ECE 4213/5213
Test 1

Monday, November 6, 2023
4:30 PM - 5:45 PM

Fall 2023 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is open book and open notes. You may also use your calculator and the course formula sheet. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points. Below, circle the numbers of the four problems you wish to have graded.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ______
2. (25/20) ______
3. (25/20) ______
4. (25/20) ______
5. (25/20) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Signed:______________ Date:______________

1. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system are related by

$$y[n] = nx[-n].$$

(a) **8/6 pts.** Is the system linear? Justify your answer.

Let $x_1[n]$ and $x_2[n]$ be two input signals and let $a, b \in \mathbb{C}$ be constants.

Then $y_1[n] = H\{x_1[n]\} = nx_1[-n]$,
$y_2[n] = H\{x_2[n]\} = nx_2[-n]$,

and $ay_1[n] + by_2[n] = anx_1[-n] + bnx_2[-n]$.

Let $x_3[n] = ax_1[n] + bx_2[n]$.

Then $y_3[n] = nx_3[-n] = n\{ax_1[-n] + bx_2[-n]\}$
$= anx_1[-n] + bnx_2[-n]$.

Since $y_3[n] = ay_1[n] + by_2[n]$,

$\boxed{\text{The system } \underline{\text{IS}} \ \underline{\text{LINEAR}}}$

Problem 1, cont...

(b) 8/7 pts. Is the system time invariant? Justify your answer.

I guess "no" because of the $\underline{\underline{n}}$ times $x[-n]$.

Let $x_1[n] = \delta[n]$.

Then $y_1[n] = n x_1[-n] = n\delta[-n] = n\delta[n] = 0 \quad \forall n \in \mathbb{Z}$.

Now let $n_0 = 2$.

Then $y_1[n-n_0] = 0 \quad \forall n \in \mathbb{Z}$.

Let $x_2[n] = x_1[n-n_0] = x_1[n-2] = \delta[n-2]$.

Then $y_2[n] = n x_2[-n] = n\,\delta[-n-2]$

$$= -2\,\delta[-n-2] = \begin{cases} -2, & n=-2 \\ 0, & \text{otherwise} \end{cases}$$

$\rightarrow$ Since $y_2[n] \neq y_1[n-2]$,

The System is $\underline{\underline{\text{NOT TIME INVARIANT}}}$

Problem 1, cont...

(c) 9/7 pts. Is the system BIBO stable? Justify your answer.

I guess "no", again because of the "n" times $x[-n]$.

Let $x[n] = u[-n]$. Then $x[n]$ is a bounded input signal because $|x[n]| \le 1 \quad \forall n \in \mathbb{Z}$.

Now suppose that $y[n] = H\{x[n]\}$ is also bounded.

Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|y[n]| \le B \quad \forall n \in \mathbb{Z}$.

Let $n_0 = \lceil B \rceil + 1$

$$\begin{aligned} \text{Then } |y[n_0]| &= |n_0 x[-n_0]| = |n_0 u[n_0]| \\ &= |(\lceil B \rceil + 1) u[\lceil B \rceil + 1]| \\ &= (\lceil B \rceil + 1) \cdot 1 \\ &= \lceil B \rceil + 1 > B \end{aligned}$$

$\rightarrow$ This contradicts the hypothesis that $y[n]$ is bounded by B.

Therefore, no such B can exist and $y[n]$ is unbounded.

Since a bounded input signal produced an unbounded output signal,

$\boxed{\text{The System is NOT STABLE}}$

2. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a **stable** discrete-time LTI system H are related by

$$y[n] + \frac{3}{2}y[n-1] - y[n-2] = x[n] - 2x[n-1].$$

(a) **3/2 pts.** Is the system H an FIR filter or an IIR filter? Justify your answer.

It is an $\boxed{\text{IIR filter}}$ – because the difference equation has nontrivial shifts of $y[n]$.

(b) **6/5 pts.** Find the transfer function $H(z)$, specify the ROC, and give a pole-zero plot.

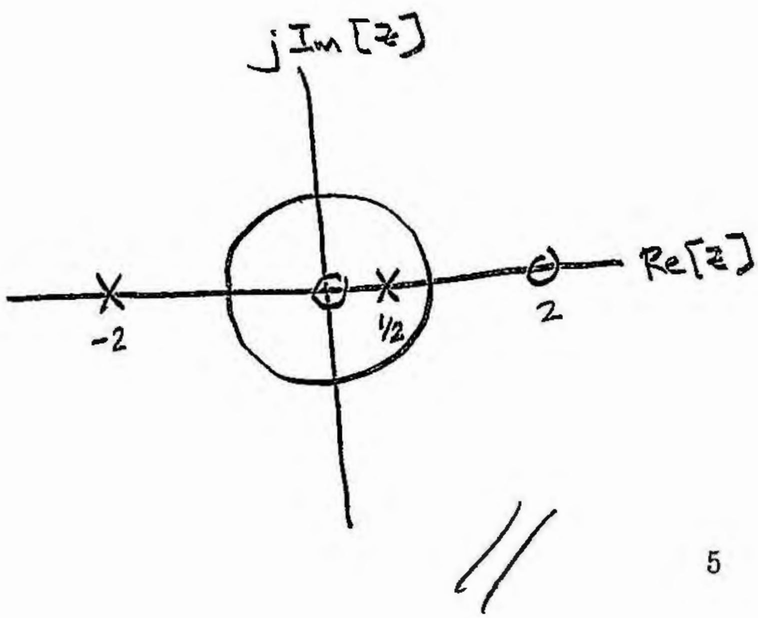
$$\mathcal{Z}: \quad Y(z)\left[1 + \tfrac{3}{2}z^{-1} - z^{-2}\right] = X(z)\left[1 - 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}} = \frac{1-2z^{-1}}{(1+2z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$H(z) \cdot \frac{z^2}{z^2} = \frac{z(z-2)}{(z+2)(z-\frac{1}{2})}$$

poles: $z = -2, \frac{1}{2}$

zeros: $z = 0, 2$



Since the system is stable, the ROC must include the unit circle.

$$\boxed{\text{ROC: } \tfrac{1}{2} < |z| < 2}$$

Problem 2, cont...

(c) **6/5 pts.** Find the impulse response $h[n]$.

$$H(z) = \frac{1-2z^{-1}}{(1+2z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A}{1+2z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{1-2\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{2}} = \frac{1+1}{1+\frac{1}{4}} = \frac{2}{5/4} = \frac{4\cdot 2}{5} = \frac{8}{5}$$

$$B = \left.\frac{1-2\theta}{1+2\theta}\right|_{\theta=2} = \frac{1-4}{1+4} = \frac{-3}{5}$$

$$H(z) = \underbrace{\frac{8/5}{1+2z^{-1}}}_{|z|<2} - \underbrace{\frac{3/5}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

Table:

$$\boxed{h[n] = -\frac{8}{5}(-2)^n u[-n-1] - \frac{3}{5}\left(\frac{1}{2}\right)^n u[n]}$$

(d) **4/3 pts.** Find the frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

$$\boxed{H(e^{j\omega}) = \frac{1-2e^{-j\omega}}{(1+2e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}}$$

Problem 2, cont...

(e) **6/5 pts.** Find the system output $y[n]$ when the input is given by

$$x[n] = \cos\left(\frac{\pi}{8}n\right).$$

Hint 1: $x[n]$ is a sum of two eigenfunctions and the difference equation (I/O equation) has all *real* coefficients.

Hint 2: make sure your calculator is set for radians, not degrees!

Notes p. 3-114: $y[n] = |H(e^{j\pi/8})| \cos\left[\frac{\pi}{8}n + \arg H(e^{j\pi/8})\right]$

$$H(e^{j\pi/8}) = \frac{1-2e^{-j\pi/8}}{(1+2e^{-j\pi/8})(1-\frac{1}{2}e^{-j\pi/8})}$$

$$= \frac{(1-2\cos\frac{\pi}{8}) + j2\sin\frac{\pi}{8}}{\left[(1+2\cos\frac{\pi}{8}) - j2\sin\frac{\pi}{8}\right]\left[(1-\frac{1}{2}\cos\frac{\pi}{8}) + j\frac{1}{2}\sin\frac{\pi}{8}\right]}$$

$$= \frac{-0.84776 + j0.76537}{[2.84776 - j0.76537][0.53806 + j0.19134]}$$

Must add π to the angle returned by atan!!

atan will work

atan will work

$$= \frac{1.14214e^{j2.40723}}{[2.94882e^{-j0.262557}][0.57107e^{j0.341667}]}$$

$$= \frac{1.14214e^{j2.40723}}{1.68398e^{j0.0791106}} = \underbrace{0.678238}_{|H(e^{j\pi/8})|} e^{j\overbrace{2.32812}^{\arg H(e^{j\pi/8})}}$$

$$\boxed{y[n] = 0.678238\cos\left(\frac{\pi}{8}n + 2.32812\right)}$$

3. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by

$$y[n] = x[n] + 2x[n-1] - 2x[n-2] - x[n-3].$$

(a) **3/2 pts.** Is the system H an FIR filter or an IIR filter? Justify your answer.

$\boxed{\text{FIR}}$ – because the difference equation has no shifts of $y[n]$.

(b) **4/3 pts.** Find the impulse response $h[n]$.

From the convolution equation $y[n] = x[n] * h[n]$, or equivalently by letting $x[n] = \delta[n]$:

$$\boxed{h[n] = \delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3]}$$

(c) **4/3 pts.** Find the frequency response $H(e^{j\omega})$.

Time shift Property:

$$\boxed{H(e^{j\omega}) = 1 + 2e^{-j\omega} - 2e^{-j2\omega} - e^{-j3\omega}}$$

Problem 3, cont...

(d) **9/8 pts.** Find functions $A(\omega)$ and $\theta(\omega)$ such that $H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$.

Highest power of the character = $e^{-j3\omega}$: Factor out $e^{-j\frac{3}{2}\omega}$

$$H(e^{j\omega}) = \left[e^{j\frac{3}{2}\omega} + 2e^{j\frac{1}{2}\omega} - 2e^{-j\frac{1}{2}\omega} - e^{-j\frac{3}{2}\omega}\right] e^{-j\frac{3}{2}\omega}$$

$$= \left[2j\left(\frac{e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}}{2j}\right) + 2(2j)\left(\frac{e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}}{2j}\right)\right] e^{-j\frac{3}{2}\omega}$$

$$= \left[2j\sin\left(\tfrac{3}{2}\omega\right) + 4j\sin\left(\tfrac{1}{2}\omega\right)\right] e^{-j\frac{3}{2}\omega}$$

$$= \left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right] j e^{-j\frac{3}{2}\omega}$$

$j = e^{j\pi/2}$

$$= \left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right] e^{j\pi/2} e^{-j\frac{3}{2}\omega}$$

$$= \underbrace{\left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right]}_{A(\omega)} e^{j\overbrace{\left(\frac{\pi}{2} - \frac{3}{2}\omega\right)}^{\theta(\omega)}}$$

$$\boxed{A(\omega) = 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) \; ; \quad \theta(\omega) = \tfrac{\pi}{2} - \tfrac{3}{2}\omega}$$

(e) **5/4 pts.** Find the system group delay $\tau(\omega)$.

$$\tau(\omega) = -\frac{d}{d\omega}\theta(\omega) = -\frac{d}{d\omega}\left[\frac{\pi}{2} - \frac{3}{2}\omega\right] = \frac{3}{2}$$

$$\boxed{\tau(\omega) = \frac{3}{2}}$$

4. **25/20 pts.** The input $x[n]$ and output $y[n]$ of a causal discrete-time LTI system H are related by

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1].$$

The input signal is given by

$$x[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$$

and initial conditions are given as $y[-1] = 1$ and $x[-1] = -2$.

Use the unilateral z-transform to find $y[n]$ for $n \geq 0$.

Table: $\mathcal{X}_u(z) = \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} = \frac{3(1-\frac{1}{3}z^{-1}) - 2(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$

$$= \frac{3 - z^{-1} - 2 + z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$\mathcal{U}_z$: $\mathcal{Y}_u(z) - \frac{1}{2}\{z^{-1}\mathcal{Y}_u(z) + \underbrace{y[-1]}_{1}\} = \mathcal{X}_u(z) - \frac{1}{2}\{z^{-1}\mathcal{X}_u(z) + \underbrace{x[-1]}_{-2}\}$

$$\mathcal{Y}_u(z) - \frac{1}{2}z^{-1}\mathcal{Y}_u(z) - \frac{1}{2} = \mathcal{X}_u(z) - \frac{1}{2}z^{-1}\mathcal{X}_u(z) + 1$$

$$\mathcal{Y}_u(z)\left[1 - \frac{1}{2}z^{-1}\right] - \frac{1}{2} = \mathcal{X}_u(z)\left[1 - \frac{1}{2}z^{-1}\right] + 1$$

$$\mathcal{Y}_u(z)\left[1 - \frac{1}{2}z^{-1}\right] = \mathcal{X}_u(z)\left[1 - \frac{1}{2}z^{-1}\right] + \frac{3}{2}$$

$$\mathcal{Y}_u(z) = \mathcal{X}_u(z)\cancel{\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}} + \frac{3/2}{1-\frac{1}{2}z^{-1}}$$

$$\mathcal{Y}_u(z) = \mathcal{X}_u(z) + \frac{3/2}{1-\frac{1}{2}z^{-1}}$$

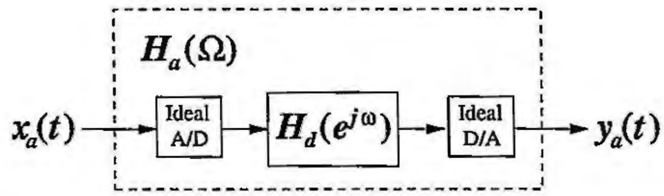
$$y[n] = x[n] + \mathcal{U}_{z^{-1}}\left\{\frac{3/2}{1-\frac{1}{2}z^{-1}}\right\}$$

Table: $y[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n] + \frac{3}{2}\left(\frac{1}{2}\right)^n u[n], \quad n \geq 0$

$$\boxed{y[n] = \frac{9}{2}\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n], \quad n \geq 0}$$

More Workspace for Problem 4. . .

5. **25/20 pts.** The continuous-time LTI system $H_a(\Omega)$ shown in the figure below is implemented using an ideal A/D converter, a discrete-time LTI system $H_d(e^{j\omega})$, and an ideal D/A converter.



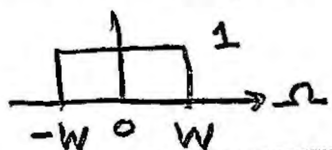
As in professional compact disc (CD) audio, the sampling frequency of the A/D and D/A converters is $F_T = 44.1$ kHz. So $\Omega_T = 88{,}200\pi$ rad/sec. All input signals $x_a(t)$ are bandlimited to $|\Omega| < \Omega_T/2$ (this simply ensures there will be no aliasing so that the overall structure shown in the figure is an LTI system).

The impulse response of the continuous-time filter H_a is given by

$$h_a(t) = \frac{\sin(20{,}000\pi t)}{\pi t}.$$

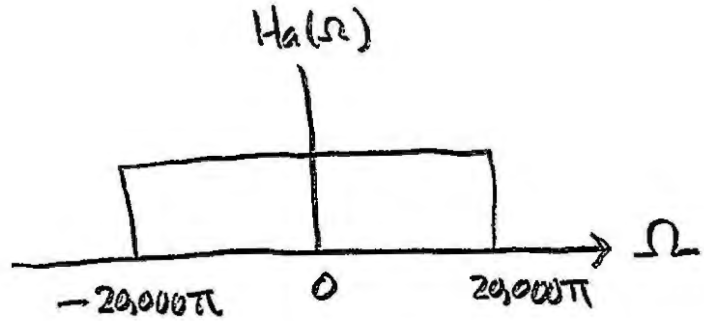
(a) **8/6 pts.** Find the continuous-time frequency response $H_a(\Omega)$.

Table: $\frac{\sin Wt}{\pi t} \xleftrightarrow{\mathcal{F}}$



$W = 20{,}000\pi \ \frac{\text{rad}}{\text{sec}}$

$$\boxed{H_a(\Omega) = \begin{cases} 1, & |\Omega| \le 20{,}000\pi \\ 0, & |\Omega| > 20{,}000\pi \end{cases}}$$



Problem 5, cont...

(b) 8/7 pts. Find the discrete-time frequency response $H_d(e^{j\omega})$.

$F_T = 44{,}100$ Hz

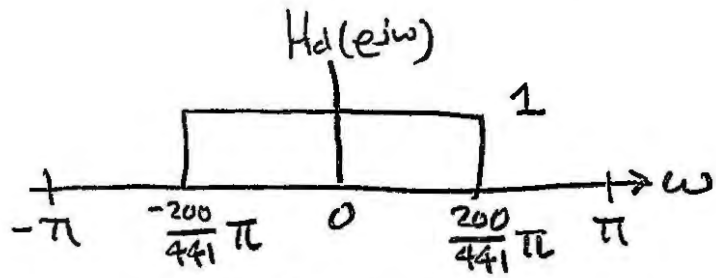
$T = \frac{1}{44{,}100}$ sec

$\Omega_T = 2\pi F_T = 88{,}200\pi \ \frac{\text{rad}}{\text{sec}}$

$\omega = \Omega T = \frac{\Omega}{44{,}100}$

$H_d(e^{j\omega}) = H_a\left(\frac{\omega}{T}\right), \ |\omega| \le \pi$

$$\boxed{H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{20{,}000\pi}{44{,}100} = \frac{200}{441}\pi \\ 0, & \frac{200}{441}\pi < |\omega| < \pi \end{cases}}$$



(c) 9/7 pts. Find the discrete-time impulse response $h_d[n]$.

Table: $\boxed{h_d[n] = \dfrac{\sin\left(\frac{200}{441}\pi n\right)}{\pi n}}$