ECE 4213/5213
Test 2

Tuesday, November 26, 2002
5:00 PM - 8:00 PM

Fall 2002

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is open book and open notes. You have 180 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work the first four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

1. A nonminimum phase causal digital filter H has transfer function

$$H(z) = \frac{(1-2z^{-2})(1+0.4z^{-1})}{1-0.85z^{-1}}.$$

Find a factorization of $H(z)$ of the form

$$H(z) = H_{\min}(z)H_{\mathrm{ap}}(z),$$

where $H_{\min}(z)$ has minimum phase and $H_{\mathrm{ap}}(z)$ is an allpass filter.

- The term $1+0.4z^{-1}$ gives a zero at $z=0.4$, inside the unit circle.
- The term $1-2z^{-2}$ gives two zeros at $z=\pm\sqrt{2}$ outside unit circle.
- For minimum phase, the term $(1-2z^{-2})$ must be replaced with equivalent minimum phase terms using an allpass section.
- Three Methods to factor $(1-2z^{-2})$:

Method I:

All pass: $\dfrac{z^{-1}-\alpha^*}{1-\alpha z^{-1}}$

$\alpha = \pm\dfrac{1}{\sqrt{2}}$

$$(1-2z^{-2}) = (1-\sqrt{2}z^{-1})(1-(-\sqrt{2})z^{-1})$$

$$= (-\sqrt{2})\left(z^{-1}-\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(z^{-1}-\frac{-1}{\sqrt{2}}\right)$$

$$=(-2)\left(z^{-1}-\frac{1}{\sqrt{2}}\right)\frac{1-\frac{1}{\sqrt{2}}z^{-1}}{1-\frac{1}{\sqrt{2}}z^{-1}}\left(z^{-1}-\frac{-1}{\sqrt{2}}\right)\frac{1-\frac{-1}{\sqrt{2}}z^{-1}}{1-\frac{-1}{\sqrt{2}}z^{-1}}$$

$$=(-2)\underbrace{\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)\left(1-\frac{-1}{\sqrt{2}}z^{-1}\right)}_{\text{two zeros inside circle}}\underbrace{\frac{z^{-1}-\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}z^{-1}}}_{\text{allpass}}\underbrace{\frac{z^{-1}-\frac{-1}{\sqrt{2}}}{1-\frac{-1}{\sqrt{2}}z^{-1}}}_{\text{allpass}}$$

$$=(-2)\left(1-\frac{1}{2}z^{-1}\right)\frac{z^{-2}-\frac{1}{2}}{1-\frac{1}{2}z^{-2}}$$

$$=(z^{-2}-2)\frac{-\frac{1}{2}(1-2z^{-2})}{-\frac{1}{2}(z^{-2}-2)} = (z^{-2}-2)\frac{1-2z^{-2}}{z^{-2}-2}$$

Method II: All pass: $\dfrac{1-\alpha z^{-1}}{z^{-1}-\alpha^*}$, $\alpha=\pm\sqrt{2}$

$$1-2z^{-2} = (1-\sqrt{2}z^{-1})(1-(-\sqrt{2})z^{-1})$$

$$=(1-\sqrt{2}z^{-1})\frac{z^{-1}-\sqrt{2}}{z^{-1}-\sqrt{2}}(1-(-\sqrt{2})z^{-1})\frac{z^{-1}-(-\sqrt{2})}{z^{-1}-(-\sqrt{2})}$$

$$=\underbrace{(z^{-1}-\sqrt{2})(z^{-1}-(-\sqrt{2}))}_{\text{two zeros inside circle}}\underbrace{\frac{1-\sqrt{2}z^{-1}}{z^{-1}-\sqrt{2}}}_{\text{allpass}}\underbrace{\frac{1-(-\sqrt{2})z^{-1}}{z^{-1}-(-\sqrt{2})}}_{\text{all pass}}$$

$$=(z^{-2}-2)\frac{1-2z^{-2}}{z^{-2}-2}$$

Method III: All pass: $\dfrac{1-\alpha z^{-2}}{z^{-2}-\alpha}$

$\alpha=\pm\sqrt{2}$

$$1-2z^{-2} = (1-2z^{-2})\frac{z^{-2}-2}{z^{-2}-2}$$

$$=\underbrace{(z^{-2}-2)}_{\text{two zeros inside circle}}\underbrace{\frac{1-2z^{-2}}{z^{-2}-2}}_{\text{allpass}}$$

$$H(z) = \underbrace{\frac{(1+0.4z^{-1})(z^{-2}-2)}{1-0.85z^{-1}}}_{H_{\min}(z)}\cdot\underbrace{\frac{1-2z^{-2}}{z^{-2}-2}}_{H_{ap}(z)}$$

2. An analog signal $x_a(t)$ is to be filtered to remove frequency components in the range 5 kHz $\leq f \leq$ 10 kHz. The maximum frequency present in $x_a(t)$ is 20 kHz. The filtering is to be done by sampling $x_a(t)$ with an ideal sampler, filtering the sampled signal, and reconstructing the analog output signal with an ideal interpolator.

Find the minimum sampling frequency that may be used to avoid aliasing, and for this minimum sampling rate, find the frequency response of the ideal digital bandstop filter $H(e^{j\omega})$ that will remove the desired frequencies from $x_a(t)$.

$X_a(t)$ is bandlimited to $\Omega_M = 20\text{kHz} \cdot 2\pi \text{rad} = 40{,}000\pi \frac{\text{rad}}{\text{sec}}$

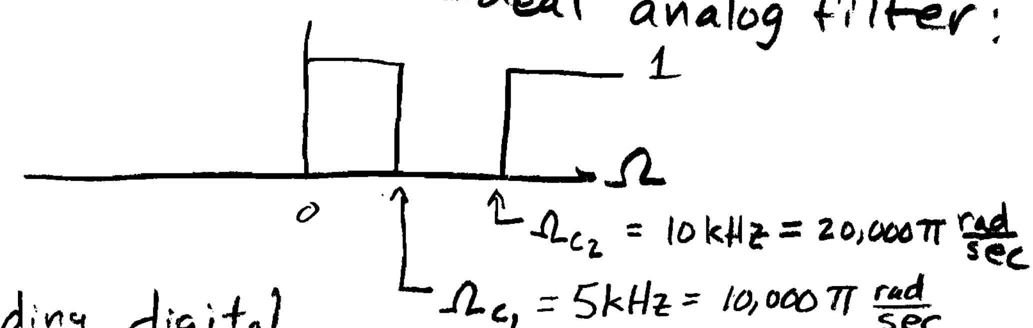
Minimum sampling frequency:

$$\Omega_T = 2\Omega_M = 80{,}000\pi \frac{\text{rad}}{\text{sec}} \quad (f_T = 40\text{kHz}).$$

- From Eq. 5.11, p. 302 of Mitra: $\Omega_T = 2\pi/T \Rightarrow T = 2\pi/\Omega_T$.
- Relationship between analog & digital frequency:

Eq. 5.15: $\Omega = \frac{\omega}{T} \Rightarrow \omega = \Omega T = 2\pi \frac{\Omega}{\Omega_T}$

- Frequency response of desired Ideal analog filter:

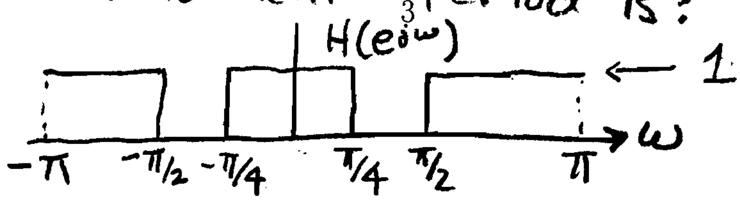


- The corresponding digital cutoff frequencies are:

$$\omega_{c_1} = 2\pi \frac{\Omega_{c_1}}{\Omega_T} = 2\pi \frac{10{,}000\pi}{80{,}000\pi} = \pi/4 \text{ rad/sample}$$

$$\omega_{c_2} = 2\pi \frac{\Omega_{c_2}}{\Omega_T} = 2\pi \frac{20{,}000\pi}{80{,}000\pi} = \pi/2 \text{ rad/sample}.$$

- The ideal digital bandstop filter frequency response is 2π-periodic. The fundamental period is:



3. Consider a third-order type I Chebyshev lowpass filter $H_a(s)$ with a cutoff frequency $\Omega_p = 1$. The maximum passband attenuation is $\frac{1}{\sqrt{1+\varepsilon^2}}$, where $\varepsilon = 0.1$. Find an expression for the frequency response squared-magnitude

$$|H_a(j\Omega)|^2 = H_a(s)H_a(-s)|_{s=j\Omega}.$$

$N=3,\ \varepsilon=0.1,\quad \Omega_p = 1 \implies \Omega/\Omega_p = \Omega.$

<u>Method I</u>: Good because it gives a closed form solution.

$\rightarrow$ Use book (5.39), (5.37)

$T_0(\Omega) = 1,\quad T_1(\Omega) = \Omega,$

$$T_2(\Omega) = 2\Omega T_1(\Omega) - T_0(\Omega) = 2\Omega^2 - 1.$$

$$T_3(\Omega) = 2\Omega T_2(\Omega) - T_1(\Omega) = 2\Omega(2\Omega^2-1) - \Omega = 4\Omega^3 - 2\Omega - \Omega = 4\Omega^3 - 3\Omega$$

$$T_3^2\left(\frac{\Omega}{\Omega_p}\right) = T_3^2(\Omega) = (4\Omega^3-3\Omega)(4\Omega^3-3\Omega) = 16\Omega^6 - 24\Omega^4 + 9\Omega^2$$

$$|H_a(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_3^2(\Omega/\Omega_p)} = \frac{1}{1+0.01(16\Omega^6-24\Omega^4+9\Omega^2)} = \frac{1}{0.16\Omega^6 - 0.24\Omega^4 + 0.09\Omega^2 + 1}$$ //

<u>Method II</u>: Poor because it does not give a closed form solution.

$\rightarrow$ Use book (5.37), (5.38).

$$T_3(\Omega) = \begin{cases} \cos[3\cos^{-1}(\Omega)], & |\Omega| \le 1 \\ \cosh[3\cosh^{-1}(\Omega)], & |\Omega| > 1 \end{cases} = T_3(\Omega/\Omega_p)$$

$$|H_a(\Omega)|^2 = \left[1+\varepsilon^2 T_3^2(\Omega/\Omega_p)\right]^{-1} = \begin{cases} \{1+0.01\cos^2[3\cos^{-1}(\Omega)]\}^{-1}, & |\Omega| \le 1 \\ \{1+0.01\cosh^2[3\cosh^{-1}(\Omega)]\}^{-1}, & |\Omega| > 1 \end{cases}$$

4. The transfer function of a second-order analog elliptic lowpass filter with a passband edge at 1 Hz and a passband ripple of 1 dB is given by

$$H_{LP}(s) = \frac{0.056(s^2 + 17.95)}{s^2 + 1.06s + 1.13}.$$

Use an analog-to-analog transformation to determine the transfer function $H_{BP}(s)$ of an analog bandpass filter with a center frequency at 3 Hz and a bandwidth of 0.5 Hz.

$f_p = 1\text{ Hz} \Rightarrow \Omega_p = 2\pi f_p = 2\pi$ rad/sec

$\hat{f}_o = 3\text{ Hz} \Rightarrow \hat{\Omega}_o = 2\pi \hat{f}_o = 6\pi$ rad/sec

$\hat{f}_{p2} - \hat{f}_{p1} = 0.5\text{ Hz} \Rightarrow \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi(\hat{f}_{p1} - \hat{f}_{p2}) = \pi$ rad/sec

Book (5.61): $s = \Omega_p \dfrac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})} = 2\pi \dfrac{\hat{s}^2 + 36\pi^2}{\pi \hat{s}} = 2\hat{s} + 72\pi^2 \hat{s}^{-1}$

$$H_{BP}(\hat{s}) = \frac{0.056\left[(2\hat{s} + 72\pi^2\hat{s}^{-1})^2 + 17.95\right]}{(2\hat{s}+72\pi^2\hat{s}^{-1})^2 + 1.06(2\hat{s}+72\pi^2\hat{s}^{-1}) + 1.13}$$

$$= \frac{0.056\left[(4\hat{s}^2 + 288\pi^2 + 5184\pi^4\hat{s}^{-2}) + 17.95\right]}{4\hat{s}^2 + 288\pi^2 + 5184\pi^4\hat{s}^{-2} + 2.12\hat{s} + 76.32\pi^2\hat{s}^{-1} + 1.13}$$

$$= \frac{0.224\hat{s}^2 + (16.128\pi^2 + 1.0052) + 290.304\pi^4\hat{s}^{-2}}{4\hat{s}^2 + 2.12\hat{s} + (288\pi^2 + 1.13) + 76.32\pi^2\hat{s}^{-1} + 5184\pi^4\hat{s}^{-2}}$$

$$= \frac{0.224\hat{s}^4 + (16.128\pi^2 + 1.0052)\hat{s}^2 + 290.304\pi^4}{4\hat{s}^4 + 2.12\hat{s}^3 + (288\pi^2 + 1.13)\hat{s}^2 + 76.32\pi^2\hat{s} + 5184\pi^4}$$

(CALCULATOR) $\approx \dfrac{0.056\hat{s}^4 + 40.0455\hat{s}^2 + 7069.5622}{\hat{s}^4 + 0.53\hat{s}^3 + 710.894\hat{s}^2 + 188.3121\hat{s} + 126{,}242.1821}$

5. Consider a causual LTI digital filter with transfer function

$$H(z) = \frac{1 + 0.875z^{-1}}{(1 + 0.2z^{-1} + 0.9z^{-2})(1 - 0.7z^{-1})}.$$

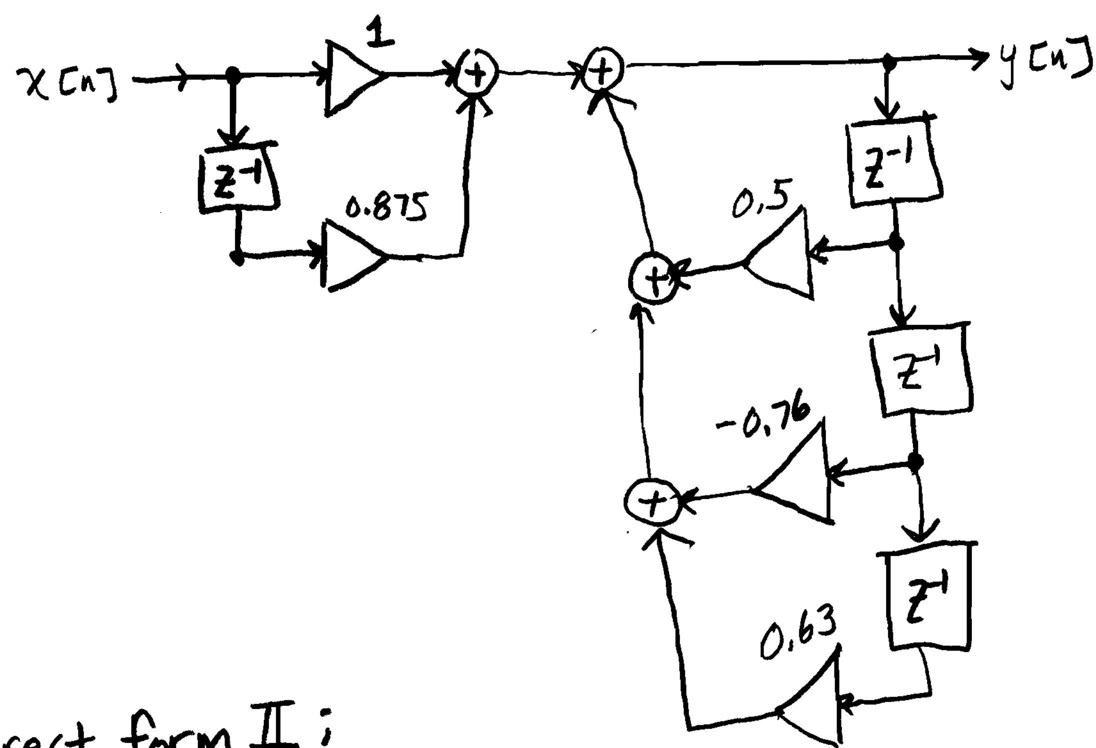
Show the signal flow graphs (block diagrams) for the Direct form I and Direct form II implementations of this system.

$p_0 = 1$; $p_1 = 0.875$

$$H(z) = \frac{1 + 0.875z^{-1}}{1 - 0.5z^{-1} + 0.76z^{-2} - 0.63z^{-3}}$$

$d_1 = -0.5$; $d_2 = 0.76$; $d_3 = -0.63$

Direct Form I :



Direct form II ;