ECE 4213/5213
Test 1

Wednesday, October 26, 2022
4:30 PM - 5:45 PM

Fall 2022 Name: SOLUTION

Dr. Havlicek Student Num: ____________

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed: ____________ Date: ____________

1. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system are related by

$$y[n] = x[2n] + 5.$$

(a) **8/7 pts.** Is the system linear? Justify your answer.

When the input is $x_1[n]$, the output is $y_1[n] = x_1[2n] + 5$
When the input is $x_2[n]$, the output is $y_2[n] = x_2[2n] + 5$
Let $a, b \in \mathbb{C}$ be constants, such that $a+b \neq 1$.
Let $x_3[n] = a x_1[n] + b x_2[n]$.

Then $y_3[n] = x_3[2n] + 5 = a x_1[n] + b x_2[n] + 5$.

But $a y_1[n] + b y_2[n] = a x_1[n] + b x_2[n] + (a+b)5$.

$\rightarrow$ since $(a+b) \neq 1$ by hypothesis,

we have $y_3[n] \neq a x_1[n] + b x_2[n]$.

Therefore, the system is <u>NOT LINEAR</u>.

(b) **8/6 pts.** Is the system causal? Justify your answer.

Let $n = 1$.

Then $y[1] = x[2] + 5$.

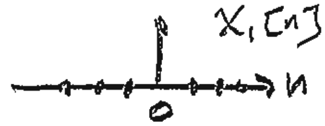
Since the value of the output signal at time $n=1$ depends on the <u>future</u> value $x[2]$ of the input signal,

the system is <u>NOT CAUSAL</u>.

Problem 1, cont…

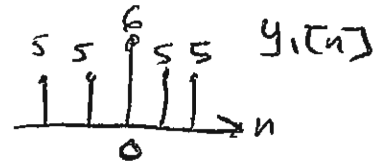
(c) **9/7 pts.** Is the system time invariant? Justify your answer.

Let $x_1[n] = \delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{other} \end{cases}$



Then $y_1[n] = x_1[2n]+5$

$= \delta[2n]+5 = \begin{cases} 6, & n=0 \\ 5, & \text{other} \end{cases}$

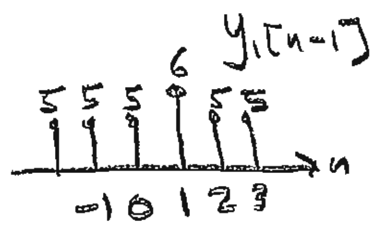


Let $n_0 = 1$

Then $y_1[n-n_0] = y_1[n-1]$

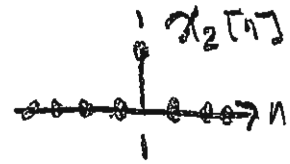
$= x_1[2(n-1)] + 5$

$= \delta[2n-2]+5 = \begin{cases} 6, & n=1 \\ 5, & \text{other} \end{cases}$



Now let $x_2[n] = x_1[n-n_0] = x_1[n-1]$

$= \delta[n-1] = \begin{cases} 1, & n=1 \\ 0, & \text{other} \end{cases}$



Then $y_2[n] = x_2[2n]+5$

$= \underbrace{\delta[2n-1]}_{\text{always zero}}+5 = 5 \ \forall n.$

why?

n	2n − 1	δ[2n−1]
−2	−5	0
−1	−3	0
0	−1	0
1	1	0
2	3	0

$\Rightarrow$ Since $y_1[n-1] \neq y_2[n]$,

the system is

<u>NOT TIME</u>

<u>INVARIANT</u>

2. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by

$$y[n] = x[n] + 3x[n-1] + 3x[n-2] + x[n-3].$$

(a) **8/6 pts.** Find the system frequency response $H(e^{j\omega})$.

$$Y(e^{j\omega}) = X(e^{j\omega})[1 + 3e^{-j\omega} + 3e^{-j2\omega} + e^{-j3\omega}]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{\underline{1 + 3e^{-j\omega} + 3e^{-j2\omega} + e^{-j3\omega}}}$$

(b) **9/7 pts.** Find functions $A(\omega)$ and $\theta(\omega)$ such that $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$.

Highest power of $e^{-j\omega} = 3$. Factor out $e^{-j\frac{3}{2}\omega}$.

$$H(e^{j\omega}) = \left[e^{j\,3/2\,\omega} + 3e^{j\,1/2\,\omega} + 3e^{-j\,1/2\,\omega} + e^{-j\,3/2\,\omega}\right]e^{-j\,3/2\,\omega}$$

$$= \underbrace{\left[2\cos\left(\tfrac{3}{2}\omega\right) + 6\cos\left(\tfrac{1}{2}\omega\right)\right]}_{A(\omega)}\underbrace{e^{-j\,3/2\,\omega}}_{e^{j\theta(\omega)}}$$

$$\boxed{\begin{aligned} A(\omega) &= 2\cos\left(\tfrac{3}{2}\omega\right) + 6\cos\left(\tfrac{1}{2}\omega\right) \\ \theta(\omega) &= -\tfrac{3}{2}\omega \end{aligned}}$$

Problem 2, cont…

(c) **8/7 pts.** Find the system group delay $\tau(\omega)$.

$$\tau(\omega) = -\frac{d}{d\omega}\theta(\omega)$$

$$= -\frac{d}{d\omega}\left[-\tfrac{3}{2}\omega\right]$$

$$= 3/2$$

3. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a **causal** discrete-time LTI system H are related by

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

(a) **12/10 pts.** Find the frequency response $H(e^{j\omega})$.

$$Y(e^{j\omega})\left[1 + \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}\right] = X(e^{j\omega})\left[1 - \frac{1}{2}e^{-j\omega}\right]$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{1 + \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}}$$

Problem 3, cont...

(b) 13/10 pts. Use the DTFT to find the output signal $y[n]$ when the input signal is given by

$$x[n] = (n+1)\left(\frac{1}{2}\right)^n u[n].$$

Table: $X(e^{j\omega}) = \dfrac{1}{(1-\frac{1}{2}e^{-j\omega})^2}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1-\frac{1}{2}e^{-j\omega}}{(1+\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-j2\omega})(1-\frac{1}{2}e^{-j\omega})^2}$$

$$= \frac{1}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= \frac{A}{1+\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}} + \frac{C}{1-\frac{1}{2}e^{-j\omega}}$$

$$A = \frac{1}{(1-\frac{1}{4}\theta)(1-\frac{1}{2}\theta)}\bigg|_{\theta=-2} = \frac{1}{(1+\frac{1}{2})(1+1)} = \frac{1}{\frac{3}{2}\cdot 2} = \frac{1}{3}$$

$$B = \frac{1}{(1+\frac{1}{2}\theta)(1-\frac{1}{2}\theta)}\bigg|_{\theta=4} = \frac{1}{(1+2)(1-2)} = \frac{1}{3\cdot(-1)} = -\frac{1}{3}$$

$$C = \frac{1}{(1+\frac{1}{2}\theta)(1-\frac{1}{4}\theta)}\bigg|_{\theta=2} = \frac{1}{(1+1)(1-\frac{1}{2})} = \frac{1}{2(\frac{1}{2})} = 1$$

$$Y(e^{j\omega}) = \frac{1/3}{1+\frac{1}{2}e^{-j\omega}} - \frac{1/3}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

Table: $\boxed{y[n] = \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n]}$

4. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a **causal** discrete-time LTI system H are related by

$$y[n] + \frac{9}{20}y[n-1] + \frac{1}{20}y[n-2] = x[n] - 2x[n-1].$$

(a) **10/8 pts.** Find the transfer function $H(z)$, specify the ROC, and give a pole-zero plot.

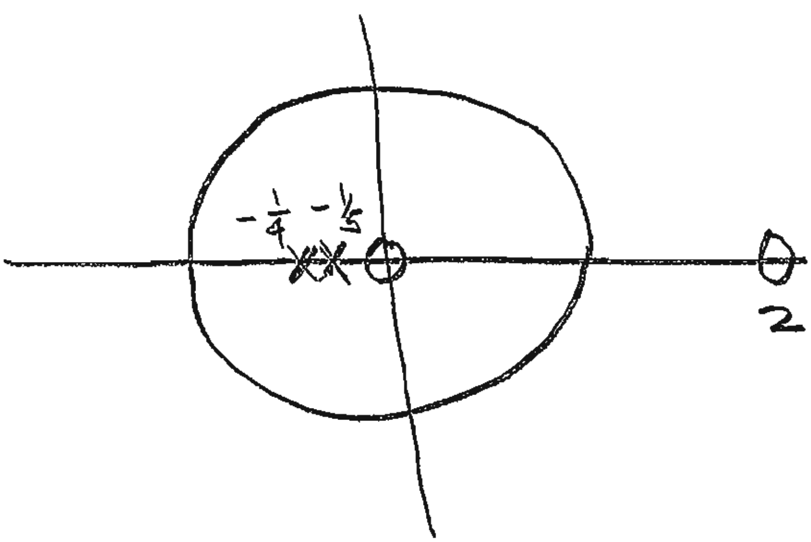
$$Y(z)\left[1 + \frac{9}{20}z^{-1} + \frac{1}{20}z^{-2}\right] = X(z)\left[1 - 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1+\frac{9}{20}z^{-1}+\frac{1}{20}z^{-2}} = \underline{\underline{\frac{1-2z^{-1}}{(1+\frac{1}{4}z^{-1})(1+\frac{1}{5}z^{-1})}}}$$

$$H(z)\cdot\frac{z^2}{z^2} = \frac{z(z-2)}{(z+\frac{1}{4})(z+\frac{1}{5})}$$

zeros: $0, 2$

poles: $-\frac{1}{4}, -\frac{1}{5}$



Because H is causal, the ROC must be exterior to the largest magnitude pole.

$$\underline{\underline{\text{ROC}: |z| > \frac{1}{4}}}$$

Problem 4, cont...

(b) **4/3 pts.** Is the system H stable? Justify your answer.

YES - because the ROC of $H(z)$ includes the unit circle.

(c) **11/9 pts.** Find the impulse response $h[n]$.

$$H(z) = \frac{1-2z^{-1}}{(1+\frac{1}{4}z^{-1})(1+\frac{1}{5}z^{-1})} = \frac{A}{1+\frac{1}{4}z^{-1}} + \frac{B}{1+\frac{1}{5}z^{-1}}$$

$$A = \left.\frac{1-2\theta}{1+\frac{1}{5}\theta}\right|_{\theta=-4} = \frac{1+8}{1-4/5} = \frac{9}{1/5} = 45$$

$$B = \left.\frac{1-2\theta}{1+\frac{1}{4}\theta}\right|_{\theta=-5} = \frac{1+10}{1-5/4} = \frac{11}{-1/4} = -44$$

$$H(z) = \underbrace{\frac{45}{1+\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}} - \underbrace{\frac{44}{1+\frac{1}{5}z^{-1}}}_{|z|>1/5}$$

Table: $\boxed{h[n] = 45(-\tfrac{1}{4})^n u[n] - 44(-\tfrac{1}{5})^n u[n]}$

5. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a causal discrete-time LTI system H are related by

$$y[n] + \frac{1}{4}y[n-1] = x[n] + \frac{1}{2}x[n-1].$$

The input signal is given by

$$x[n] = 2\left(\frac{1}{3}\right)^n u[n]$$

and initial conditions are given as $y[-1] = 1$ and $x[-1] = 2$.

Use the unilateral z-transform to find $y[n]$ for $n \geq 0$.

UZ: $\mathcal{Y}_u(z) + \frac{1}{4}\left[z^{-1}\mathcal{Y}_u(z) + \underbrace{y[-1]}_{1}\right] = \mathcal{X}_u(z) + \frac{1}{2}\left[z^{-1}\mathcal{X}_u(z) + \underbrace{x[-1]}_{2}\right]$

$$\mathcal{Y}_u(z) + \frac{1}{4}z^{-1}\mathcal{Y}_u(z) + \frac{1}{4} = \mathcal{X}_u(z) + \frac{1}{2}z^{-1}\mathcal{X}_u(z) + 1$$

$$\mathcal{Y}_u(z)\left[1 + \frac{1}{4}z^{-1}\right] = \mathcal{X}_u(z)\left[1 + \frac{1}{2}z^{-1}\right] + 3/4$$

Table: $\mathcal{X}_u(z) = \dfrac{2}{1 - \frac{1}{3}z^{-1}}$

$$\mathcal{Y}_u(z)\left[1 + \frac{1}{4}z^{-1}\right] = \frac{2}{1 - \frac{1}{3}z^{-1}}\left[1 + \frac{1}{2}z^{-1}\right] + \frac{3}{4}$$

$$= \frac{2\left[1 + \frac{1}{2}z^{-1}\right]}{1 - \frac{1}{3}z^{-1}} + \frac{3}{4}$$

$$\mathcal{Y}_u(z) = \frac{2(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{3}z^{-1})(1 + \frac{1}{4}z^{-1})} + \frac{3/4}{1 + \frac{1}{4}z^{-1}}$$

$$= \frac{2 + z^{-1} + \frac{3}{4}(1 - \frac{1}{3}z^{-1})}{(1 - \frac{1}{3}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{2 + z^{-1} + \frac{3}{4} - \frac{1}{4}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{1}{4}z^{-1})}$$

$$\mathcal{Y}_u(z) = \frac{\frac{11}{4} + \frac{3}{4}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{1}{4}z^{-1})}$$

$\longrightarrow$

More Workspace for Problem 5...

$$\mathcal{Y}_u(z) = \frac{\frac{11}{4} + \frac{3}{4}z^{-1}}{(1-\frac{1}{3}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{A}{1-\frac{1}{3}z^{-1}} + \frac{B}{1+\frac{1}{4}z^{-1}}$$

$$A = \left.\frac{\frac{11}{4} + \frac{3}{4}\theta}{1+\frac{1}{4}\theta}\right|_{\theta=3} = \frac{\frac{11}{4} + \frac{9}{4}}{1+\frac{3}{4}} = \frac{\frac{20}{4}}{7/4} = \frac{20}{7}$$

$$B = \left.\frac{\frac{11}{4} + \frac{3}{4}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=-4} = \frac{\frac{11}{4} - 3}{1+\frac{4}{3}} = \frac{-1/4}{7/3} = -\left(\frac{1}{4}\right)\left(\frac{3}{7}\right) = -\frac{3}{28}$$

$$\mathcal{Y}_u(z) = \frac{20}{7}\,\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{3}{28}\,\frac{1}{1+\frac{1}{4}z^{-1}}$$

Table:

$$\boxed{y[n] = \frac{20}{7}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{28}\left(-\frac{1}{4}\right)^n u[n], \quad n \geq 0}$$