[n-1].$$

The input signal is given by

$$x[n] = 4\left(\frac{1}{2}\right)^n u[n]$$

and initial conditions are given as $y[-1] = 1$ and $x[-1] = \frac{9}{2}$.

Use the unilateral z-transform to find $y[n]$ for $n \geq 0$.

Since $x[n]=0 \ \forall n<0$, we can use the regular bilateral z-transform table to look up $\mathcal{X}_u(z)$... since it's the same as $X(z)$: $\mathcal{X}_u(z) = \dfrac{4}{1-\frac{1}{2}z^{-1}}$

Applying the UZ-transform to both sides of the difference equation, we get

$$\mathcal{Y}_u(z) + \tfrac{1}{2}\Big[z^{-1}\mathcal{Y}_u(z) + \overbrace{y[-1]}^{1}\Big] = \mathcal{X}_u(z) + \Big[z^{-1}\mathcal{X}_u(z) + \overbrace{x[-1]}^{9/2}\Big]$$

$$\mathcal{Y}_u(z)\left(1+\tfrac{1}{2}z^{-1}\right) + \tfrac{1}{2} = \mathcal{X}_u(z)\left(1+z^{-1}\right) + 9/2$$

$$\mathcal{Y}_u(z)\left(1+\tfrac{1}{2}z^{-1}\right) = \mathcal{X}_u(z)\left(1+z^{-1}\right) + 4$$

$$= \frac{4(1+z^{-1})}{(1-\frac{1}{2}z^{-1})} + 4 = \frac{4(1+z^{-1}) + 4(1-\frac{1}{2}z^{-1})}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{4+4z^{-1}+4-2z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{8+2z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\mathcal{Y}_u(z) = \frac{8+2z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}$$

More Workspace for Problem 5...

$$\mathcal{Y}_u(\theta) = \frac{8+2\theta}{(1-\frac{1}{2}\theta)(1+\frac{1}{2}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1+\frac{1}{2}\theta}$$

$$A = \left.\frac{8+2\theta}{1+\frac{1}{2}\theta}\right|_{\theta=2} = \frac{8+4}{1+1} = \frac{12}{2} = 6$$

$$B = \left.\frac{8+2\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-2} = \frac{8-4}{1+1} = \frac{4}{2} = 2$$

$$\mathcal{Y}_u(z) = \underbrace{\frac{6}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{2}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

Table: $\boxed{y[n] = 6\left(\tfrac{1}{2}\right)^n u[n] + 2\left(-\tfrac{1}{2}\right)^n u[n], \; n \geq 0}$