ECE 4213/5213
Test 2

Tuesday, November 27, 2007
12:00 PM - 1:15 PM

Fall 2007 /

Name: SOLUTION

Dr. Havlicek

Student Num:

Directions: This test is open book. You may also use "clean" copies of the course notes and a calculator, as well as a "clean" copy of the formula sheet from the course web site. Other materials are not allowed. You have 75 minutes to complete the test. All work must be your own.

Students enrolled for undergraduate credit: work any four problems. Each problem counts 25 points.

Students enrolled for graduate credit: work all five problems. Each problem counts 20 points.

SHOW ALL OF YOUR WORK for maximum partial credit! **GOOD LUCK!**

SCORE:

1. (25/20) ________
2. (25/20) ________
3. (25/20) ________
4. (25/20) ________
5. (25/20) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________ Date: ________

1. A causal discrete-time LTI system H has input-output relation

$$y[n] - 7y[n-1] + 12y[n-2] = x[n] + 5x[n-1].$$

Find a new transfer function $G(z)$ such that

(a) $|G(e^{j\omega})| = |H(e^{j\omega})| \ \forall\, \omega \in \mathbb{R}$.

(b) The system G has minimum phase.

(c) The system G is both causal and stable.

$$Y(z)\left[1 - 7z^{-1} + 12z^{-2}\right] = X(z)\left[1 + 5z^{-1}\right]$$

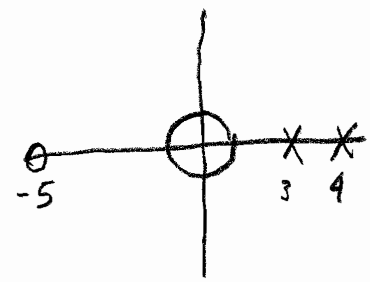
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+5z^{-1}}{1-7z^{-1}+12z^{-2}} = \frac{1+5z^{-1}}{(1-4z^{-1})(1-3z^{-1})}$$

one zero at $z = -5$

two poles at $z = 4$ and $z = 3$

→ All outside the unit circle:

- H is <u>not</u> minimum phase.
- H <u>cannot</u> be causal & stable.



→ Use three allpass sections to trade these for a new zero & two poles that are inside the unit circle.

$$H(z) = (1+5z^{-1}) \cdot 1 \cdot \frac{1}{1-4z^{-1}} \cdot 1 \cdot \frac{1}{1-3z^{-1}} \cdot 1$$

$$= (1+5z^{-1})\,\frac{5+z^{-1}}{5+z^{-1}} \cdot \frac{1}{1-4z^{-1}} \cdot \frac{-4+z^{-1}}{-4+z^{-1}} \cdot \frac{1}{1-3z^{-1}} \cdot \frac{-3+z^{-1}}{-3+z^{-1}}$$

→

More Workspace for Problem 1...

$$\ldots H(z) = (5+z^{-1}) \underbrace{\frac{1+5z^{-1}}{5+z^{-1}}}_{\text{All Pass}} \frac{1}{-4+z^{-1}} \underbrace{\frac{-4+z^{-1}}{1-4z^{-1}}}_{\text{All pass}} \frac{1}{-3+z^{-1}} \underbrace{\frac{-3+z^{-1}}{1-3z^{-1}}}_{\text{All Pass}}$$

$$= \underbrace{\frac{5+z^{-1}}{(-4+z^{-1})(-3+z^{-1})}}_{G(z)} \underbrace{\left[\frac{1+5z^{-1}}{5+z^{-1}} \cdot \frac{-4+z^{-1}}{1-4z^{-1}} \cdot \frac{-3+z^{-1}}{1-3z^{-1}} \right]}_{\text{Allpass}} \quad (*)$$

$$\boxed{G(z) = \frac{5+z^{-1}}{(4-z^{-1})(3-z^{-1})} = \frac{5}{12} \frac{1+\frac{1}{5}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}}$$

Check:

- It follows from $(*)$ that $|G(e^{j\omega})| = |H(e^{j\omega})| \quad \forall \omega \in \mathbb{R}$ ✓
- $G(z)$ has one zero at $z = -\frac{1}{5}$
  two poles at $z = \frac{1}{4}$ and $z = \frac{1}{3}$

  → All inside the unit circle

  → G is minimum phase ✓

  → G can be causal & stable ✓

2. A causal discrete-time system G has transfer function

$$G(z) = \frac{5 + z^{-1}}{(3 - z^{-1})(4 - z^{-1})}.$$

Give a Parallel Form I realization of $G(z)$.

PFE in z^{-1};

$$\frac{5+\theta}{(3-\theta)(4-\theta)} = \frac{A}{3-\theta} + \frac{B}{4-\theta}$$

$$A = \left.\frac{5+\theta}{4-\theta}\right|_{\theta=3} = \frac{8}{1} = 8$$

$$B = \left.\frac{5+\theta}{3-\theta}\right|_{\theta=4} = \frac{9}{-1} = -9$$

$$\underline{\underline{G(z) = \frac{8}{3-z^{-1}} - \frac{9}{4-z^{-1}}}}$$

$$= \frac{8/3}{1-\frac{1}{3}z^{-1}} + \frac{-9/4}{1-\frac{1}{4}z^{-1}}$$

$$= \sum_{l=1}^{2} \frac{\rho_l}{1-\lambda_l z^{-1}}$$ (6.39) p. 315 of text

$\rho_1 = 8/3$ $\qquad$ $\rho_2 = -9/4$

$\lambda_1 = \frac{1}{3}$ $\qquad$ $\lambda_2 = \frac{1}{4}$

$\longrightarrow$

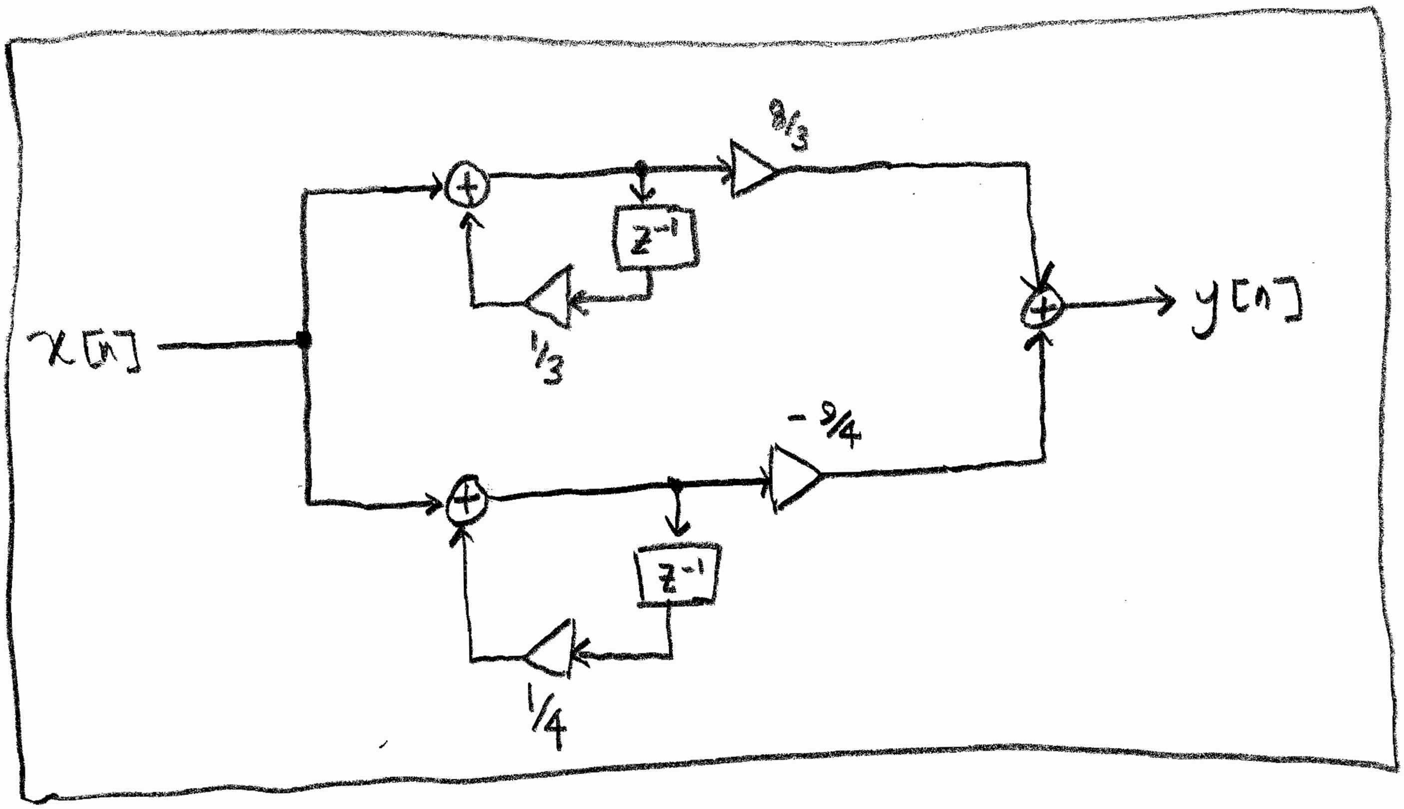
More Workspace for Problem 2...

In terms of (8.30) on p. 442 of the text, we have

$$G(z) = \gamma_0 + \sum_{k=1}^{2} \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

$\gamma_0 = 0$ | $\gamma_{01} = 8/3$ $\gamma_{11} = 0$ | $\gamma_{02} = -9/4$ $\gamma_{12} = 0$

| $\alpha_{11} = -\frac{1}{3}$ $\alpha_{21} = 0$ | $\alpha_{12} = -\frac{1}{4}$ $\alpha_{22} = 0$



(This follows from Fig. 8.20(a) on p. 442 of the text)

3. Use the bilinear transform to design a lowpass digital Butterworth filter with the following specifications:

3 dB cutoff freq.	$\omega_c = 0.2\pi$ rad/sample
passband edge freq.	$\omega_p = \pi/8$ rad/sample
stopband edge freq.	$\omega_s = 2\pi/5$ rad/sample
max. passband ripple	$1/\sqrt{1+\epsilon^2} = 0.9$
min. stopband atten.	$1/A = 0.2$

Be sure to give explicit expressions for both $H_a(s)$ and $H(z)$.

$$\frac{1}{\sqrt{1+\varepsilon^2}} = .9 = \frac{9}{10} \Rightarrow \sqrt{1+\varepsilon^2} = \frac{10}{9} \Rightarrow 1+\varepsilon^2 = \frac{100}{81}$$

calculator register

$$\Rightarrow \varepsilon^2 = \frac{100}{81} - \frac{81}{81} = \frac{19}{81} = 0.234568 \quad \boxed{0}$$

$$\frac{1}{A} = .2 = \frac{2}{10} = \frac{1}{5} \Rightarrow A = 5 \Rightarrow A^2 = 25$$

Prewarp the Critical Frequencies using (9.18) on p. 494 with $T=2$ as suggested just above (9.33) on p. 499:

$$\Omega_c = \tan\left(\frac{\omega_c}{2}\right) = \tan(.1\pi) = 0.324920 \quad \boxed{1}$$

$$\Omega_p = \tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{\pi}{16}\right) = 0.198912 \quad \boxed{2}$$

$$\Omega_s = \tan\left(\frac{\omega_s}{2}\right) = \tan(.2\pi) = 0.726543 \quad \boxed{3}$$

Estimate the order N using (4.35) on p. 190 of the text:

$$N = \left\lceil \frac{1}{2} \frac{\log_{10}[(A^2-1)/\varepsilon^2]}{\log_{10}(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{1}{2} \frac{\log_{10}[24/0.234568]}{\log_{10}\left(\frac{0.726543}{0.198912}\right)} \right\rceil$$

$$= \lceil 1.78630 \rceil = 2 \Rightarrow \boxed{N=2}$$

From (4.37) on p. 190: $p_\ell = \Omega_c \exp\{j[\pi(1+2\ell)/4]\}$ $\ell = 1,2$

More Workspace for Problem 3...

Again from (4.37): $p_1 = \Omega_c e^{j3\pi/4}$, $p_2 = \Omega_c e^{j5\pi/4}$

(4.36) on p. 190: $H_a(s) = \dfrac{\Omega_c^2}{(s-p_1)(s-p_2)} = \dfrac{\Omega_c^2}{(s-\Omega_c e^{j3\pi/4})(s-\Omega_c e^{j5\pi/4})}$

$$H_a(s) = \frac{\Omega_c^2}{s^2 - \Omega_c(e^{j3\pi/4}+e^{j5\pi/4})s + \Omega_c^2 \underbrace{e^{j8\pi/4}}_{1}}$$

$$= \frac{\Omega_c^2}{s^2 - \Omega_c\left(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)s + \Omega_c^2} = \frac{\Omega_c^2}{s^2+\sqrt{2}\Omega_c s + \Omega_c^2}$$

$$\boxed{H_a(s) = \frac{\Omega_c^2}{s^2+\sqrt{2}\Omega_c s+\Omega_c^2} = \frac{0.105573}{s^2+0.459506s+0.105573}}$$

From (9.15) on p. 494 w/ T=2 as in (9.33) on p. 499:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\Omega_c^2}{\frac{(1-z^{-1})^2}{(1+z^{-1})^2} + \sqrt{2}\Omega_c\frac{1-z^{-1}}{1+z^{-1}} + \Omega_c^2}$$

$$= \frac{(1+z^{-1})^2\Omega_c^2}{(1-z^{-1})^2 + \sqrt{2}\Omega_c(1+z^{-1})(1-z^{-1}) + \Omega_c^2(1+z^{-1})^2}$$

$$= \frac{(1+2z^{-1}+z^{-2})\Omega_c^2}{(1-2z^{-1}+z^{-2}) + \sqrt{2}\Omega_c(1-z^{-2}) + \Omega_c^2(1+2z^{-1}+z^{-2})}$$

$$= \frac{\Omega_c^2 + 2\Omega_c^2 z^{-1} + \Omega_c^2 z^{-2}}{[1+\sqrt{2}\Omega_c+\Omega_c^2] + [-2+2\Omega_c^2]z^{-1} + [1-\sqrt{2}\Omega_c+\Omega_c^2]z^{-2}}$$

$$= \frac{0.105573 + 0.211146z^{-1} + 0.105573z^{-2}}{1.56508 - 1.78885z^{-1} + 0.646067z^{-2}}$$

$$\boxed{H(z) = \frac{0.0674553 + 0.134911z^{-1} + 0.0674553z^{-2}}{1 - 1.14298z^{-1} + 0.412802z^{-2}}}$$

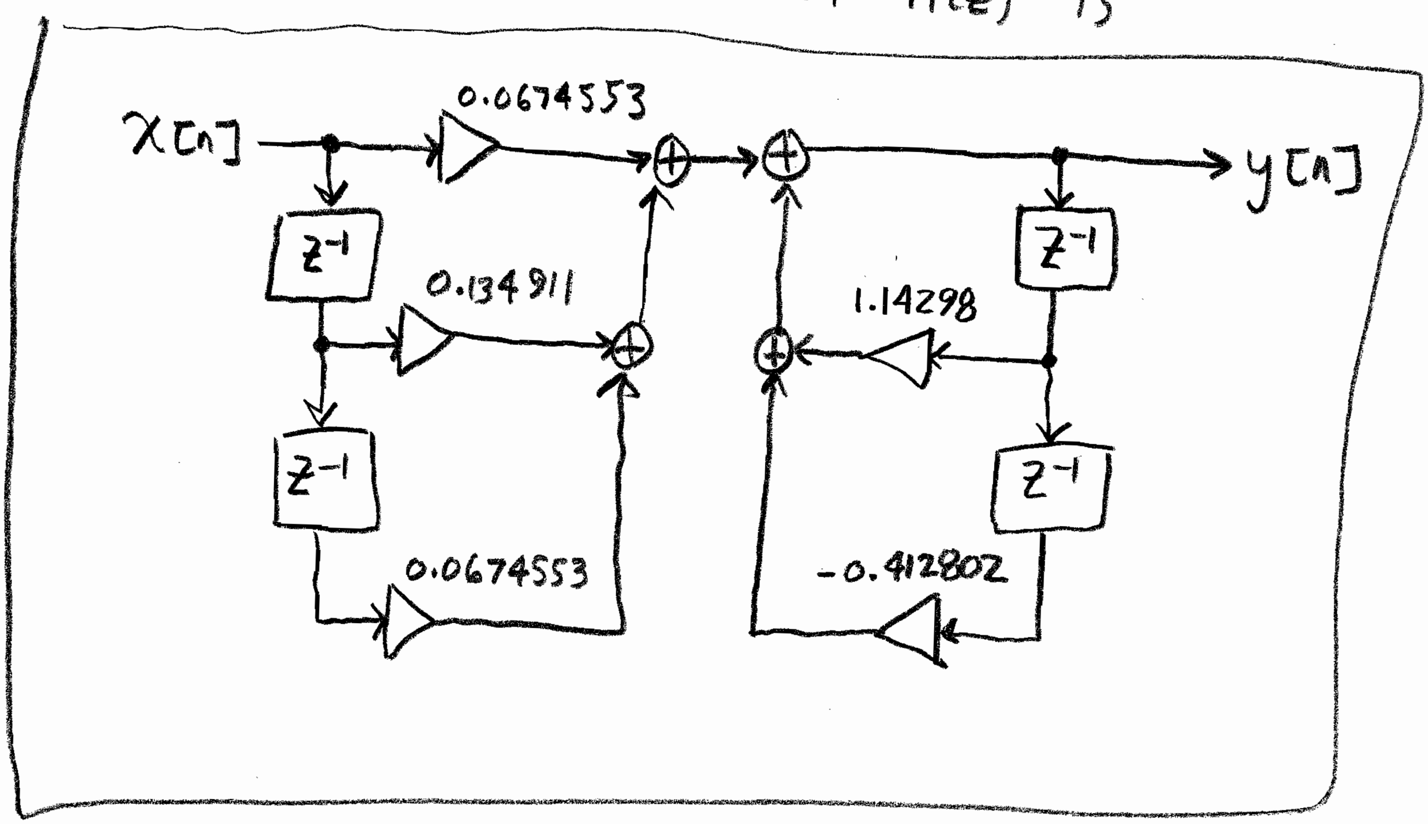
4. Give Direct Form I and Direct Form II realizations for the digital filter $H(z)$ designed in Problem 3.

Comparing the solution to P3 on p. 7 to (8.23) on p. 437 of the text, we have:

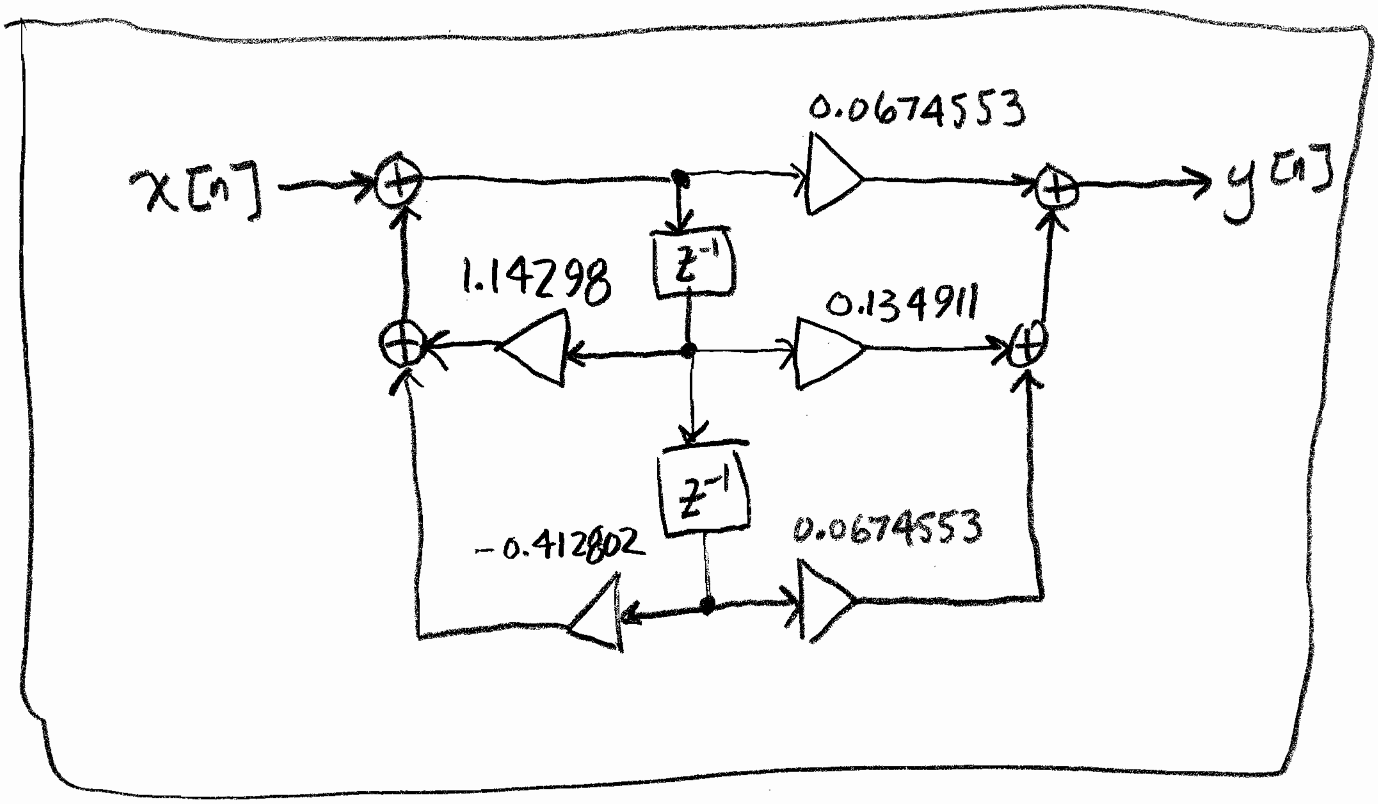
$p_0 = 0.0674553$ $\quad p_1 = 0.134911$ $\quad p_2 = p_0 = 0.0674553$

$d_1 = -1.14298$ $\quad d_2 = 0.412802$

It follows from Fig. 8.13(a) on p. 438 of the text that the Direct form I realization of $H(z)$ is



With $p_0, p_1, p_2, d_1,$ and d_2 as on the previous page, it follows from Fig. 8.14(a) on p. 439 of the text that the Direct form II realization of $H(z)$ is



5. Use the window design method with an appropriate **fixed** window from Table 10.2 (page 535 of the text) to design a lowpass FIR digital filter with the following specifications:

passband edge freq.	$\omega_p = 0.3\pi$ rad/sample
stopband edge freq.	$\omega_s = 0.35\pi$ rad/sample
max. passband ripple	$\delta_p = 0.01$
max. stopband ripple.	$\delta_s = 0.01$

Give the filter impulse response $h[n]$.

(4.30) on p. 187: $\alpha_s = -20\log_{10}\delta_s = 40$ dB (minimum stopband attenuation)

$\Delta\omega = \omega_s - \omega_p = 0.35\pi - 0.3\pi = 0.05\pi$ (transition bandwidth)

- From Table 10.2, the stopband attenuation spec can be met by a Hann, Hamming, or Blackman window.
- From the last column of Table 10.2, estimate the required order for each type of window:

Hann: $\Delta\omega = 0.05\pi = \frac{3.11\pi}{M} \Rightarrow M = \frac{3.11}{0.05} = 62.2$

order $= \lceil 2M \rceil = 125$

length $= \lceil 2M+1 \rceil = 126$

Hamming: $\Delta\omega = 0.05\pi = \frac{3.32\pi}{M} \Rightarrow M = \frac{3.32}{0.05} = 66.4$

order $= \lceil 2M \rceil = 133$

length $= \lceil 2M+1 \rceil = 134$

Blackman: $\Delta\omega = 0.05\pi = \frac{5.56\pi}{M} \Rightarrow M = \frac{5.56}{0.05} = 111.2$

order $= \lceil 2M \rceil = 223$

length $= \lceil 2M+1 \rceil = 224$

☆ Hann window is the best choice because it meets the spec with the lowest order. USE HANN.

→ Since the window formulas given on p. 533 of the book are only for an odd length $2M+1$ where $M \in \mathbb{Z}$, we must round up to $M = 63$. This gives a length of $2M+1 = 127$.

From (10.30) on p. 533, the zero-phase window is

$$w[n] = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi n}{127}\right)\right] \qquad -63 \le n \le 63.$$

From the text on p. 536, we set the cutoff frequency at

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.65\pi}{2} = 0.325\pi$$

From (10.14) on p. 528, the zero phase ideal impulse response is

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\sin(0.325\pi n)}{\pi n}$$

So the zero phase windowed impulse response is

$$h_{LP}[n]\,w[n] = \frac{\sin(0.325\pi n)}{2\pi n}\left[1 + \cos\left(\frac{2\pi n}{127}\right)\right]$$

$$-63 \le n \le 63.$$

Finally, the causal impulse response is obtained by shifting this right by 63 samples:

$$\boxed{h[n] = \frac{\sin[0.325\pi(n-63)]}{2\pi(n-63)}\left\{1 + \cos\left[\frac{2\pi(n-63)}{127}\right]\right\} \qquad 0 \le n \le 127}$$