

ECE 4213/5213

Test 2

Wednesday, November 29, 2006
3:30 PM - 5:30 PM

Fall 2006

Name: SOLUTION

Dr. Havlicek

Student Num: _____

Directions: This test is open book and closed notes. You may also use a calculator and the formula sheet from the course web site. You have 120 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) _____

2. (25/20) _____

3. (25/20) _____

4. (25/20) _____

5. (25/20) _____

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: _____

Date: _____

1. Design an analog lowpass Butterworth filter $H_a(s)$ having a 3 dB cutoff frequency $\Omega_c = 2\pi \times 1.5$ kHz, a stopband edge frequency $\Omega_s = 2\pi \times 3.0$ kHz, and a stopband attenuation of at least 24 dB.

Hint: The stopband requirement is for $\mathcal{G}(\Omega) = 10 \log_{10} |H_a(\Omega)|^2 < -24$ at $\Omega = \Omega_s = 2\pi \times 3.0 \times 10^3$. Use this fact with Eq. (4.33) on page 189 of the text to solve for N . You will then know both of the two parameters N and Ω_c that completely specify the Butterworth transfer function in Eq. (4.36).

$|H_a(\Omega)|$ is monotonically decreasing in Ω . So the stopband spec will be met throughout the entire stop band if it is met at the left edge where $\Omega = \Omega_s$, i.e., if

$$10 \log_{10} |H_a(\Omega_s)|^2 \leq -24$$

$$10 \log_{10} \left(\frac{1}{1 + (\Omega_s/\Omega_c)^{2N}} \right) \leq -24$$

$$\log_{10} \left(\frac{1}{1 + 2^{2N}} \right) \leq -2.4$$

$$\frac{1}{1 + 2^{2N}} \leq 10^{-2.4} \approx 3.98107 \times 10^{-3}$$

$$1 \leq 10^{-2.4} + 10^{-2.4} \cdot 2^{2N}$$

$$1 - 10^{-2.4} \leq 10^{-2.4} \cdot 2^{2N}$$

$$\frac{1 - 10^{-2.4}}{10^{-2.4}} \leq 2^{2N}$$

$$\log_2 \frac{1 - 10^{-2.4}}{10^{-2.4}} = \frac{\ln \left(\frac{1 - 10^{-2.4}}{10^{-2.4}} \right)}{\ln 2} \leq 2N$$

$$\frac{\ln \left(\frac{1 - 10^{-2.4}}{10^{-2.4}} \right)}{2 \ln 2} = 3.98344... \leq N$$

Take N = 4

(4.37):

$$\Omega_c = 2\pi \times 1.5 \times 10^3$$

$$P_1 = \Omega_c e^{j5\pi/8}$$

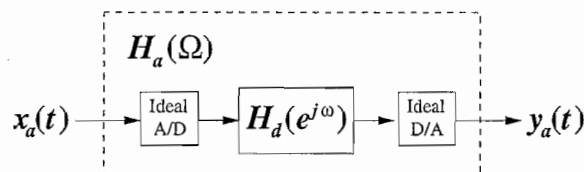
$$P_2 = \Omega_c e^{j7\pi/8}$$

$$P_3 = \Omega_c e^{j9\pi/8}$$

$$P_4 = \Omega_c e^{j11\pi/8}$$

$$(4.36): H_a(s) = \frac{\Omega_c^4}{(s - \Omega_c e^{j5\pi/8})(s - \Omega_c e^{j7\pi/8})(s - \Omega_c e^{j9\pi/8})(s - \Omega_c e^{j11\pi/8})}$$

2. Consider a fourth-order analog Butterworth filter $H_a(\Omega)$ having a 3 dB cutoff frequency $\Omega_c = 2\pi \times 1.5$ kHz. It is desired to implement this analog filter using a digital filter $H_d(e^{j\omega})$ in combination with ideal A/D and D/A converters as shown below:



The sampling frequency of the ideal A/D and D/A converters is 44.1 kHz $= 2\pi \times 44.1 \times 10^3$ rad/sec. Design $H_d(e^{j\omega})$.

We have $N=4$ and $\Omega_c = 2\pi \times 1.5 \times 10^3$. It follows from (4.37):

$$p_1 = \Omega_c e^{j5\pi/8}, \quad p_2 = \Omega_c e^{j7\pi/8}, \quad p_3 = \Omega_c e^{j9\pi/8}, \quad p_4 = \Omega_c e^{j11\pi/8}$$

From (4.36) with $s = j\Omega$ we have

$$H_a(\Omega) = \frac{\Omega_c^4}{(j\Omega - \Omega_c e^{j5\pi/8})(j\Omega - \Omega_c e^{j7\pi/8})(j\Omega - \Omega_c e^{j9\pi/8})(j\Omega - \Omega_c e^{j11\pi/8})}$$

$$T = \frac{2\pi}{\Omega_s} = \frac{2\pi}{2\pi \times 44.1 \times 10^3} = \frac{1}{44.1} \times 10^{-3} \text{ sec}$$

$$\frac{1}{T} = 44.1 \times 10^3 = 44100 \Rightarrow \frac{\omega}{T} = 44100\omega$$

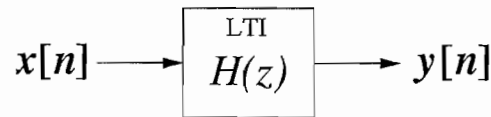
The fundamental period of $H_d(e^{j\omega})$ is given by $H_a(\frac{\omega}{T})$:

$$H_d(e^{j\omega}) = \frac{\Omega_c^4}{(44100j\omega - \Omega_c e^{j5\pi/8})(44100j\omega - \Omega_c e^{j7\pi/8})(44100j\omega - \Omega_c e^{j9\pi/8})(44100j\omega - \Omega_c e^{j11\pi/8})}, \quad |\omega| < \pi$$

For $|\omega| > \pi$, $H_d(e^{j\omega})$ periodically repeats; i.e.,

$$H_d(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(\frac{\omega - 2\pi k}{T}\right) \dots$$

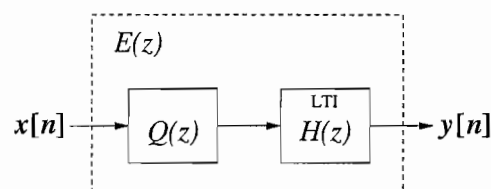
3. Consider the non-ideal digital communications channel H shown below.



It is determined by experiment that this channel can be modeled as a **causal IIR LTI** system with transfer function

$$H(z) = \frac{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{3}z^{-1})(1 + \frac{5}{3}z^{-1})}{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})}$$

Design a digital pre-equalizer $Q(z)$ to go in series with the channel as shown below



so that the overall *equalized* channel $E(z)$ is **allpass**.

It is required for your pre-equalizer $Q(z)$ to be both **causal** and **stable** and to have **minimum group delay**.

We would like to make the equalizer $\frac{1}{H(z)}$, but we can't do that: $H(z)$ has zeros at $\frac{3}{2}$ and $\frac{5}{3}$ outside the unit circle. So we can't make $\frac{1}{H(z)}$ be both causal and stable. So we need to find $H_{min}(z)$ such that $|H_{min}(z)| = |H(z)|$, but where $H_{min}(z)$ has minimum phase. Then we can take $Q(z) = \frac{1}{H_{min}(z)}$ and the overall $E(z)$ will be allpass, as required.

→ The first step is to factor $H(z)$ into a product

$$H(z) = H_{min}(z) H_{ap}(z)$$

where $H_{min}(z)$ has minimum phase and $H_{ap}(z)$ is allpass,

More Workspace for Problem 3...

$$H(z) = \frac{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{3}z^{-1})(1 + \frac{5}{3}z^{-1})}{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})}$$

$$= \frac{(1 + \frac{1}{3}z^{-1})}{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})} \cdot (1 - \frac{3}{2}z^{-1}) \frac{z^{-1} - \frac{3}{2}}{z^{-1} - \frac{3}{2}} \cdot (1 + \frac{5}{3}z^{-1}) \frac{z^{-1} + \frac{5}{3}}{z^{-1} + \frac{5}{3}}$$

$$= \frac{(1 + \frac{1}{3}z^{-1})(z^{-1} - \frac{3}{2})(z^{-1} + \frac{5}{3})}{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})} \cdot \frac{1 - \frac{3}{2}z^{-1}}{z^{-1} - \frac{3}{2}} \cdot \frac{1 + \frac{5}{3}z^{-1}}{z^{-1} + \frac{5}{3}}$$

$$= \frac{(1 + \frac{1}{3}z^{-1})(-\frac{3}{2})(1 - \frac{2}{3}z^{-1})(\frac{5}{3})(1 + \frac{3}{5}z^{-1})}{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})} \cdot \frac{(1 - \frac{3}{2}z^{-1})(1 + \frac{5}{3}z^{-1})}{(z^{-1} - \frac{3}{2})(z^{-1} + \frac{5}{3})}$$

$$= \underbrace{\left(-\frac{5}{2}\right) \frac{(1 + \frac{1}{3}z^{-1})(1 - \frac{2}{3}z^{-1})(1 + \frac{3}{5}z^{-1})}{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})}}_{H_{\min}(z) \rightarrow \text{minimum phase}} \cdot \underbrace{\frac{(1 - \frac{3}{2}z^{-1})(1 + \frac{5}{3}z^{-1})}{(z^{-1} - \frac{3}{2})(z^{-1} + \frac{5}{3})}}_{H_{\text{ap}}(z) \rightarrow \text{ALLPASS}}$$

$H_{\min}(z) \rightarrow \text{minimum phase}$

$H_{\text{ap}}(z) \rightarrow \text{ALLPASS}$

$$Q(z) = \frac{1}{H_{\min}(z)}$$

$$Q(z) = \left(-\frac{2}{5}\right) \frac{(1 - \frac{1}{2}z^{-1})^2(1 - \frac{1}{4}z^{-1})}{(1 + \frac{1}{3}z^{-1})(1 - \frac{2}{3}z^{-1})(1 + \frac{3}{5}z^{-1})}$$

4. Consider a causal, stable, minimum phase digital filter

$$Q(z) = \left(-\frac{2}{5}\right) \frac{(1 - \frac{1}{2}z^{-1})^2 (1 - \frac{1}{4}z^{-1})}{(1 + \frac{1}{3}z^{-1})(1 - \frac{2}{3}z^{-1})(1 + \frac{3}{5}z^{-1})}$$

Give Direct Form I and Direct Form II realizations for $Q(z)$.

$$\begin{aligned} Q(z) &= \left(-\frac{2}{5}\right) \frac{(1 - z^{-1} + \frac{1}{4}z^{-2})(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2})(1 + \frac{3}{5}z^{-1})} \\ &= \left(-\frac{2}{5}\right) \frac{1 - z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{16}z^{-3}}{1 - \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2} + \frac{3}{5}z^{-1} - \frac{1}{5}z^{-2} - \frac{6}{45}z^{-3}} \\ &= \left(-\frac{2}{5}\right) \frac{1 - \frac{5}{4}z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{16}z^{-3}}{1 + \frac{4}{15}z^{-1} - \frac{19}{45}z^{-2} - \frac{2}{15}z^{-3}} \\ &= \frac{-\frac{2}{5} + \frac{1}{2}z^{-1} - \frac{1}{5}z^{-2} + \frac{1}{40}z^{-3}}{1 + \frac{4}{15}z^{-1} - \frac{19}{45}z^{-2} - \frac{2}{15}z^{-3}} \end{aligned}$$

The input of $Q(z)$ is $x[n]$. Let's call the output $w[n]$:

$$x[n] \rightarrow \boxed{Q(z)} \rightarrow w[n]$$

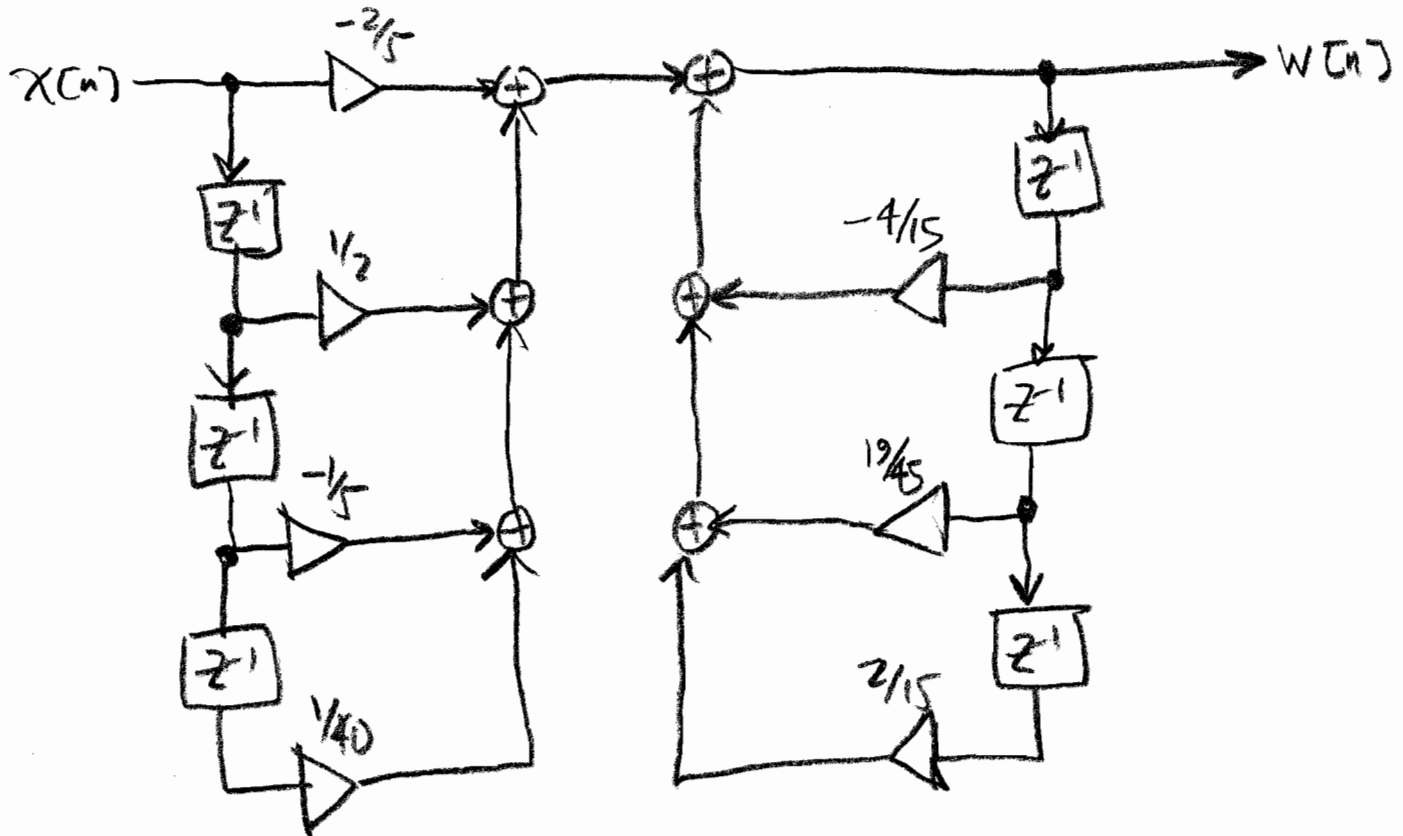
$$\text{Then } Q(z) = \frac{W(z)}{X(z)} = \frac{-\frac{2}{5} + \frac{1}{2}z^{-1} - \frac{1}{5}z^{-2} + \frac{1}{40}z^{-3}}{1 + \frac{4}{15}z^{-1} - \frac{19}{45}z^{-2} - \frac{2}{15}z^{-3}}$$

$$W(z) \left[1 + \frac{4}{15}z^{-1} - \frac{19}{45}z^{-2} - \frac{2}{15}z^{-3}\right] = X(z) \left[-\frac{2}{5} + \frac{1}{2}z^{-1} - \frac{1}{5}z^{-2} + \frac{1}{40}z^{-3}\right]$$

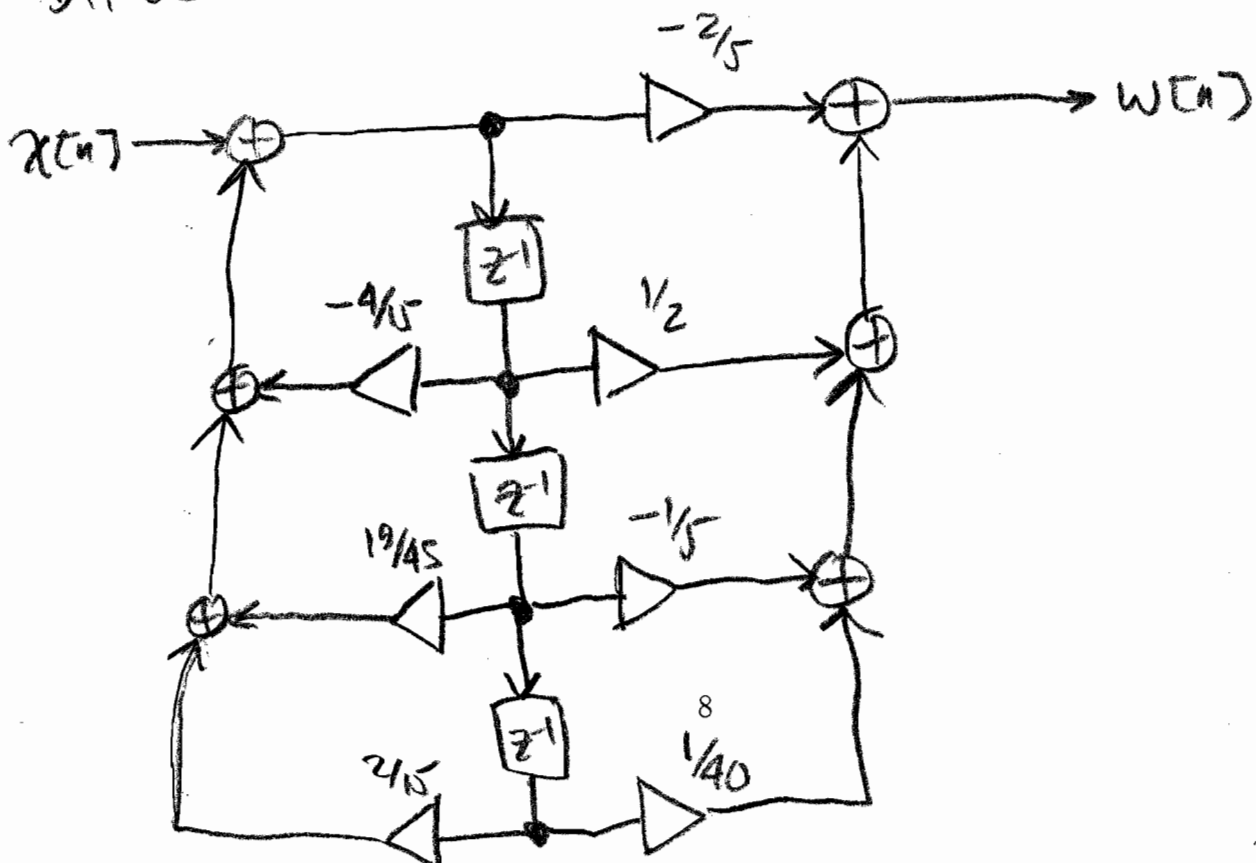
$$w[n] + \frac{4}{15}w[n-1] - \frac{19}{45}w[n-2] - \frac{2}{15}w[n-3] = -\frac{2}{5}x[n] + \frac{1}{2}x[n-1] - \frac{1}{5}x[n-2] + \frac{1}{40}x[n-3]$$



DIRECT FORM I :



DIRECT FORM II :



5. An FIR linear phase filter has a **real-valued** impulse response $h[n]$ that is nonzero only for $0 \leq n \leq 6$. In addition, it is known that $h[0] = 1$.

The transfer function $H(z)$ has a complex zero at $z = 0.4e^{-j\pi/3}$ and a real zero at $z = 3$. **Note:** these are not the *only* zeros.

Find $H(z)$. $H(z) = \sum_{n=0}^6 h[n]z^{-n} = 1 + h[1]z^{-1} + h[2]z^{-2} + \dots + h[6]z^{-6}$

$\rightarrow H(z)$ is a 6th order polynomial in z^{-1} , so it has six zeros.

so $H(z) = A \prod_{k=1}^6 (1 - z_k z^{-1})$ where z_k are the zeros and

A is a constant that gives $h[0] = 1$.

$z_1 = \frac{2}{5}e^{-j\pi/3}$ is given. Since $h[n]$ is real, we must also have a zero at $z_2 = z_1^* = \frac{2}{5}e^{+j\pi/3}$. Since $H(e^{j\omega})$ is linear phase, we must also have zeros at $z_3 = z_1^{-1} = \frac{5}{2}e^{j\pi/3}$ and $z_4 = z_2^{-1} = \frac{5}{2}e^{-j\pi/3}$.

The zero at $z_5 = 3$ is given, and b/c $H(e^{j\omega})$ is linear phase we also have $z_6 = z_5^{-1} = \frac{1}{3}$. All together we have

$$\begin{aligned} H(z) &= A(1 - \frac{2}{5}e^{-j\pi/3}z^{-1})(1 - \frac{2}{5}e^{j\pi/3}z^{-1})(1 - \frac{5}{2}e^{j\pi/3}z^{-1})(1 - \frac{5}{2}e^{-j\pi/3}z^{-1})(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1}) \\ &= A(1 - \frac{4}{5}\cos\frac{\pi}{3}z^{-1} + \frac{4}{25}z^{-2})(1 - 5\cos\frac{\pi}{3}z^{-1} + \frac{25}{4}z^{-2})(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1}) \\ &= A(1 - \frac{2}{5}z^{-1} + \frac{4}{25}z^{-2})(1 - \frac{5}{2}z^{-1} + \frac{25}{4}z^{-2})(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1}) \end{aligned}$$

\Rightarrow Clearly, if this is multiplied out, the coefficient of z^0 will be A . Since this coefficient is $h[0]$, we have $A = 1$.

$$H(z) = (1 - \frac{2}{5}z^{-1} + \frac{4}{25}z^{-2})(1 - \frac{5}{2}z^{-1} + \frac{25}{4}z^{-2})(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1})$$